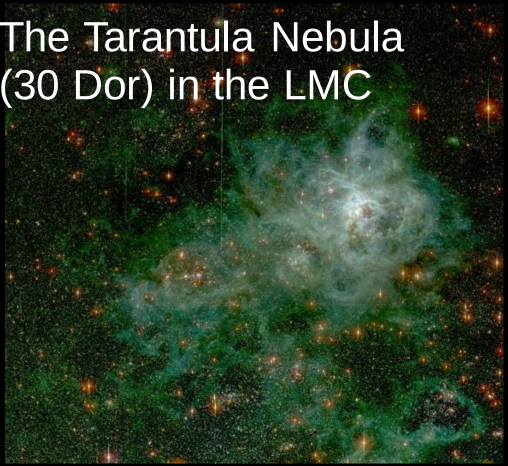
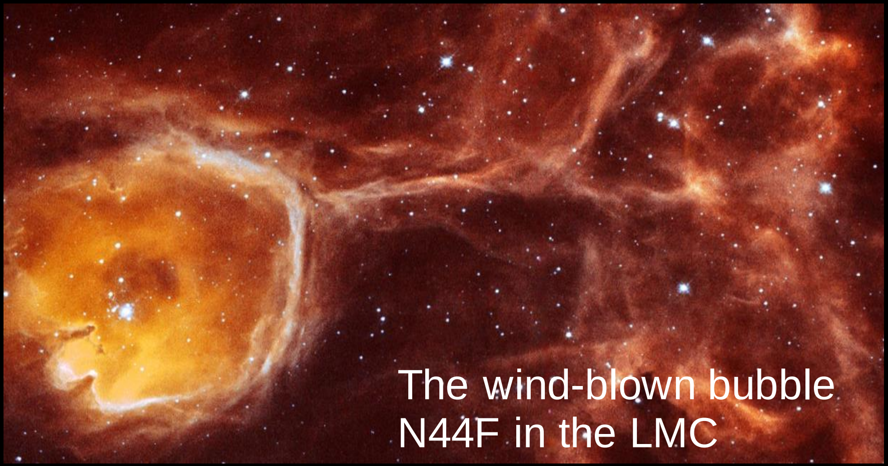
Radiative processes, stellar atmospheres and winds

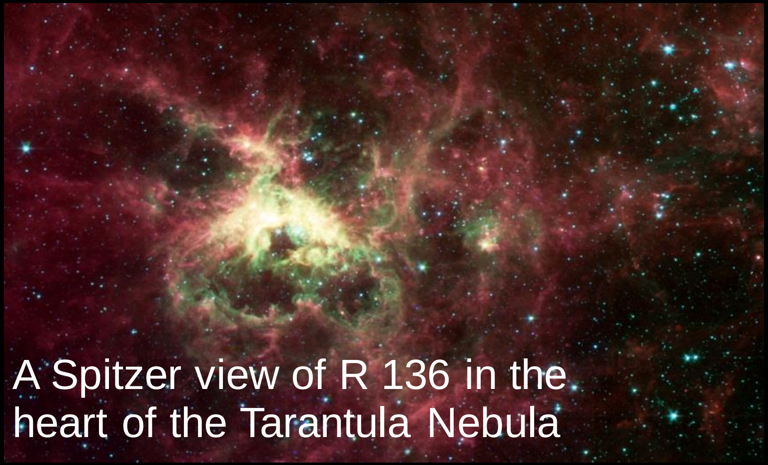
Master of Science in Astrophysics – P5.0.2
Master of Science in Physics with main focus on Astrophysics – P4.0.5, P5.2.5, P6.0.5



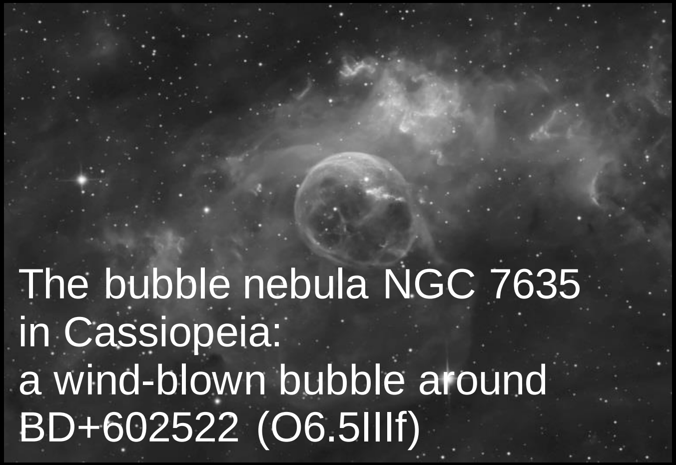
The Tarantula Nebula (30 Dor) in the LMC



The wind-blown bubble N44F in the LMC



A Spitzer view of R 136 in the heart of the Tarantula Nebula



The bubble nebula NGC 7635 in Cassiopeia:
a wind-blown bubble around BD+602522 (O6.5IIIf)

Joachim Puls, University observatory Munich (LMU)

Content

Part I

Part II

Literature

- Carroll, B.W., Ostlie, D.A., “An Introduction to Modern Astrophysics”, 2nd edition, Pearson International Edition, San Francisco, 2007, Chap. 3,5,8,9
- **Mihalas, D.**, "Stellar atmospheres"**,** 2nd edition, Freeman & Co., San Francisco, 1978
- **Hubeny, I., Mihalas, D.**, “Theory of Stellar Atmospheres”, Princeton Univ. Press, 2014
- (3rd edition - together with I. Hubeny – to appear in the near future)
- Unsöld, A., "Physik der Sternatmosphären", 2nd edition, Springer Verlag, Heidelberg, 1968
- Shu, F.H., "The physics of astrophysics, Volume I: radiation", University science books, Mill Valley, 1991
- Rybicki, G.B., Lightman, A., "Radiative Processes in Astrophysics", New York, Wiley, 1979
- Osterbrock, D.E., "Astrophysics of Gaseous Nebulae and Active Galactic Nuclei", University science books, Mill Valley, 1989
- Mihalas, D., Weibel Mihalas, B., “Foundations of Radiation Hydrodynamics”, Oxford University Press, New York, 1984
- Cercignani, C., "The Boltzmann Equation and Its Applications", Appl. Math. Sciences 67, Springer, 1987
- Kudritzki, R.-P., Hummer, D.G., "Quantitative spectroscopy of hot stars", Annual Review of Astronomy and Astrophysics, Vol. 28, p. 303, 1990
- Sobolev, V.V., “Moving envelopes of stars”, Cambridge: Harvard University Press, 1960
- Kudritzki, R.-P., Puls, J., "Winds from hot stars", Annual Review of Astronomy and Astrophysics, Vol. 38, p. 613, 2000
- Puls, J., Vink, J.S., Najarro, F., "Mass loss from hot massive stars", Astronomy & Astropyhsics Review 16, ISSUE 3, p. 209, Springer, 2008

Chap.1 – Prelude

- cosmology, galaxies, dark energy, dark matter, …

What are stars good for?

- … and who cares for radiative transfer and stellar atmospheres?

- remember
  - galaxies consist of stars (and gas, dust)
  - most of the (visible) light originates from stars
  - astronomical experiments are (mostly) observations of light:
    have to understand how it is created and transported

The cosmic circuit of matter

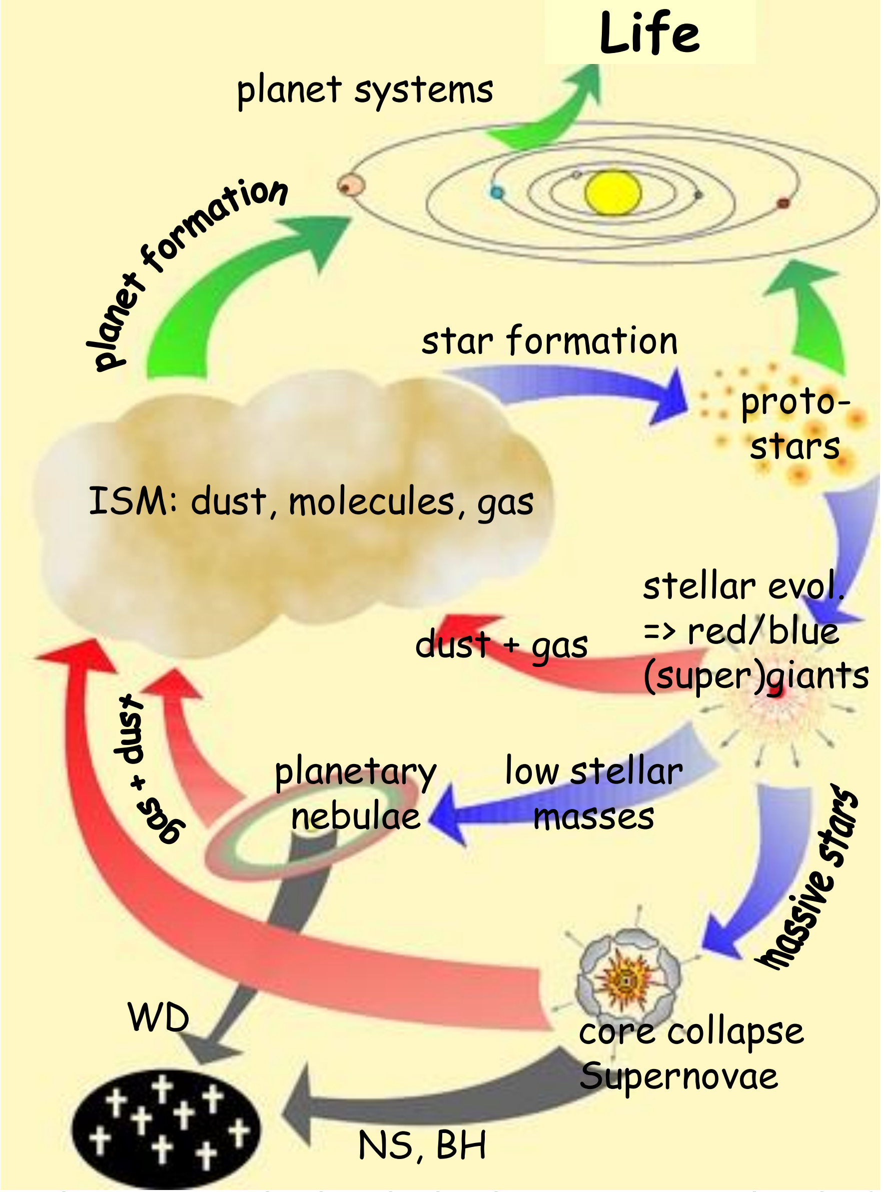
What are stars good for?

- Us!
- (whether this is *really* good, is another question...)

Joni Mitchell - Woodstock (1970!)

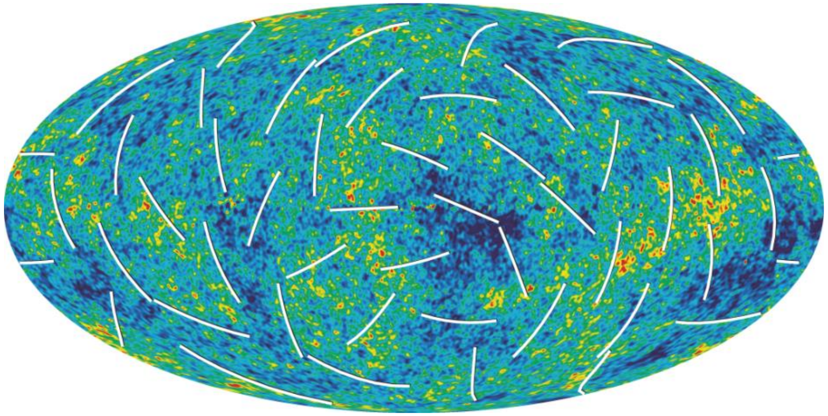
"… We are stardust

Billion year old carbon…"



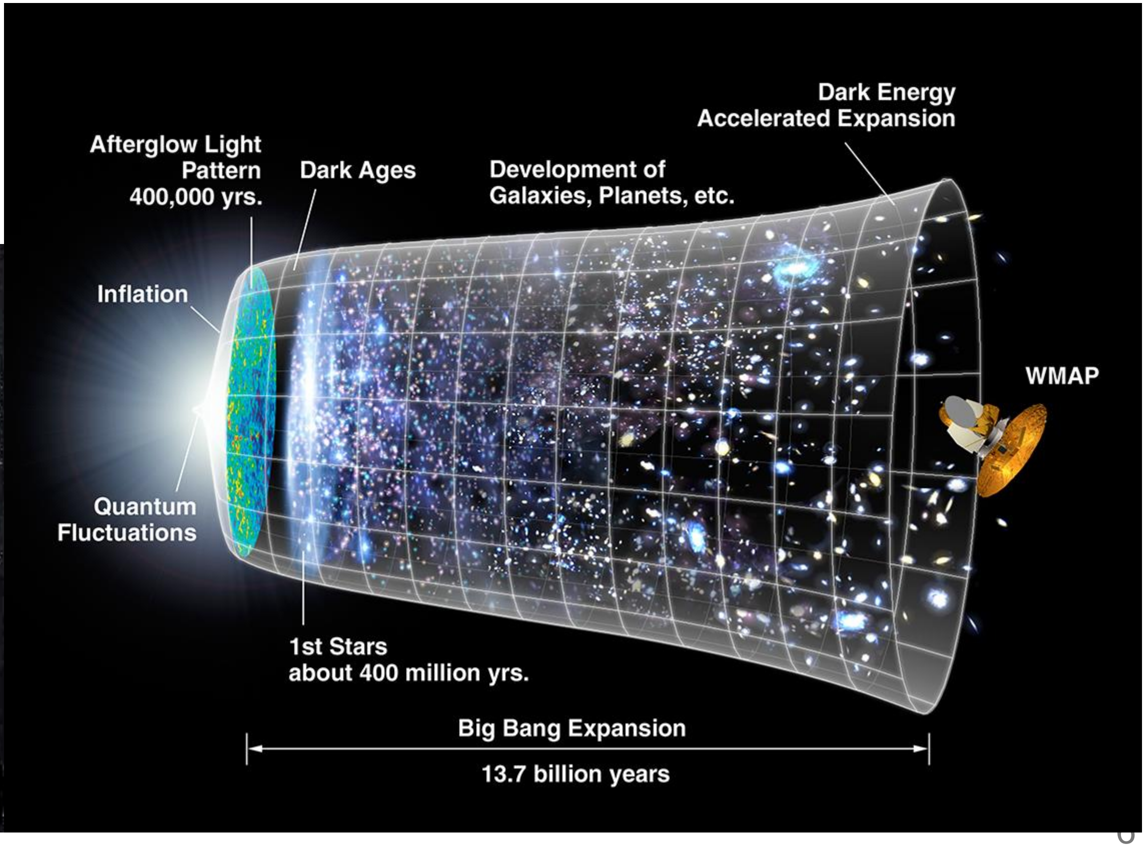
adapted from http://astro.physik.tu-berlin.de/~sonja/Materiekreislauf/index.html

First stars and reionization



credit: NASA/WMAP Science Team

WMAP = Wilkinson Microwave Anisotropy Probe
color coding: ΔT range ± 200 μK, $\Delta T/T \sim$ few 10^{-5}
=> “anisotropy” of last scattering surface (before recomb.)
white bars: polarization vector
⇒ CMB photons scattered at electrons (reionzed gas)
[NOTE: newer data from PLANCK]

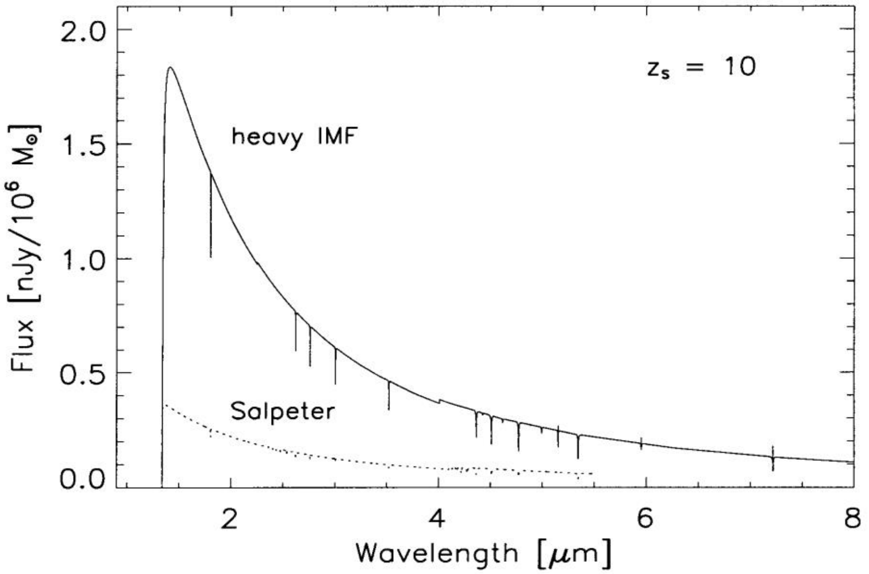


The first stars …

- begin of reionization:
    - z < 10, average redshift for reionization z=7.8 to 8.8 (from PLANCK, state 2016)
    - z ≈ 11 (from WMAP, polarization, assuming instantaneous reionization)
    - z ≈ 15 … 30 (modeling)
- complete (for hydrogen) at z ~ 6.0
- quasars alone not capable to reionize Universe at that high redshift (z > 6) , since rapid decline in space density for z > 3 (Madau et al.1999, ApJ 514, Fan et al. 2006, ARA&A 44)

Bromm et al. (2001, ApJ 552)

- (almost) metal free: Pop III
- very massive stars (VMS) with $1000 \, M_\odot > M > 100 \, M_\odot$
- hotter (≈ 10^5 K), more compact
- $L \propto M$, spectrum almost BB,
- large H/He ionizing fluxes: 10^{48} ($10^{47)}$ H (He) ionizing photons per second *and solar mass*
- assume that primordial IMF favours formation of VMS



IF heavy IMF,
then capable to reionize universe
(at least in a first step, cf. Cen 2003, ApJ 591)

see also

Abel et al. 2000, ApJ 540; Bromm et al. 2002, ApJ 564; Furnaletto & Loeb 2005, ApJ 634; Wise & Abel 2008, ApJ 684; Johnson et al. 2008, Proc IAU Symp 250 (review); Maio et al. 2009, A&A 503; Maio et al. 2010, MNRAS 407; Weber et al. 2013, A&A 555

… and many more publications

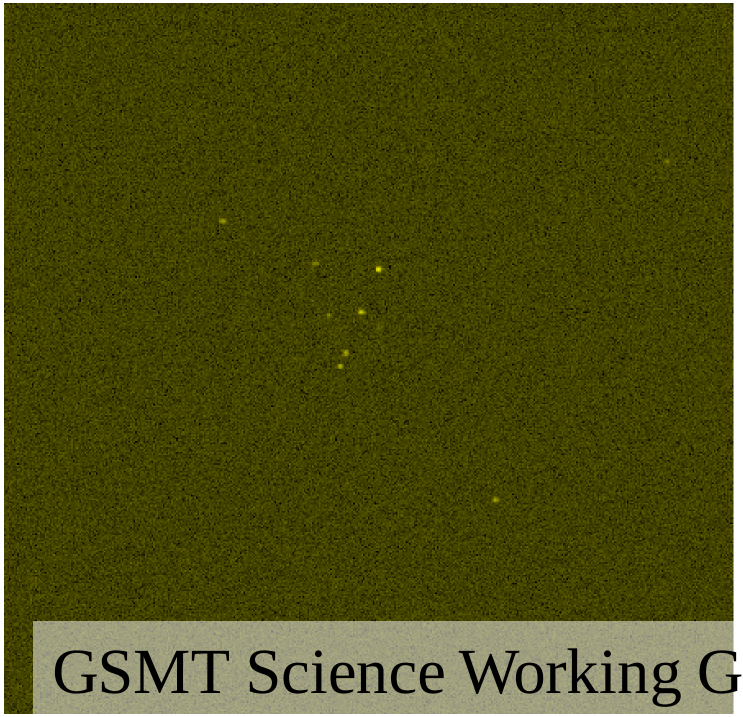
… might be observable in the NIR

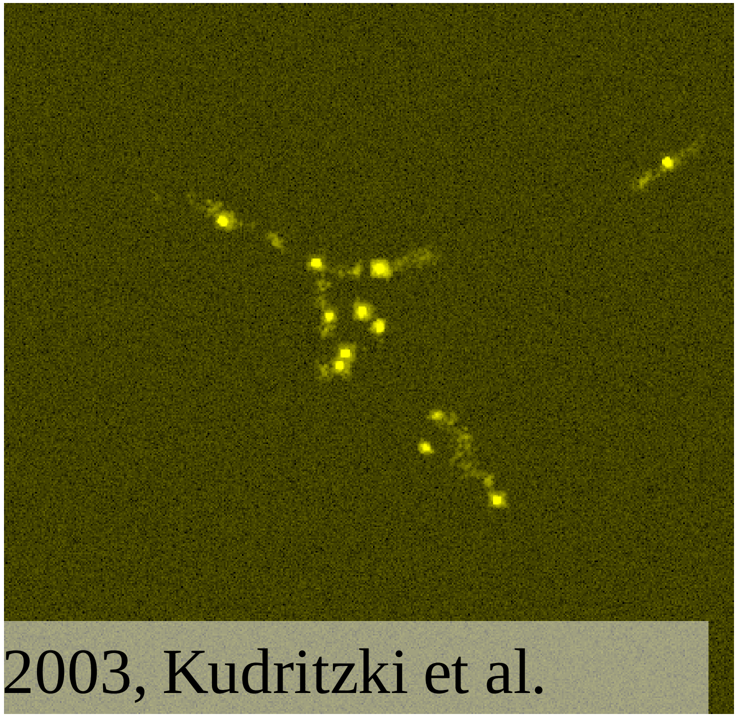
with a ≥ 30m telescope, e.g. via HeII λ1640 Å (strong ISM recomb. line)

Standard IMF

1 Mpc (comoving)

Heavy IMF, zero metallicity





GSMT Science Working Group Report, 2003, Kudritzki et al.

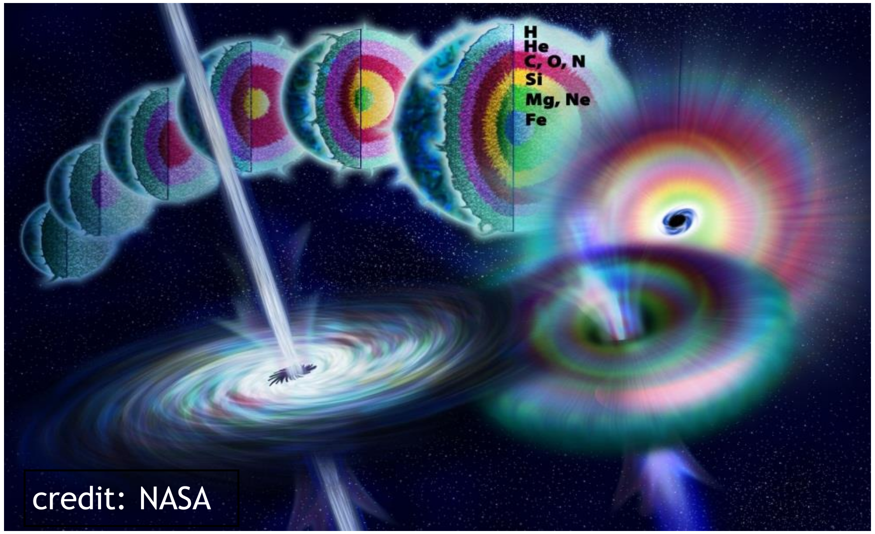
http://www.aura-nio.noao.edu/gsmt_swg/SWG_Report/SWG_Report_7.2.03.pdf

(Hydro-simulations by
Davé, Katz, & Weinberg)

As observed through 30-meter telescope
R=3000, 10^5 seconds (favourable conditions, see
also Barton et al., 2004, ApJ 604, L1)

Long Gamma Ray Bursts

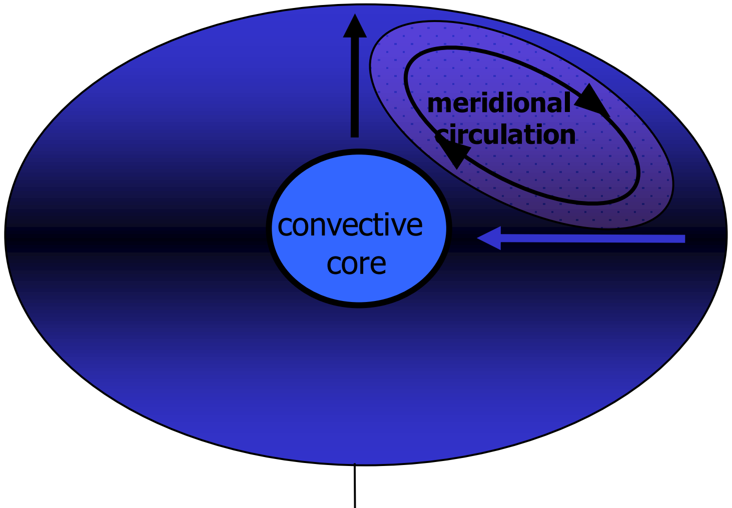
- long: >2s
- Collapsar: death of a massive star



credit: NASA

Collapsar Scenario for Long GRB (Woosley 1993)

- massive core (enough to produce a BH)
- removal of hydrogen envelope
- rapidly rotating core (enough to produce an accretion disk)



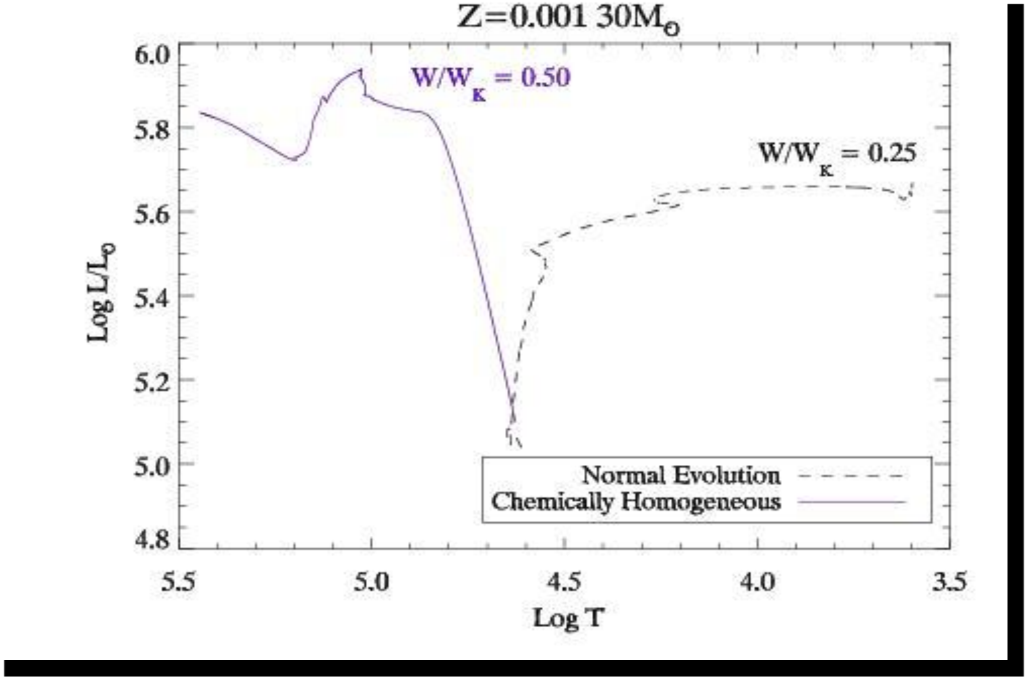
requires **chemically homogeneous evolution** of **rapidly rotating** massive star

- pole hotter than equator (von Zeipel)
- rotational mixing due to meridional circulation (Eddington-Sweet)

Chemically Homogeneous Evolution …

- …if rotational mixing during main sequence *faster than* built-up of chemical gradients due to nuclear fusion (*Maeder 1987*)
- bluewards evolution directly towards Wolf-Rayet phase (no RSG phase). Due to meridional circulation, envelope and core are mixed -> no hydrogen envelope
- since no RSG phase, higher angular momentum in the core (*Yoon & Langer 2005*)

W/W_k: rotational frequency in units of critical one

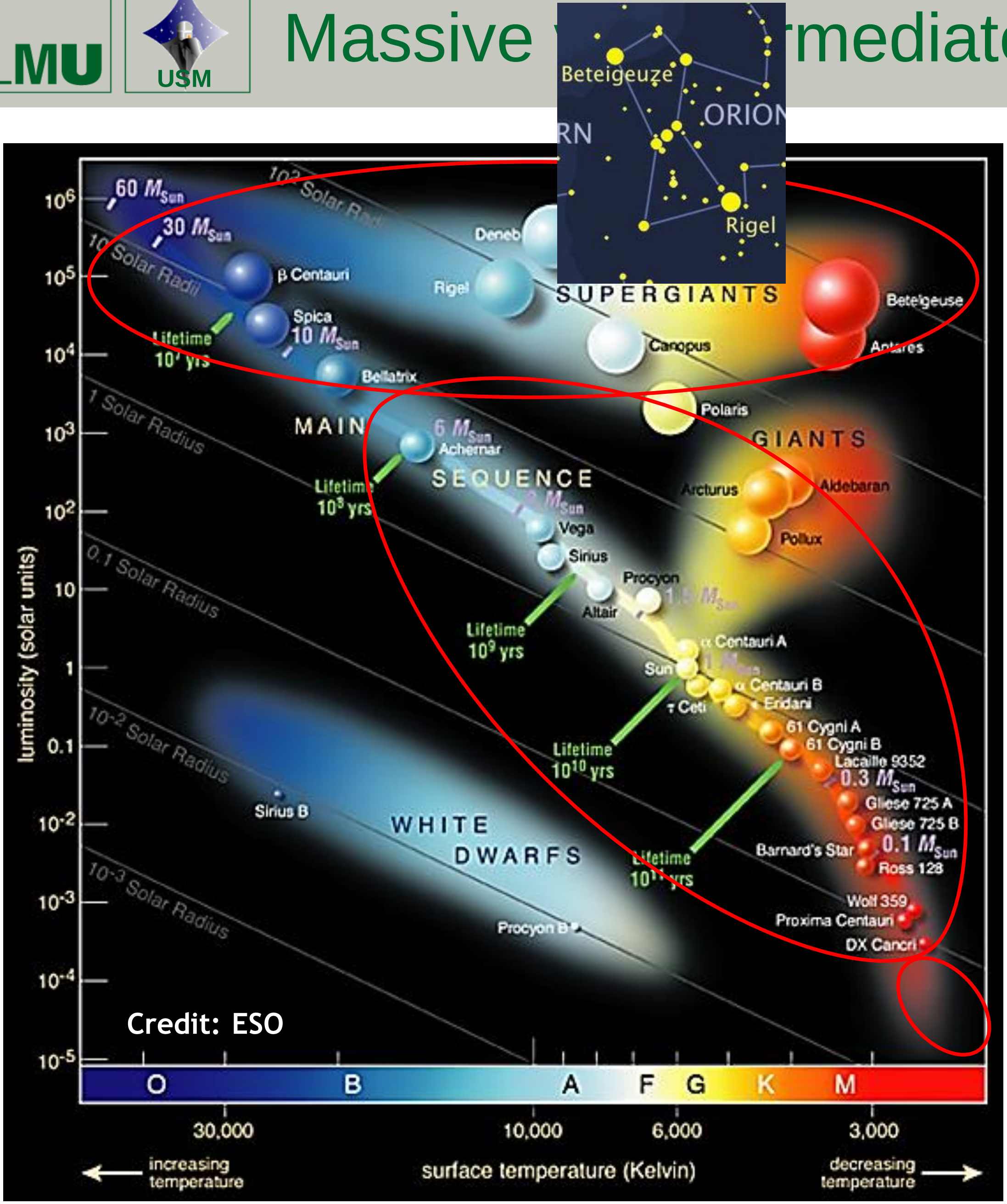


massive stars as progenitors of high redshift GRBs:

✓early work: Bromm & Loeb 2002, Ciardi & Loeb 2001, Kulkarni et al. 2000, Djorgovski et al. 2001, Lamb & Reichart 2000

✓At low metallicity stars are expected to be rotating faster because of weaker stellar winds

Massive v rmediate-/low-mass stars



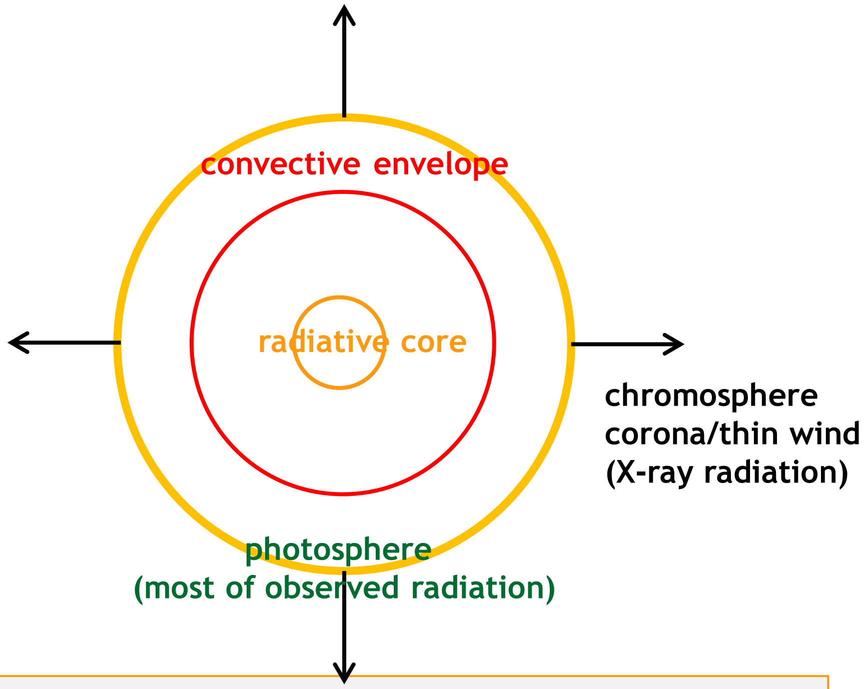
Credit: ESO

- massive stars ($M_{ZAMS} > 8\ M_{sun}$)
  - short life-times (few to 20 million years)
  - end products: core-collapse SNe (sometimes as slow GRBs) → neutron stars, black holes (or even complete disruption in case of pair-instability SNe)
- intermediate-/low-mass stars ($0.1…0.8\ M_{sun} < M_{ZAMS} < 8\ M_{sun}$)
  - long life-times (0.1 to 100 billion years)
  - end products: White dwarfs, SNIa
- brown dwarfs ($13\ M_{Jupiter} < M < 0.08\ M_{sun}$)
  - 'failed stars', core temperature not sufficient to ignite H-fusion
  - instead, Deuterium and, for higher masses, Lithium fusion

ZAMS: Zero Age Main Sequence
MS: Main sequence, core hydrogen burning

low-mass vs. massive star during the MS

low-mass star (e.g., the sun)



convective envelope

radiative core

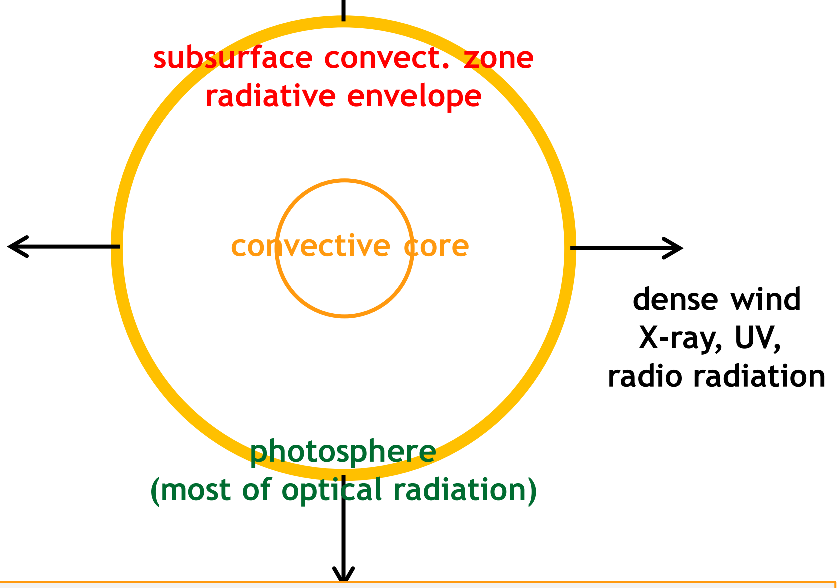
chromosphere
corona/thin wind
(X-ray radiation)

photosphere
(most of observed radiation)

- radiative core
- convective envelope
- geometrically thin photosphere
    - level populations collisionally dominated
      →Local thermodynamic equilibrium (LTE):
      Saha-Boltzmann population
- chromosphere (temp. begins to increase outwards)
- hot corona/thin wind (very low mass-loss rate)

massive star

factor 10...20 larger than sun
factor 10^4...10^6 more luminous than sun



subsurface convect. zone
radiative envelope

convective core

dense wind
X-ray, UV,
radio radiation

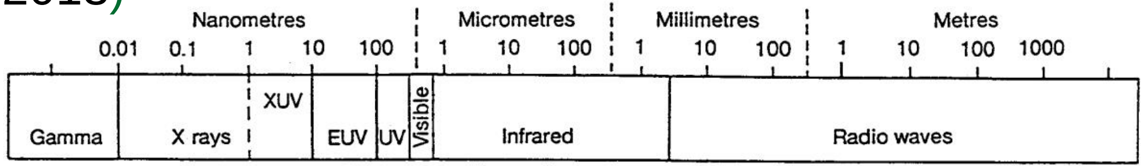
photosphere
(most of optical radiation)

- convective core
- radiative envelope with subsurface convection zone
- geometrically thin photosphere + dense wind
    - level populations radiatively dominated
      →non-LTE level population
      (all transition-probabilities need to be
      calculated explicitly)

NOTE: evolved objects (red giants and supergiants) and brown dwarfs are fully convective

Examples for current research:
Observations …

- … in all frequency bands
- both earthbound and via satellites
- Gamma-rays (Integral), X-rays (Chandra, XMM-Newton), (E)UV (IUE, HST), optical (VLT), IR (VLT, →JWST, →ELT) , (sub-) mm (ALMA) , radio (VLA, VLBI, →SKMA) …
- photometry, spectroscopy, polarimetry, interferometry, gravitational waves (aLIGO!)
- current telescopes allow for high S/N and high spatial resolution
- because of their high luminosity, massive stars can be spectroscopically observed not only in the Milky Way, but also in many Local Group (and beyond) galaxies (‘record-holder’: blue supergiants in NGC 4258 at a distance of ≈ 7.8 Mpc, Kudritzki+ 2013)



Electromagnetic spectrum

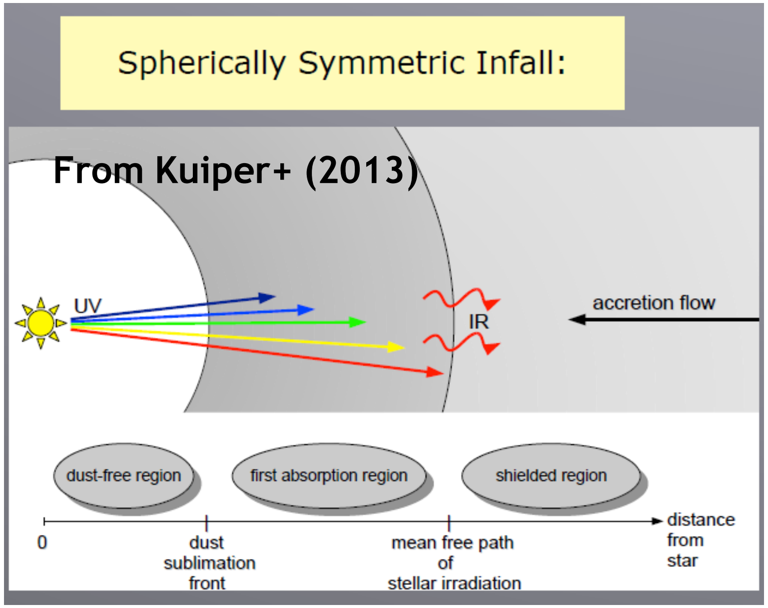
Abbreviations:

IUE – International Ultraviolet Explorer
HST – Hubbble Space Telescope
VLT – Very Large Telescope (Cerro Paranal, Chile)
JWST – James Webb SpaceTelescope
ELT – Extremely Large Telescope (Cerro Armazones, Chile, 20 km away from VLT))
ALMA – Atacama Large Millimeter/Submillimeter Array (Chajnantor-Plateau, Chile, 5000 m altitude)
VLA – Very Large Array (Socorro, New Mexico, USA)
VLBI – Very Large Baseline Interferometer
SKMA – Square Kilometer Array (South Africa and Australia)

Examples for current research:
Star formation

Star formation – formation of massive stars

- until 2010, it was not possible to ‘make’ stars with $M > 40\ M_{sun}$



Spherically Symmetric Infall:

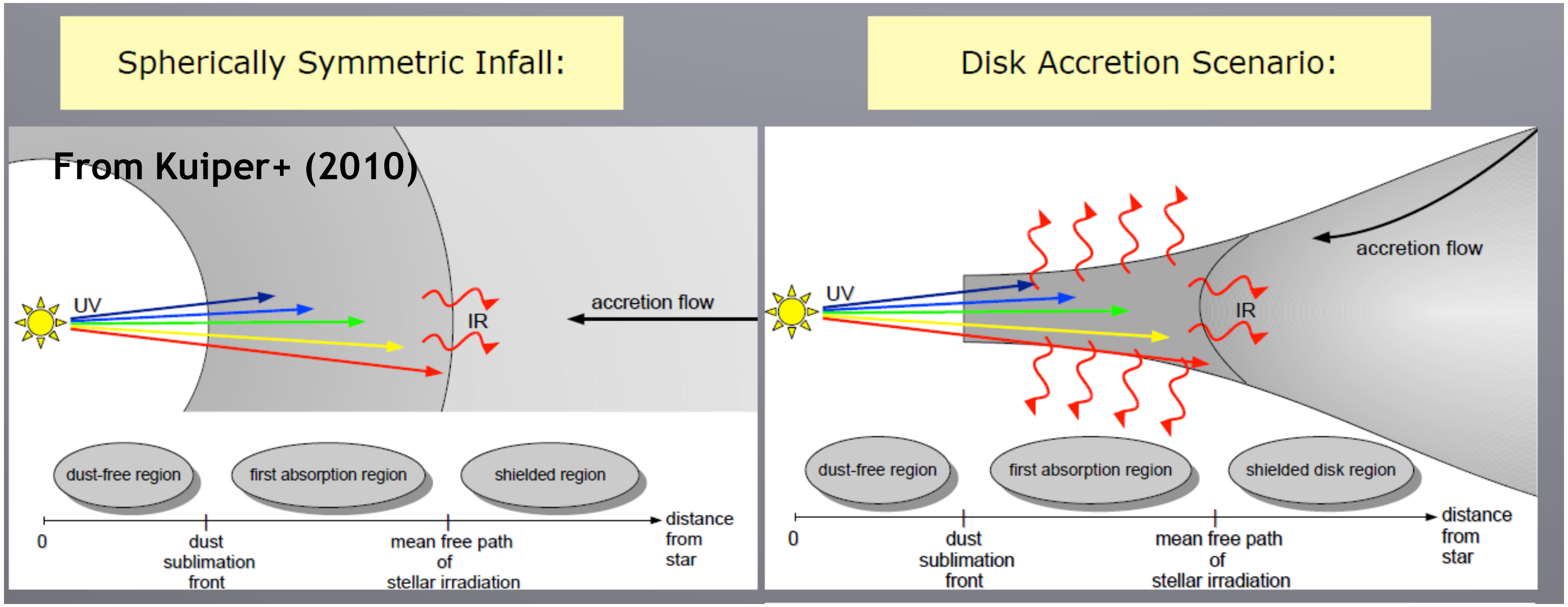
From Kuiper+ (2013)

- Radiation pressure barrier for spherical infall:
  when core becomes massive, high luminosity heats ‘first absorption region’,
  radiation pressure due to re-processed IR radiation stops and reverts accretion flow.

Examples for current research:
Star formation

Star formation – formation of massive stars

- until 2010, it was not possible to 'make' stars with $M > 40\ M_{sun}$

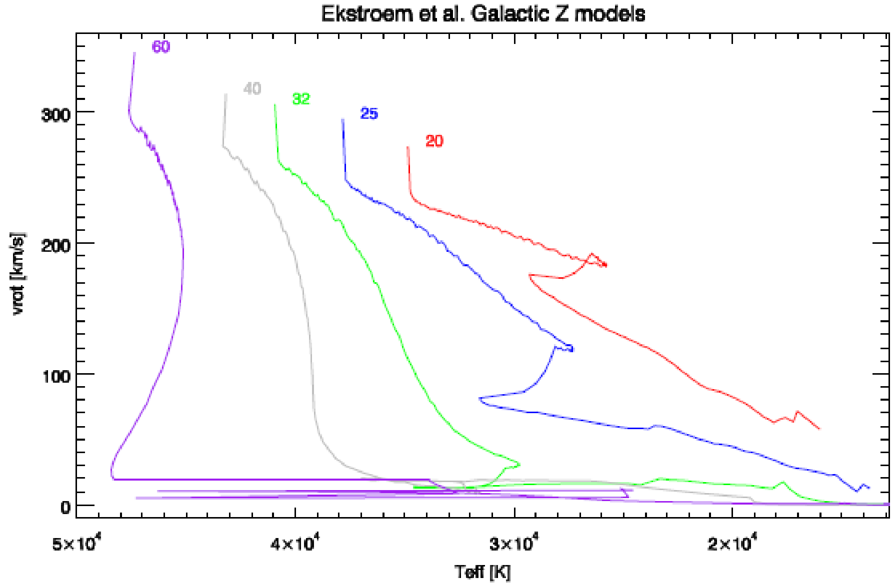


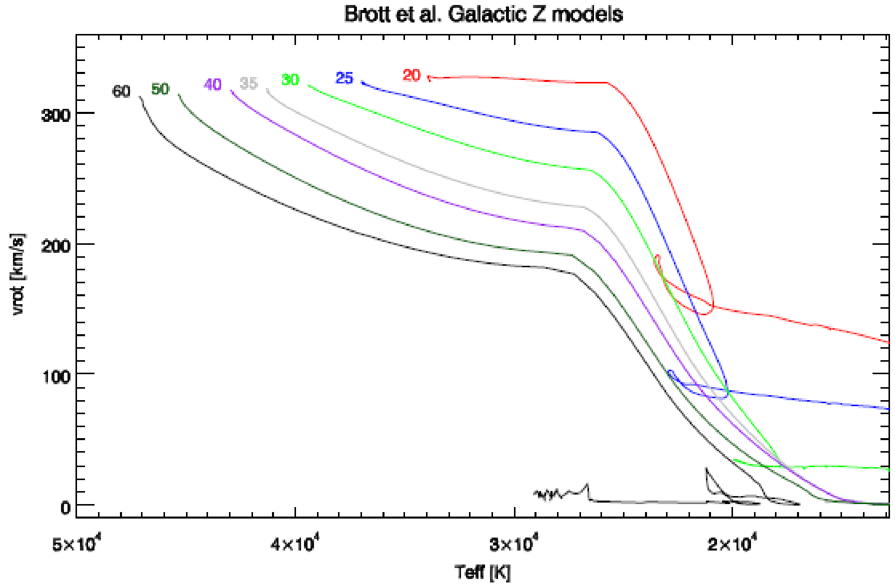
- Radiation pressure barrier for spherical infall: when core becomes massive, high luminosity heats 'first absorption region', radiation pressure due to re-processed IR radiation stops and reverts accretion flow.
- If accretion via disk, re-processes radiation-field becomes highly anisotropic, the radial component of the radiative acceleration becomes diminished, and further accretion becomes possible. Stars with $M > 40\ M_{sun}$ (… $140\ M_{sun}$) can be formed. (see work by R. Kuiper and collaborators)

Examples for current research:
Stellar structure and evolution

- **Stellar structure and evolution**
  - implementation/improved description of various processes, e.g.,
    - impact of mass-loss and rotation (mixing!) in massive stars
    - generation and impact of B-fields
    - convection, mixing processes, core-overshoot etc. still described by simplified approximations in 1-D (e.g., diffusive processes), needs to be studied in 3-D (work in progress)

Examples for current research:
Stellar structure and evolution





- *vrot* vs. *Teff* , for rotating Galactic massive-star models from Ekström+(2012, 'GENEC') and Brott+ (2011, 'STERN'), with *vrot(initial)* ≈ 300km/s
- The main difference on the MS is due to the lack (Ekström) and presence (Brott) of *assumed* internal magnetic fields and the treatment of angular momentum transport.

- NOTE: Even at main sequence, stellar evolution of massive stars unclear in many details!!!!
- Do not believe in statements such as 'stellar evolution is understood'

Examples for current research:
Stellar structure and evolution

Stellar structure and evolution

- NOTE: binarity fraction of Galactic stars

  M-stars: 25%, solar-type: 45%, A-stars: 55% (Duchene & Kraus 2013, review)

  O-stars in Galactic clusters:
  - 70% of all stars will interact with a companion during their lifetime (Sana+ 2012)

- THUS: needs to be included in evolutionary calculations
  - even more approximations regarding tidal effects, mass-transfer, merging … (e.g., ‘binary_c’ by Izzard+ 2004/06/09)

- predictions on pulsations
  - frequency spectrum of excited oscillations
  - period-luminosity relations as a function of metallicity

Asteroseismology: Revealing the internal structure

- non-radial pulsations: examples for different models

following slides adapted
from C. Aerts (Leuven)

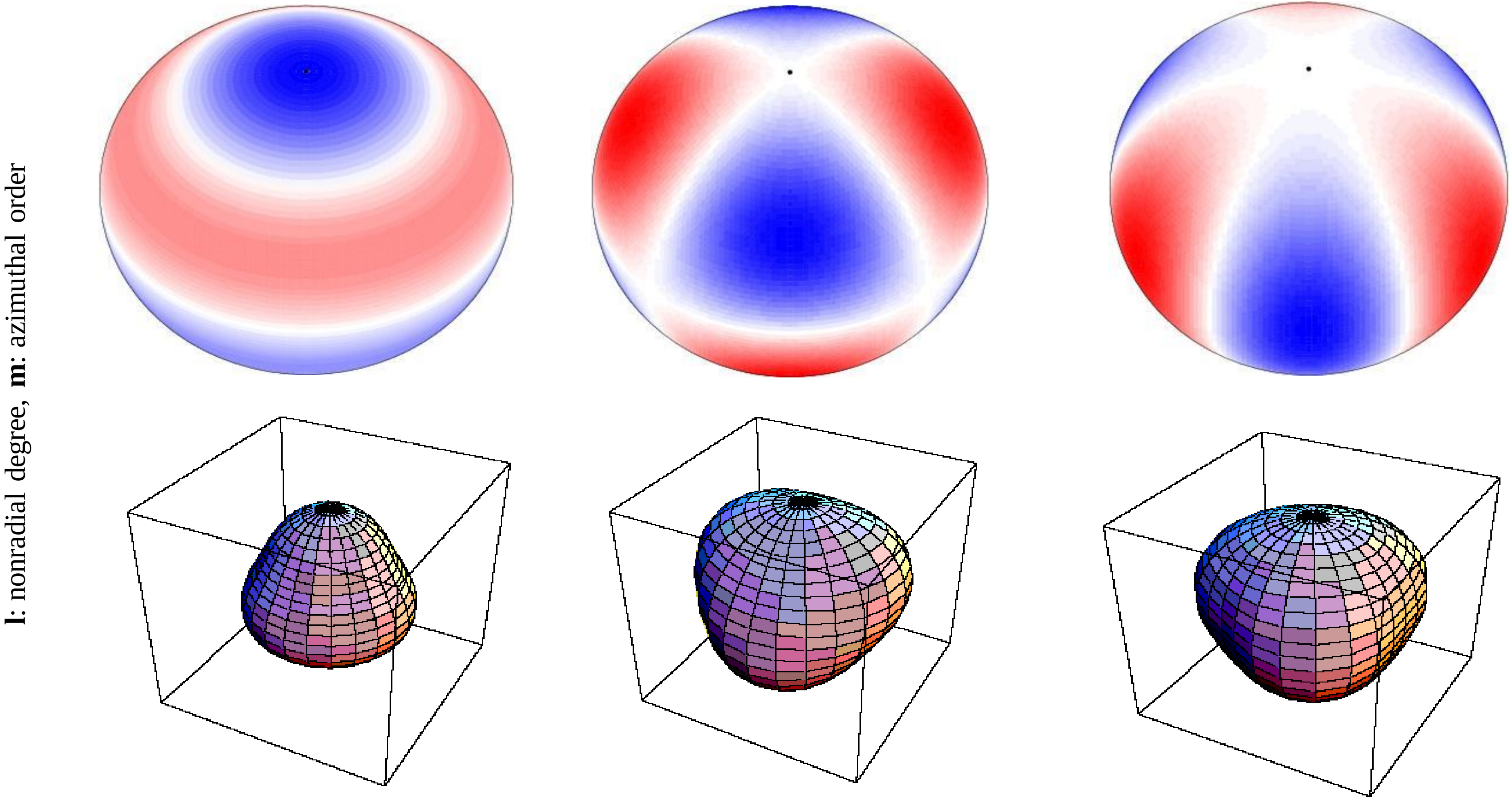
Blue: Moving towards Observer

Red: Moving away from Observer

(l,m)=(3,0)
axisymmetric

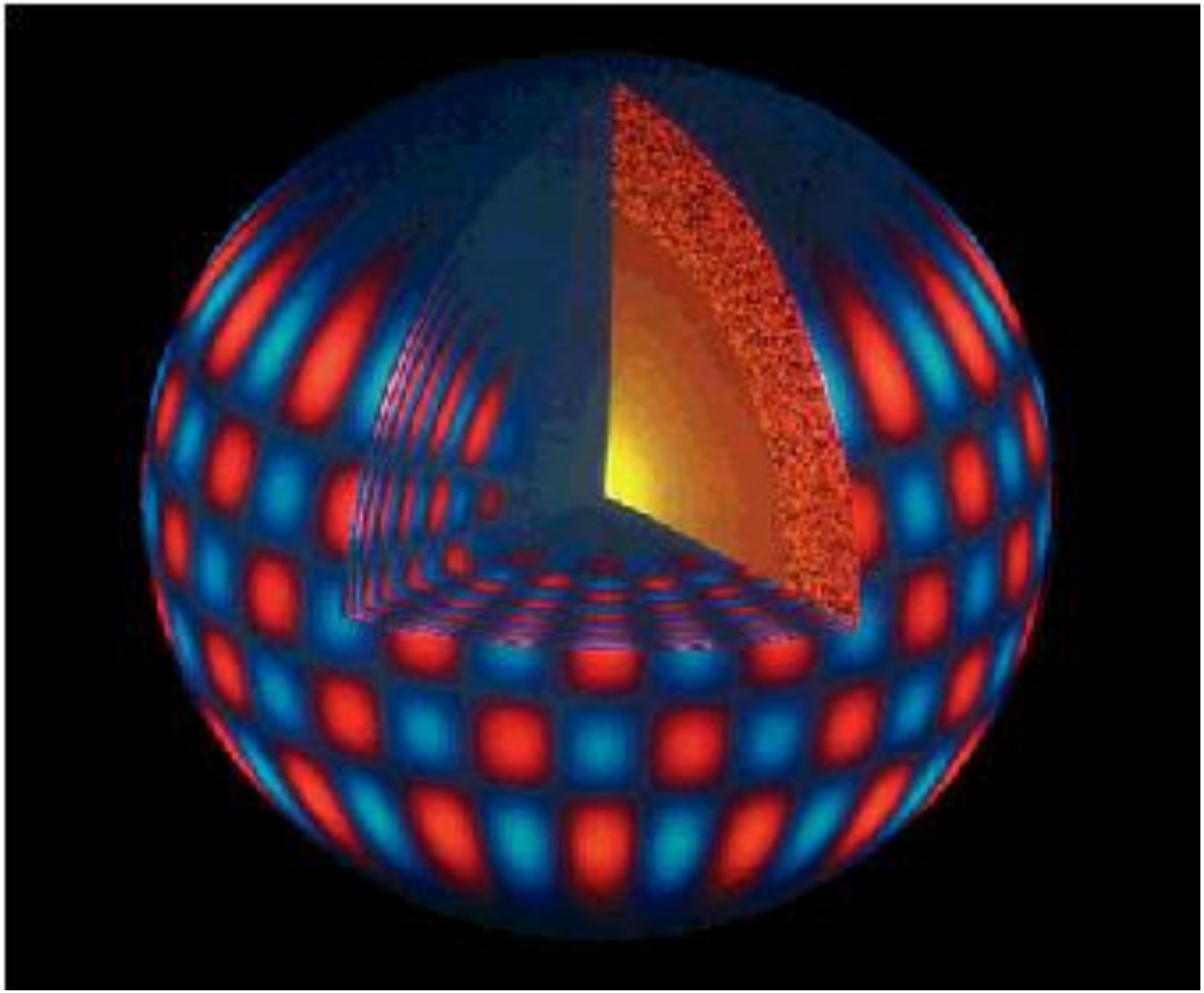
(l,m) = (3,2)
tesseral

(l,m)=(3,3)
sectoral



l: nonradial degree, **m**: azimuthal order

Internal behaviour of the oscillations



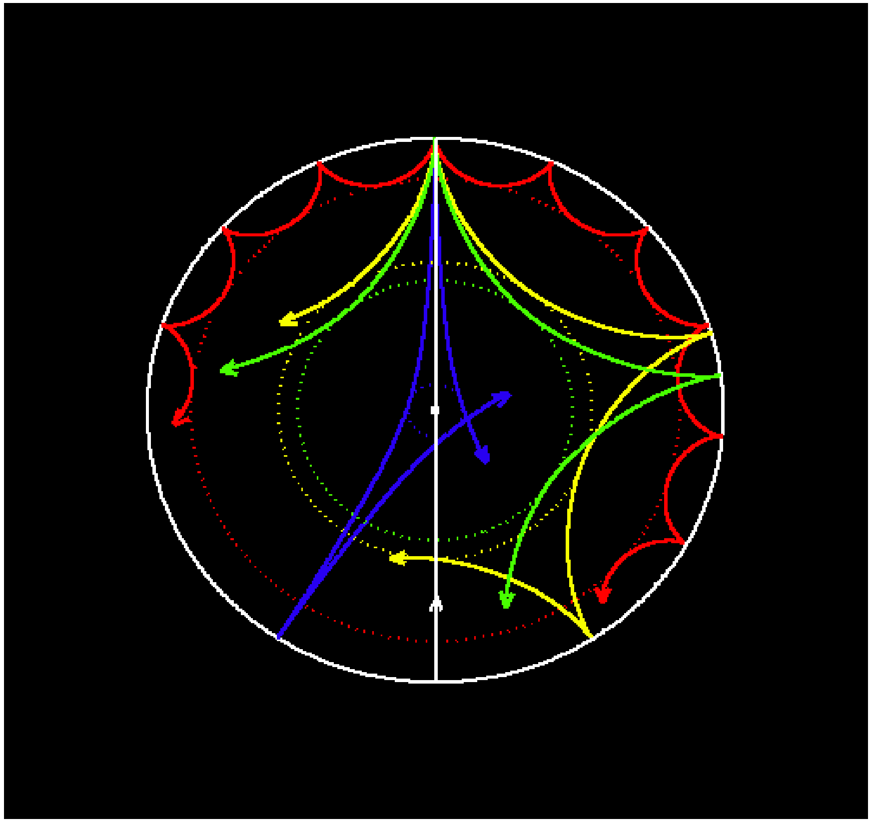
The oscillation pattern at the surface propagates in a continuous way towards the stellar centre.

Study of the surface patterns hence allows to characterize the oscillation throughout the star.

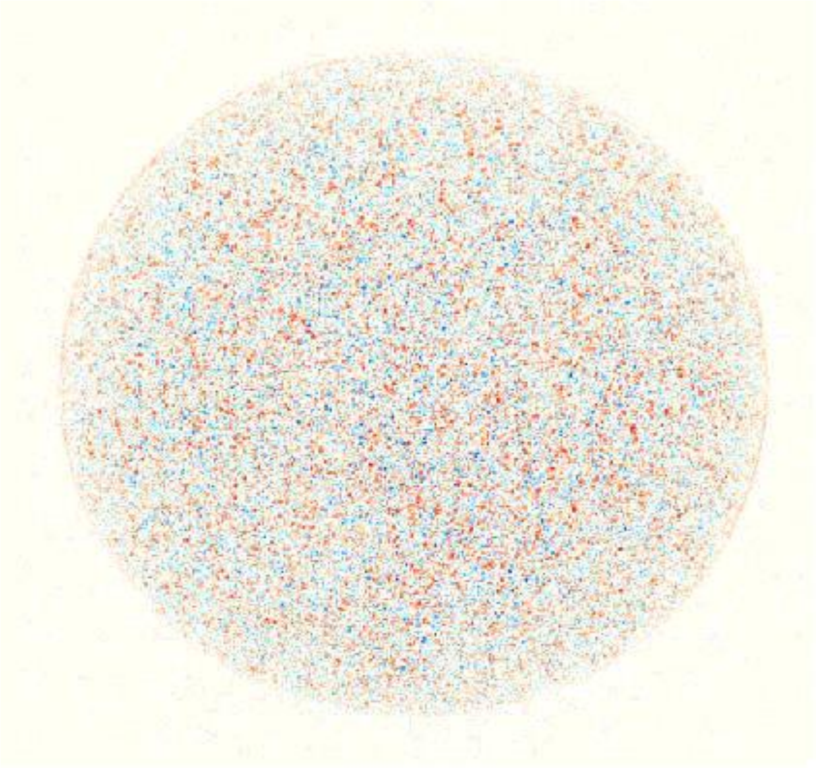
Inversion of the frequencies

The oscillations are standing sound waves that are reflected within a cavity

Different oscillations penetrate to different depths and hence probe different layers



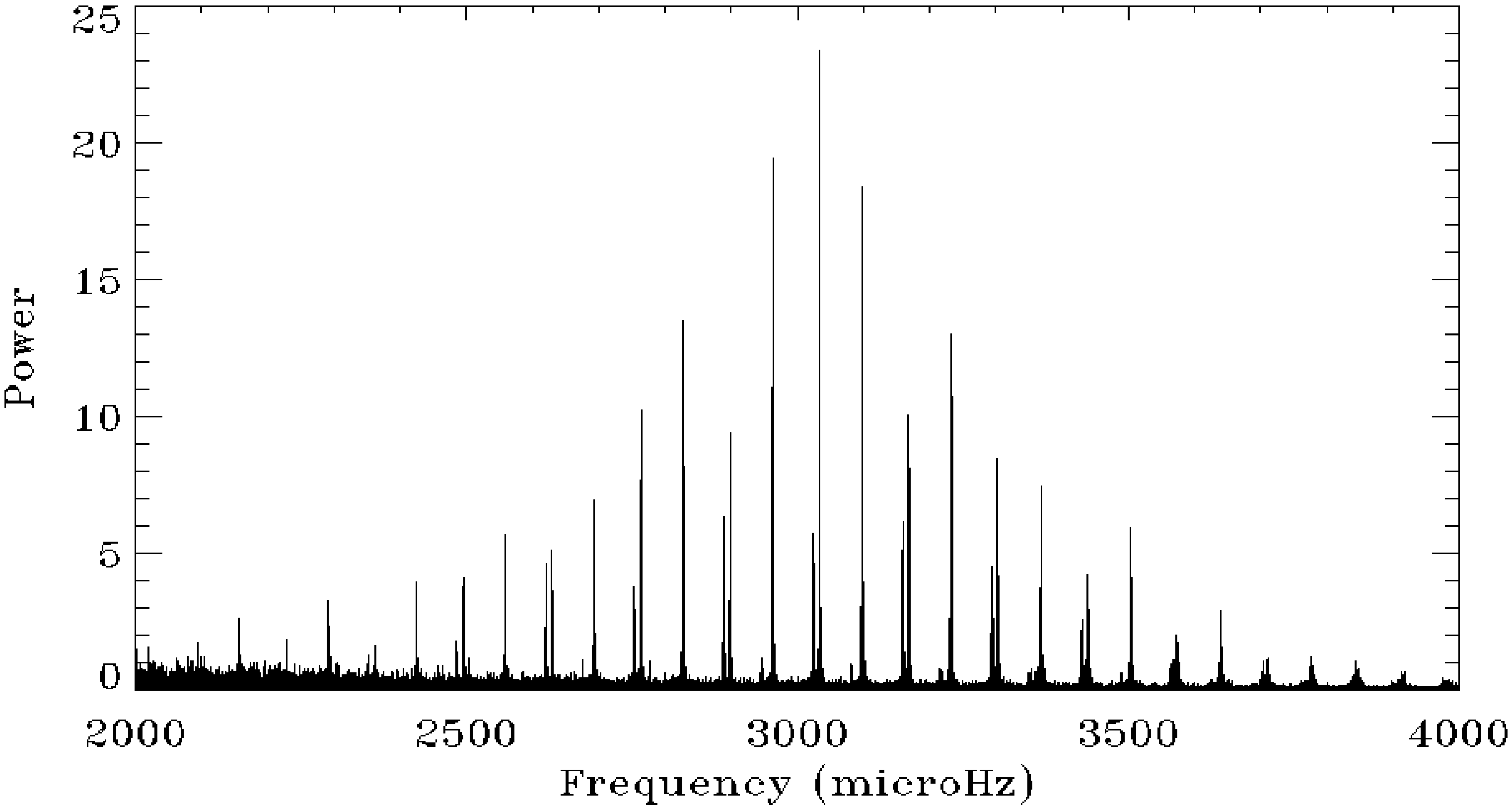
Doppler map of the Sun



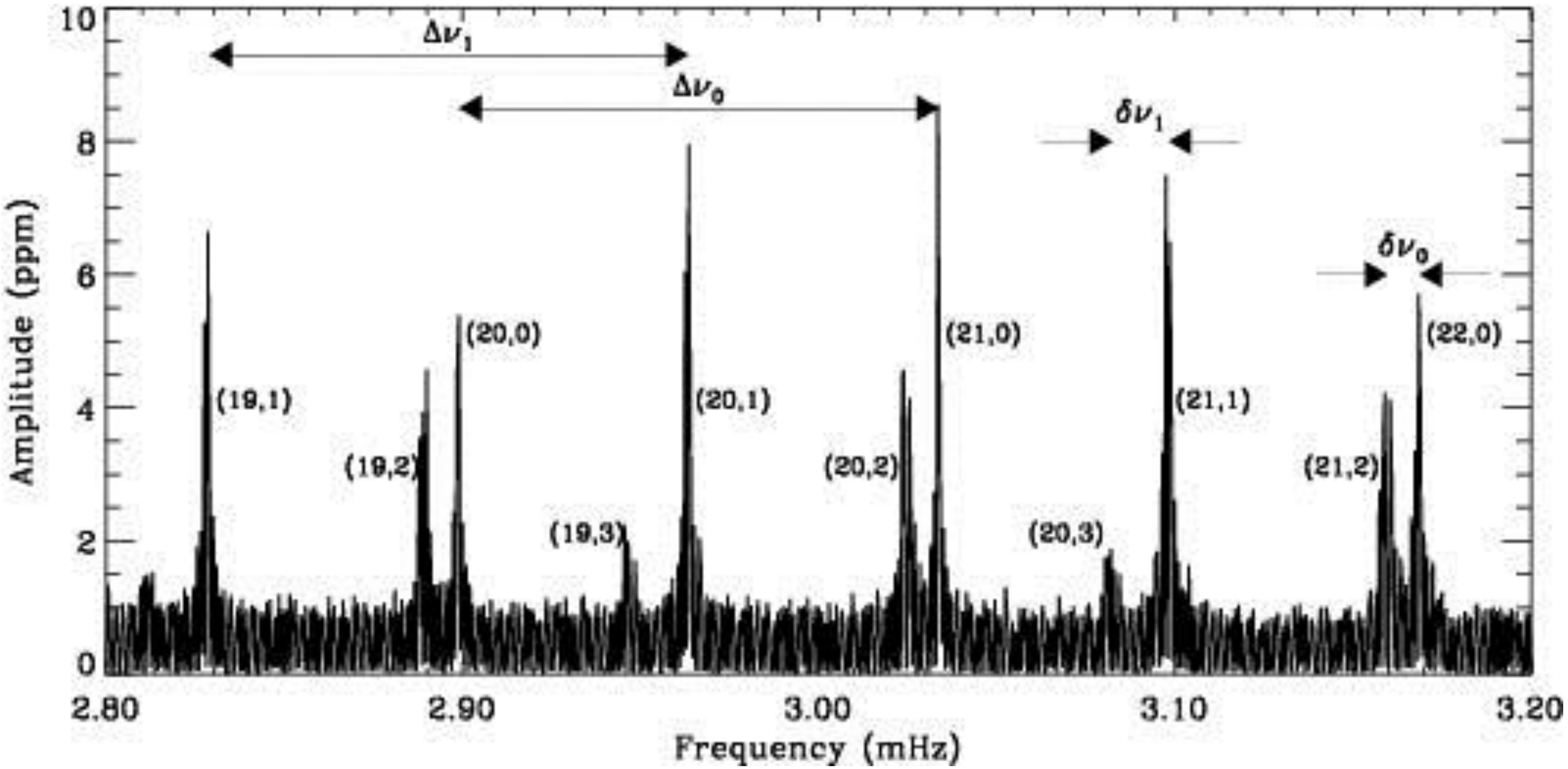
The Sun oscillates in thousands of non-radial modes with periods of ~5 minutes

The Dopplermap shows velocities of the order of some cm/s

Solar frequency spectrum from ESA/NASA satellite SoHO: systematics !

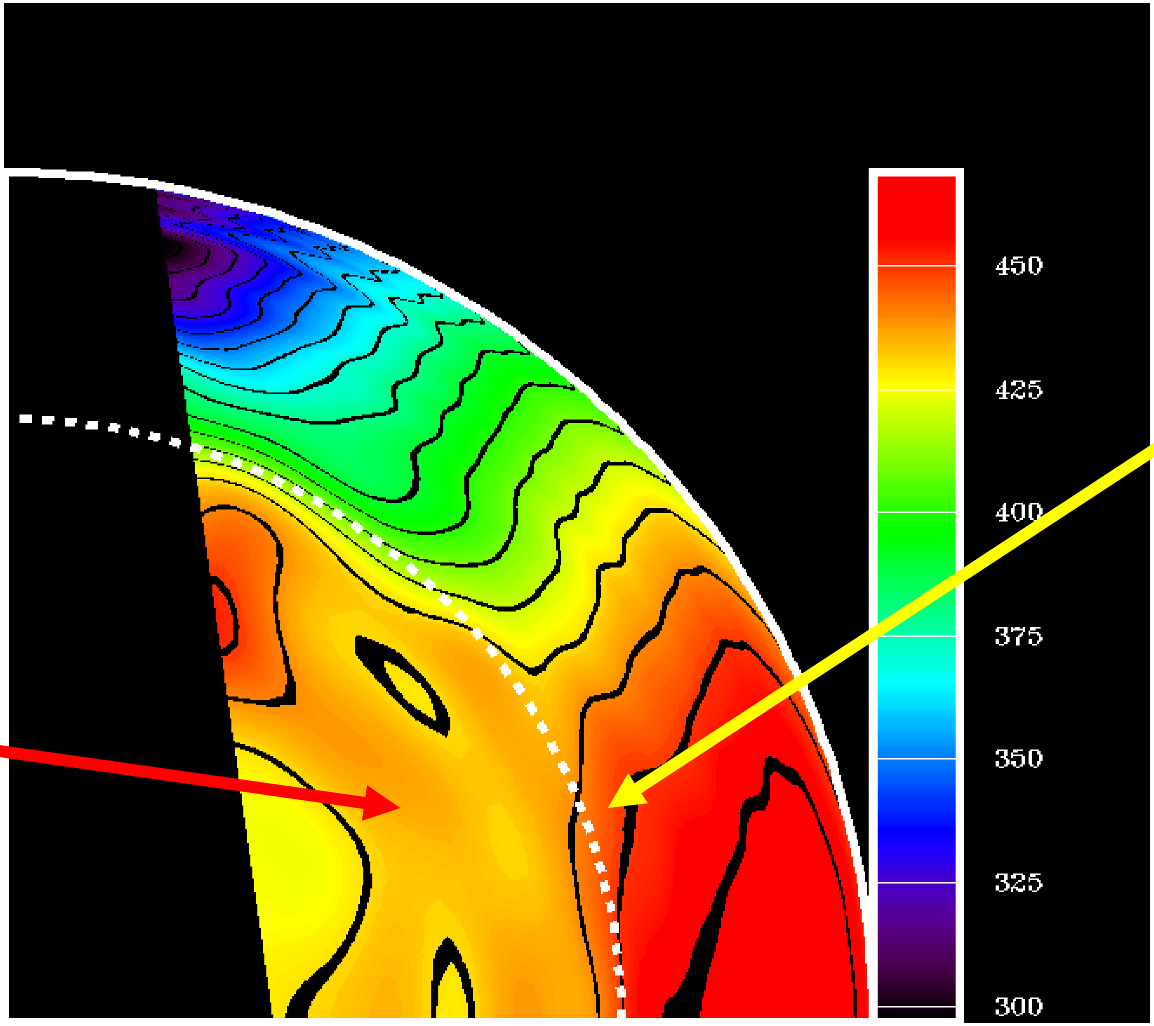


Frequency separations in the Sun



Result: internal sound speed and internal rotation could be determined very accurately by means of helioseismic data from SoHO

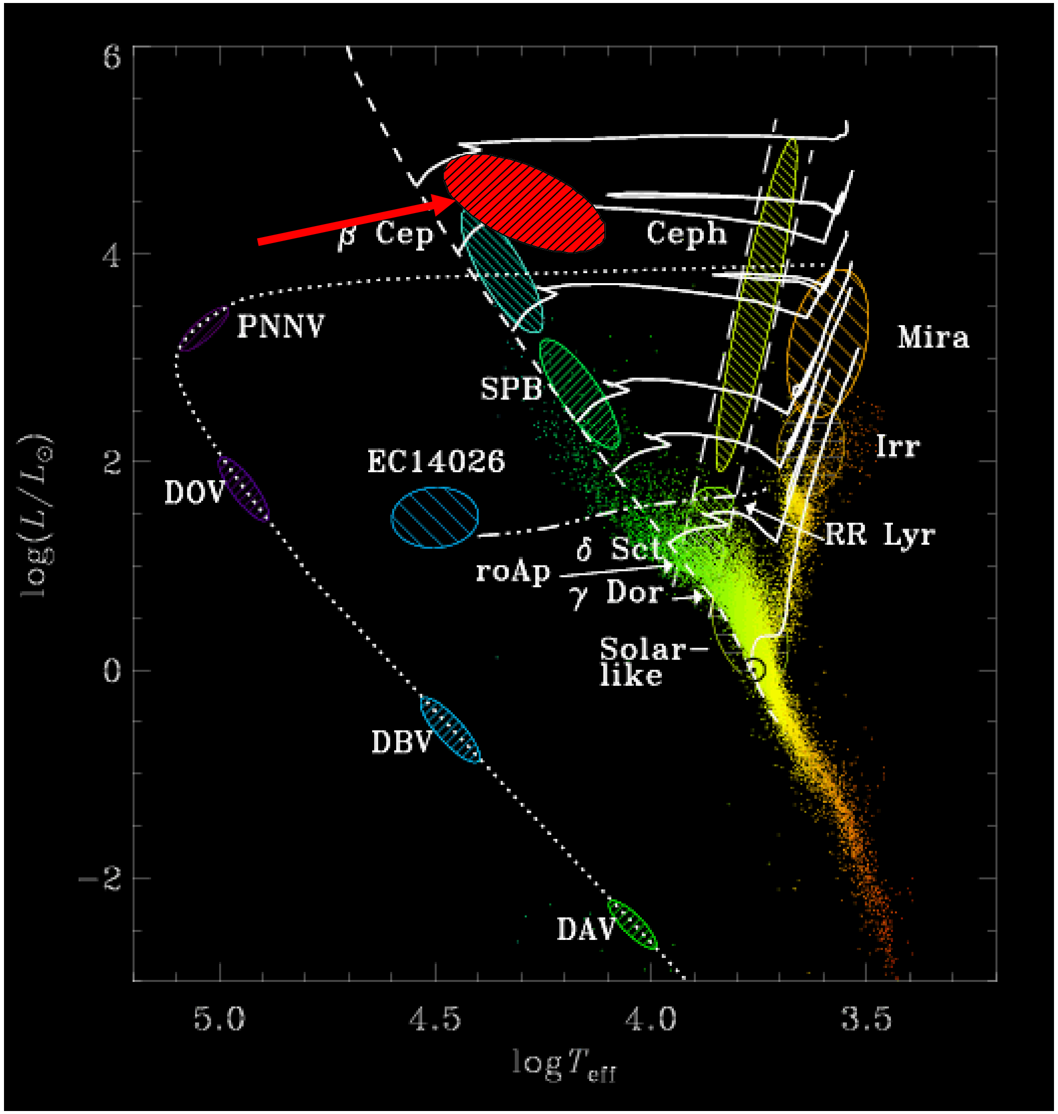
Internal rotation of the Sun



Solar interior has rigid rotation

Beginning of outer convection zone

… towards massive star seismology



- β Cep:
  low order p- and g-modes
- SPB
  slowly pulsating B-stars
  high order g-modes
- Hipparcos:
  29 periodically variable
  B-supergiants
  (Waelkens et al. 1998)
- no instability region
  predicted at that time
- nowdays: additional
  region for high order
  g-mode instability

➔ asteroseismology of evolved massive stars becomes possible

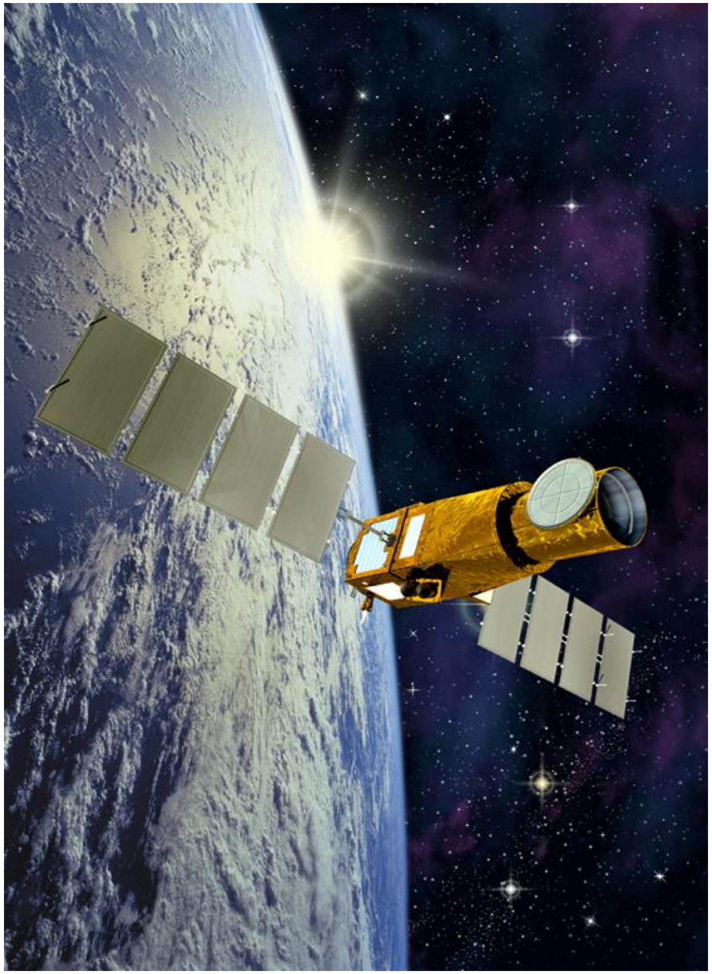
Space Asteroseismology

COROT: COnvection ROtation and planetary Transits
French-European mission (27 cm mirror)
launched December 2006

Kepler: NASA mission (1.2m mirror),
launched March 2009

MOST: Canadian mission
(65 x 65 x 30 cm, 70 kg)
launched in June 2003

BRITE-Constellation:
Canadian-Austrian-Polish mission
(six 20^3 cm nano-satellites, 7kg)
first one launched 2013
asteroseismology of bright (= massive) stars



Examples for current research:
End phases of evolution

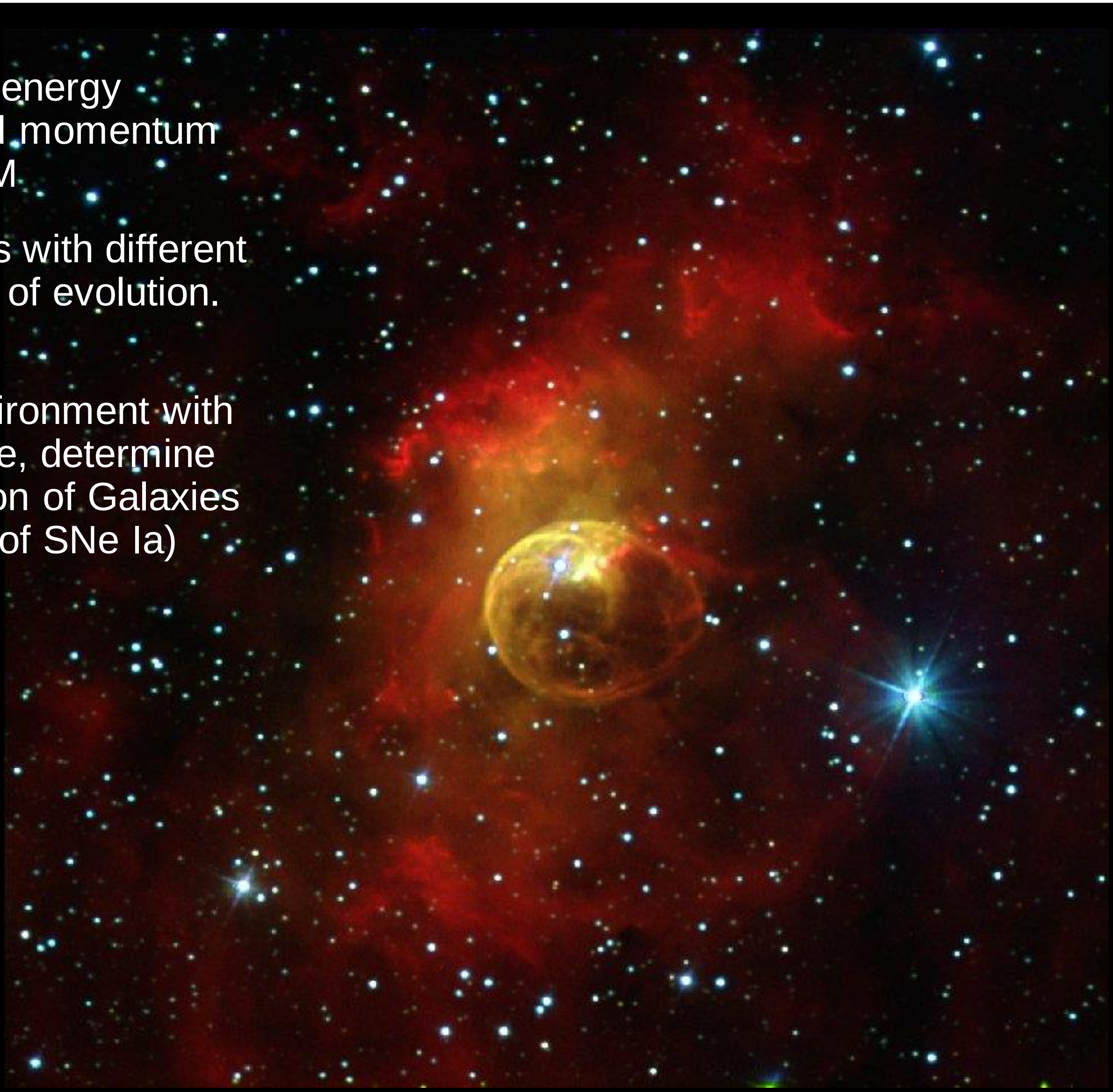
End phases

- evolutionary tracks towards ‘the end’
- models for SNe and Gamma-ray bursters
- models for neutron stars and white dwarfs
- accretion onto black holes
- X-ray binaries (‘normal’ star + white dwarf/neutron star/black hole)
- synthetic spectra of SN-remnants in various phases
- observations (now including gravitational waves) and comparison with theory
  - first detection of aLIGO was the merger of two black holes with masses around 30 M_{sun} (Abbott et al. 2016)
  - Corresponding theoretical scenario published just before announcement of detection (Marchant+ 2016), predicting one BH merger for 1000 cc-SNe, and a high detection rate with aLIGO

Examples for current research:
Impact on environment

- **Impact on environment**
  - cosmic re-ionization and chemical enrichment
  - chemical yields (due to SNe and winds)
  - ionizing fluxes (for HII regions)
  - Planetary nebulae (excited by hot central stars)
  - impact of winds on ISM (energy/momentum transfer, triggering of star formation)
  - stars and their (exo)planets

Feedback



- massive stars determine energy (kinetic and radiation) and momentum budget of surrounding ISM
- massive stars have winds with different strengths, in dependence of evolution. status
- massive stars enrich environment with metals, via winds and SNe, determine chemo-dynamical evolution of Galaxies (exclusively before onset of SNe Ia)

→“FEEDBACK”

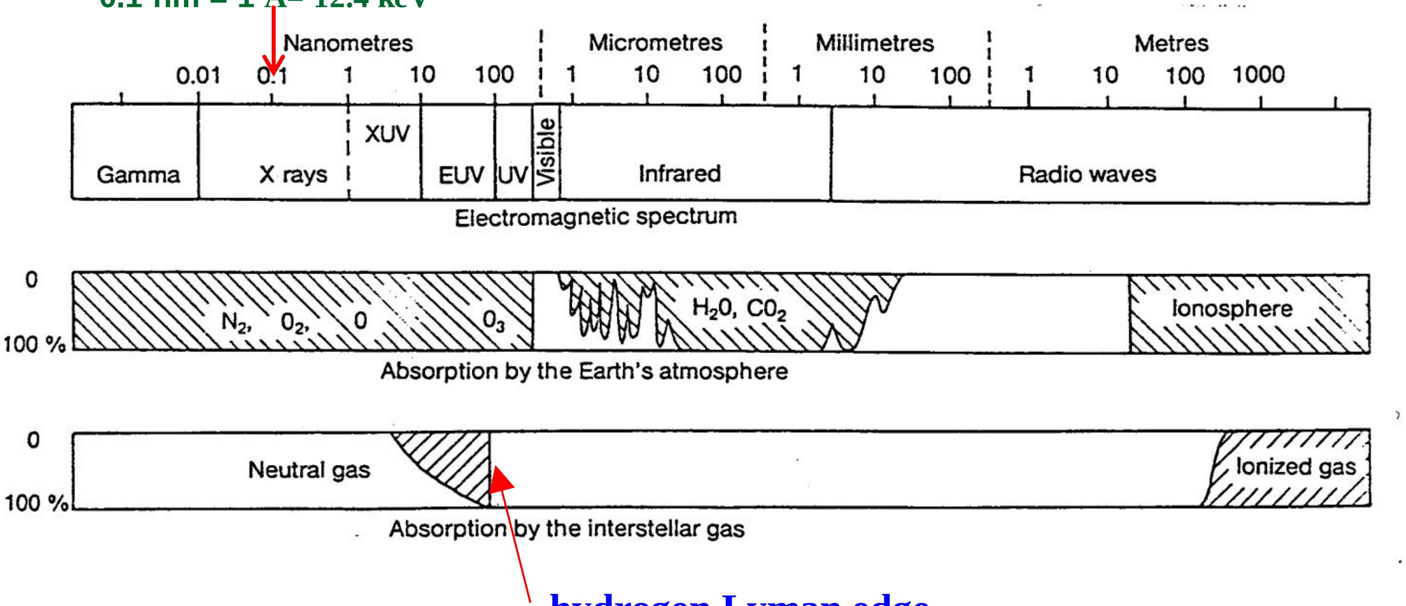
Bubble Nebula
(NGC 7635)
in Cassiopeia

wind-blown
bubble around
BD+602522
(O6.5IIIf)

Astrophysics → theory

Astrophysics → experiment

Experiment in astrophysics = Collecting photons from cosmic objects



0.1 nm = 1 Å= 12.4 keV

hydrogen Lyman edge

1 Å = 10^{-8} cm = 10^{-4} µm (micron); 1 nm = 10 Å

Collecting: earthbound and via satellites!

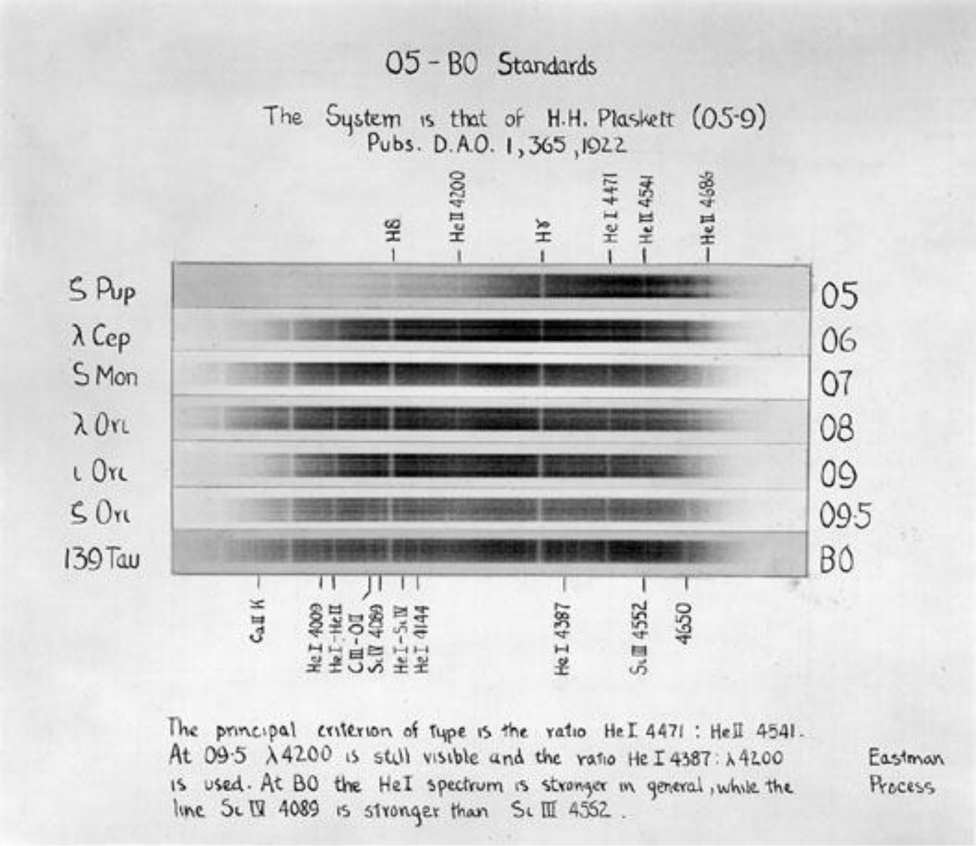
Note: Most of these photons originate from the atmospheres of stellar(-like) objects.
Even galaxies consist of stars!

Astrophys. monographs, Univ. Chicago Press (1943)

AN ATLAS OF STELLAR SPECTRA

WITH AN OUTLINE OF SPECTRAL CLASSIFICATION

Morgan, Keenan, Kellman



O5 - B0 Standards

The System is that of H.H. Plaskett (O5·9)
Pubs. D.A.O. 1, 365, 1922

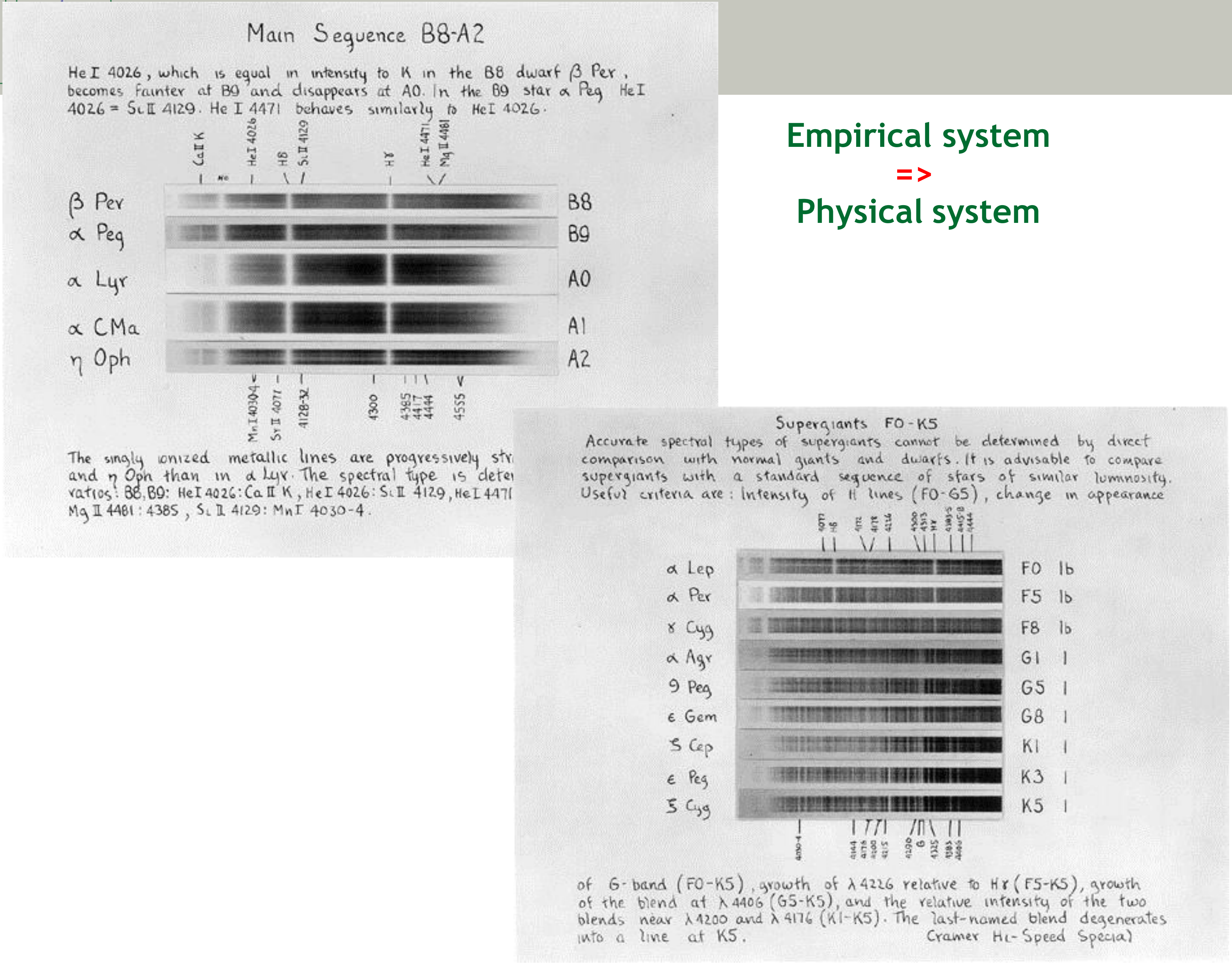
The principal criterion of type is the ratio HeI 4471 : HeII 4541. At O9·5 λ4200 is still visible and the ratio HeI 4387 : λ4200 is used. At B0 the HeI spectrum is stronger in general, while the line SiIV 4089 is stronger than SiIII 4552.

Eastman
Process

Empirical system

=>

Physical system



Main Sequence B8-A2

He I 4026, which is equal in intensity to K in the B8 dwarf β Per, becomes fainter at B9 and disappears at A0. In the B9 star α Peg He I 4026 = Si II 4129. He I 4471 behaves similarly to He I 4026.

The singly ionized metallic lines are progressively str
and η Oph than in α Lyr. The spectral type is dete
ratios: B8, B9: He I 4026 : Ca II K, He I 4026 : Si II 4129, He I 4471
Mg II 4481 : 4385, Si II 4129 : Mn I 4030-4.

Supergiants F0-K5

Accurate spectral types of supergiants cannot be determined by direct comparison with normal giants and dwarfs. It is advisable to compare supergiants with a standard sequence of stars of similar luminosity. Useful criteria are: Intensity of H lines (F0-G5), change in appearance

of G-band (F0-K5), growth of λ 4226 relative to Hγ (F5-K5), growth of the blend at λ 4406 (G5-K5), and the relative intensity of the two blends near λ 4200 and λ 4176 (K1-K5). The last-named blend degenerates into a line at K5.

Cramer Hi-Speed Special

Digitized spectra

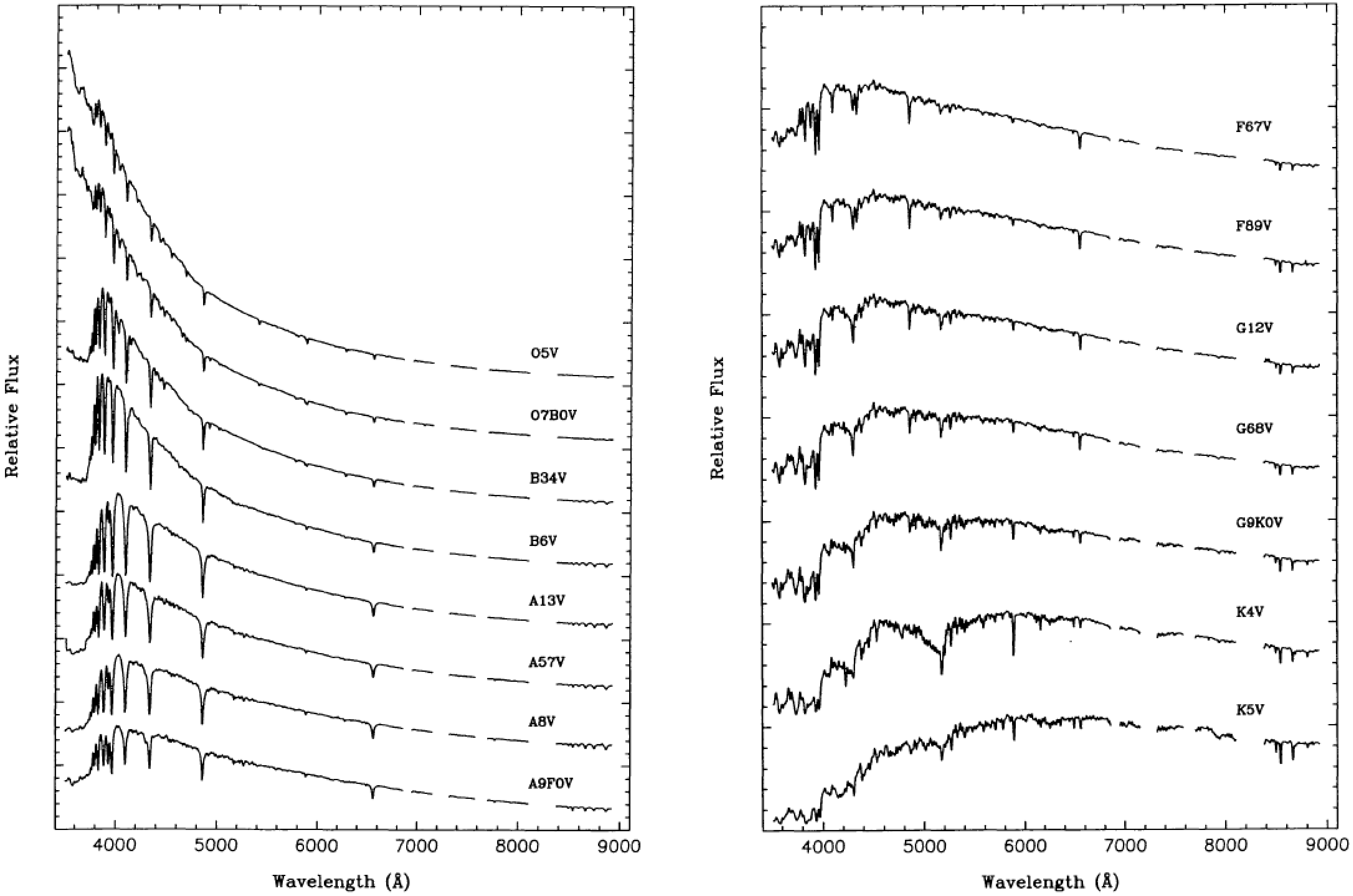


FIG. 1.—Dwarf-type library stars. Near-IR gaps are excised telluric absorption bands. All spectra have been normalized to 100 at 5450 Å. Major tick marks on "Relative Flux" axis are separated by 100 relative units. The M dwarf library stars are displayed with the M giants in Fig. 3.

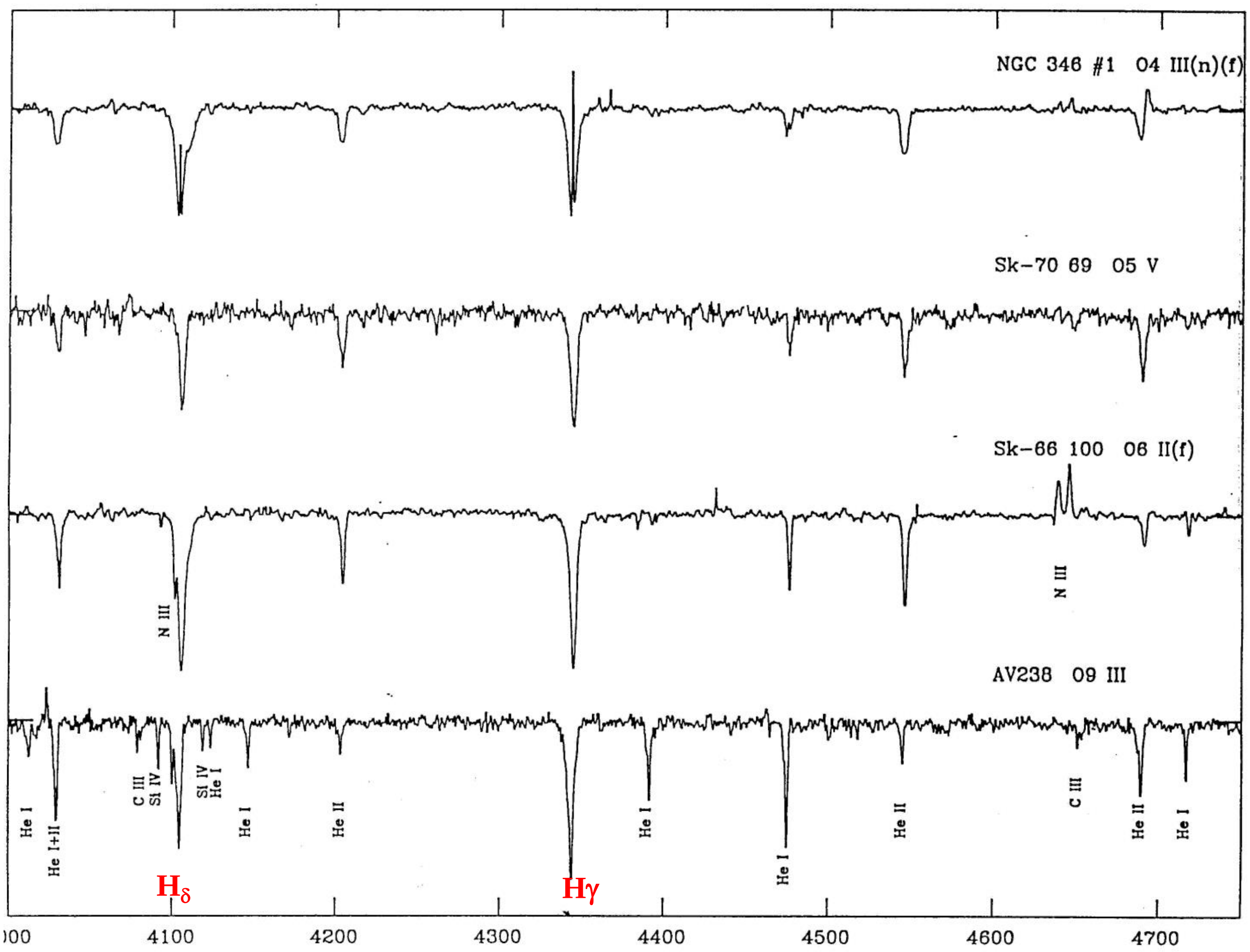
from Silva & Cornell, 1992

Spectral lines formed in (quasi-)hydrostatic atmospheres

ESO 3.6m
CASPEC

Δλ ≈ 0.5Å
S/N 30...70

(Walborn et al.,1995



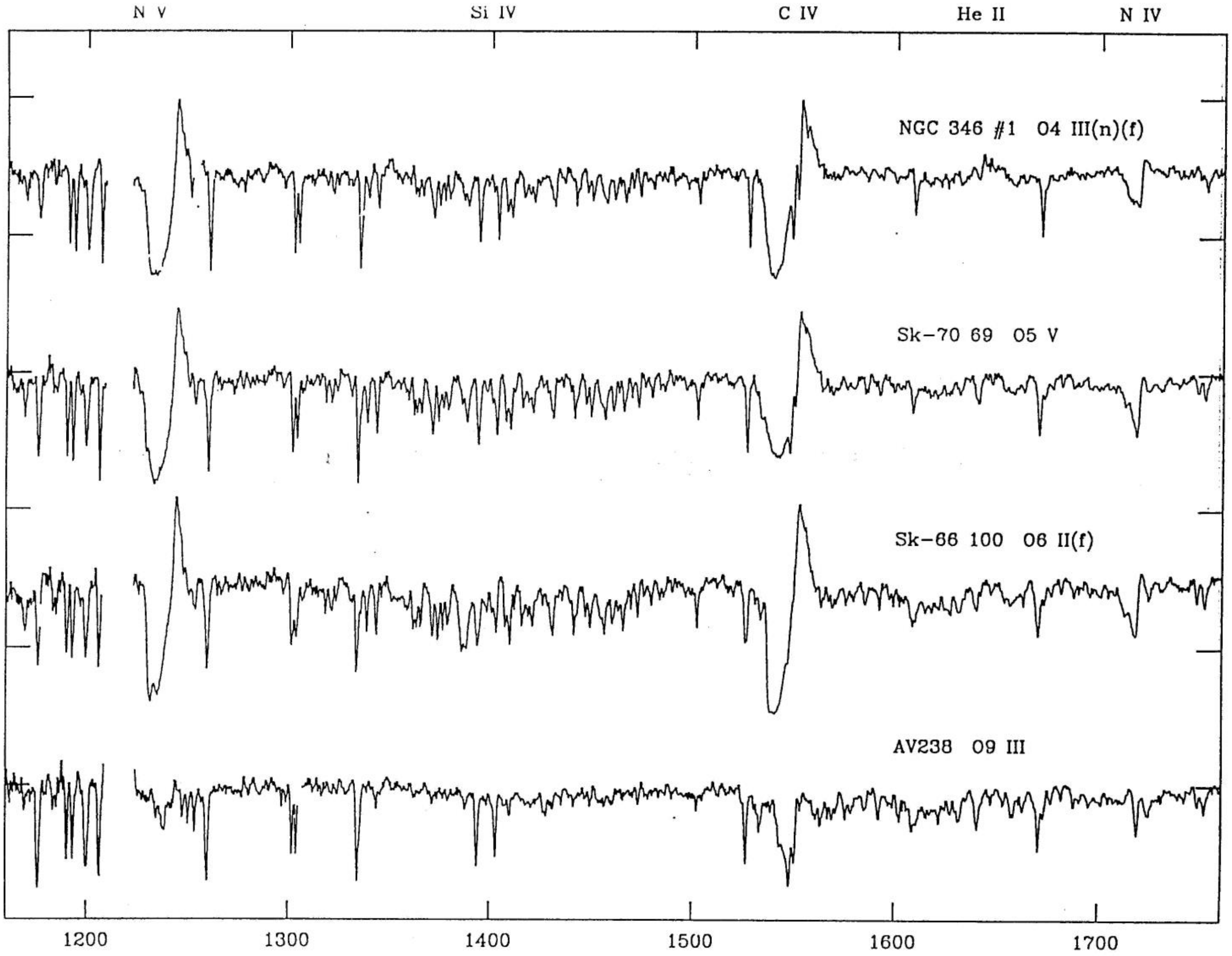
SMC

LMC

LMC

SMC

P-Cygni lines formed in hydrodynamic atmospheres



HST-FOS

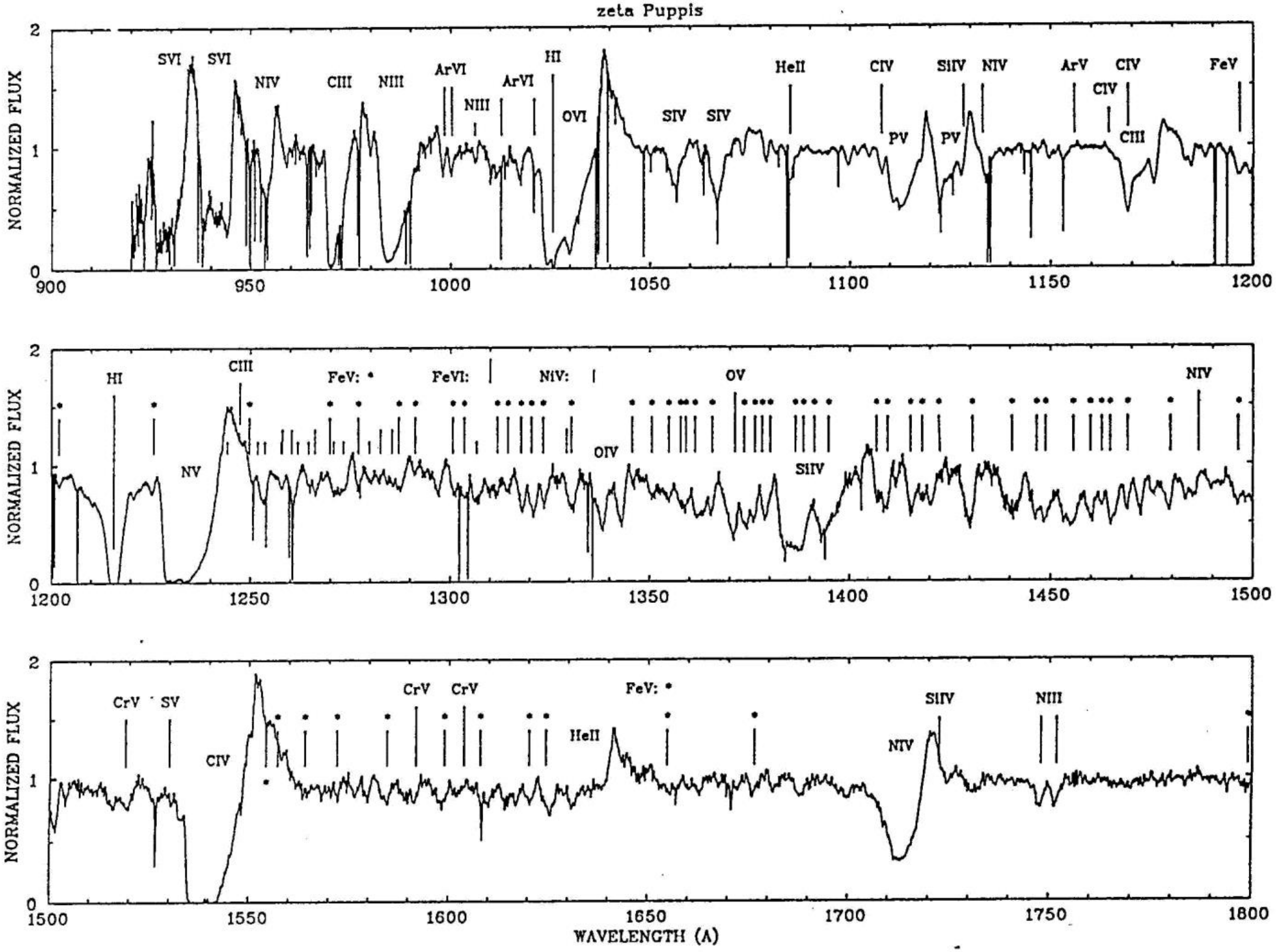
SMC

LMC

LMC

SMC

UV spectrum of the O4I(f) supergiant ζ Pup



montage of **Copernicus** **(λ < 1500 Å, high res. mode, Δλ ≈ 0.05 Å, Morton & Underhill 1977)**
and **IUE** **(Δλ ≈ 0.1 Å) observations**

Supernova Type II in different phases

photospheric phase

transition to nebular phase

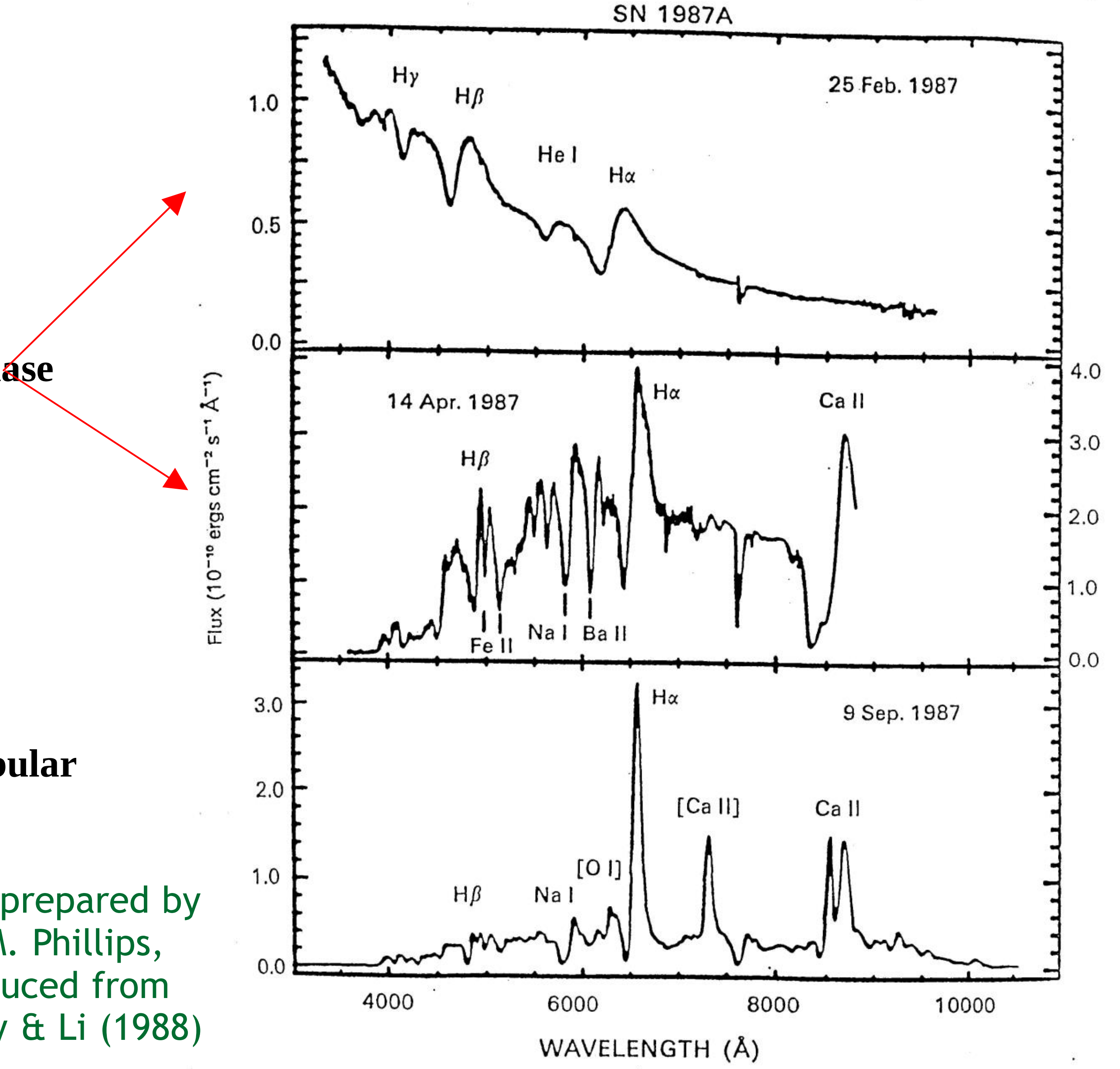
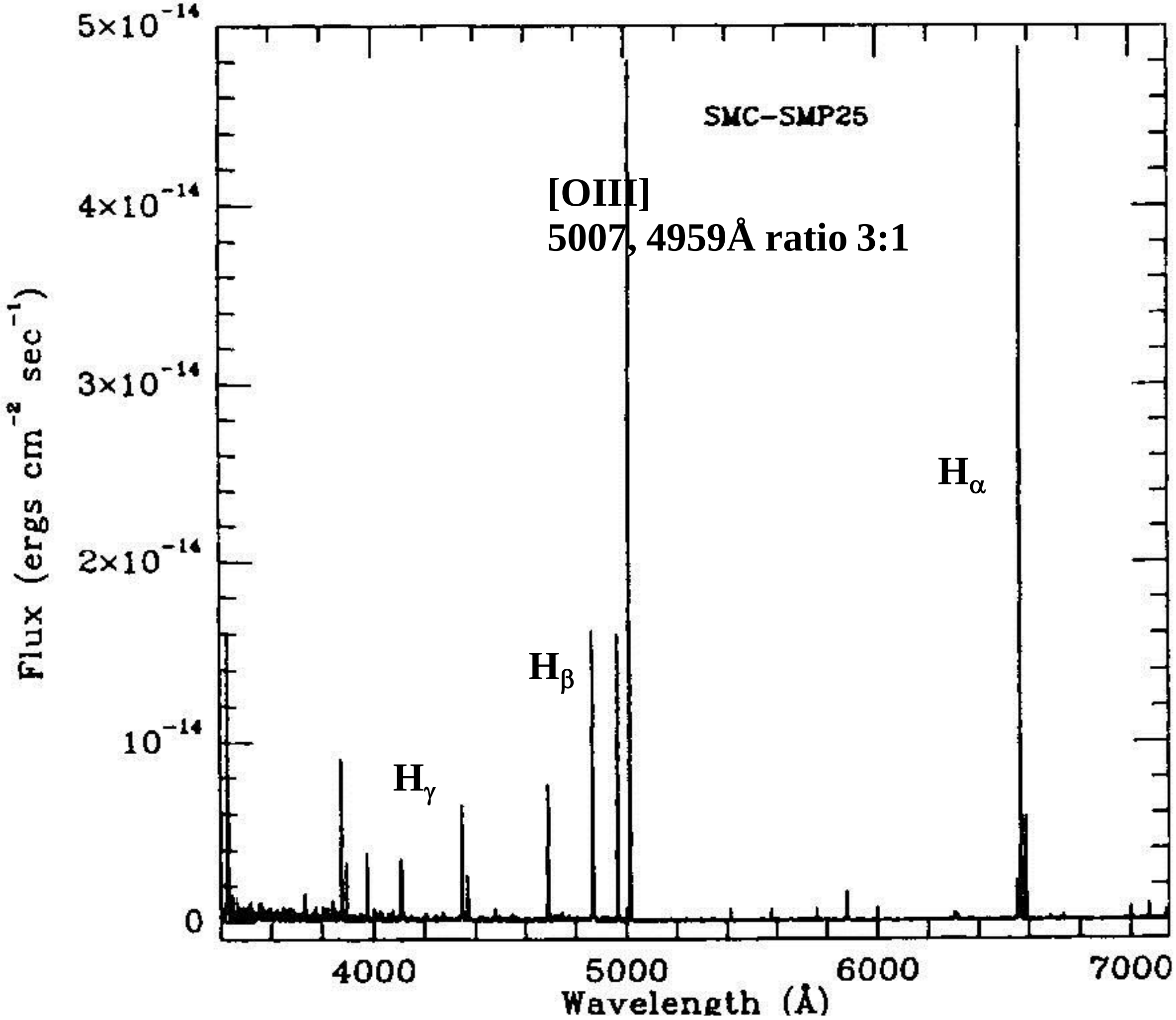


figure prepared by Mark M. Phillips, reproduced from McCray & Li (1988)

Spectrum of Planetary Nebula

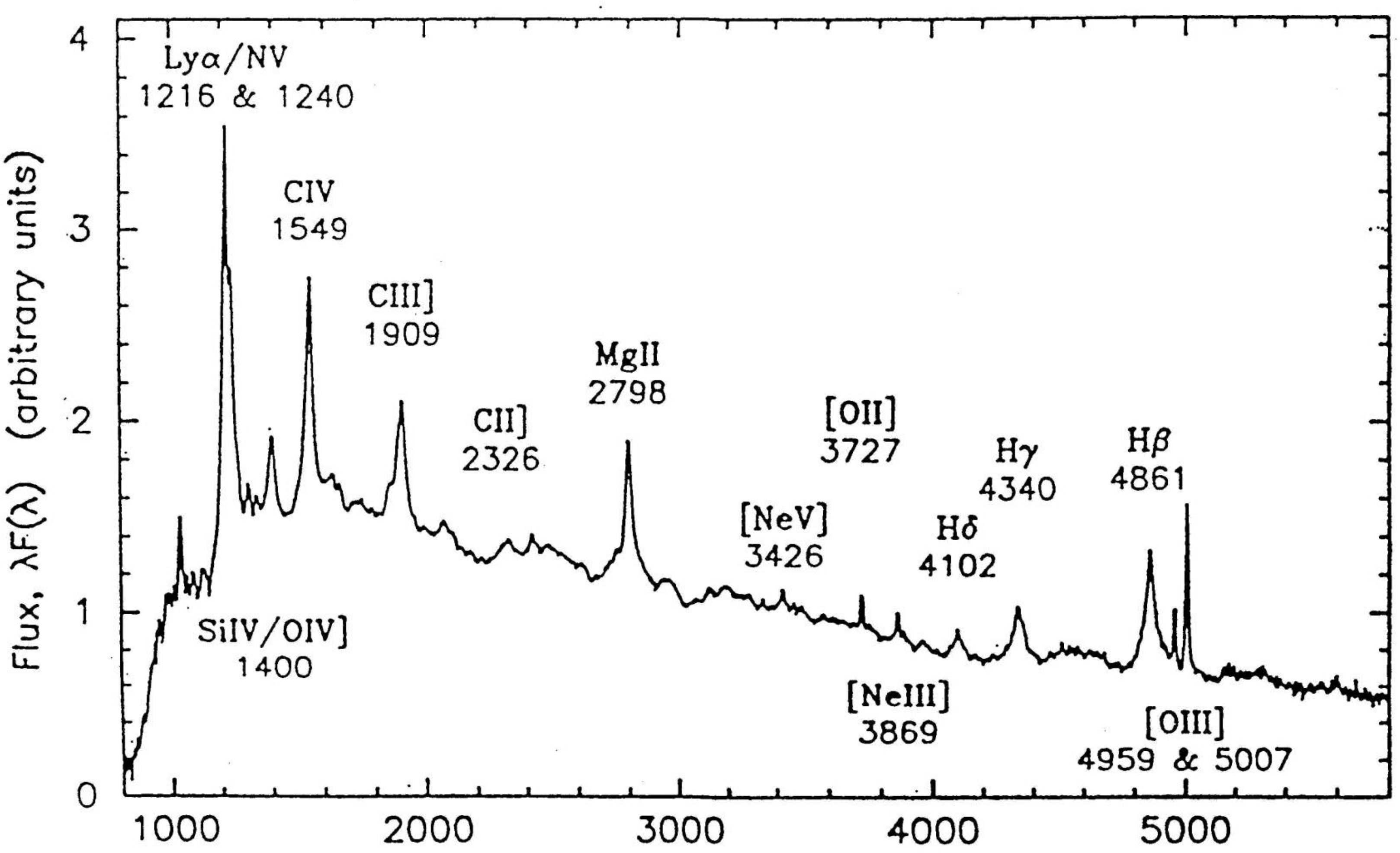
pure emission
line spectrum
with forbidden
lines of O III



From Meatheringham & Dopita, 1991, ApJS 75

FIG. 1*a*

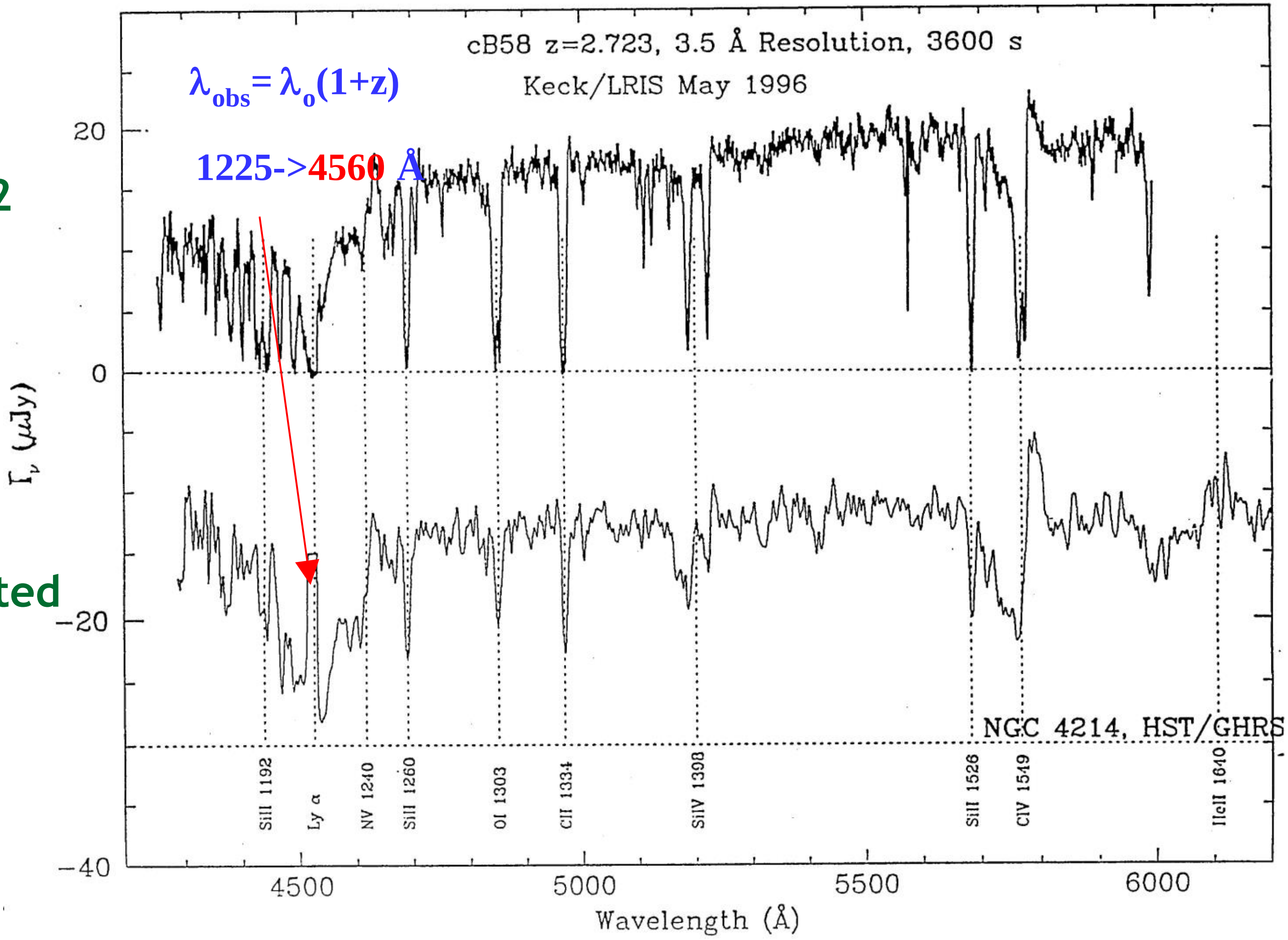
Quasar spectrum in rest frame of quasar



“UV”-spectra of starburst galaxies

galaxy at z = 2.72

local
starburst galaxy,
wavelengths shifted



$\lambda_{obs} = \lambda_o(1+z)$

1225->4560 Å

From Steidel et al. (1997)

Quantitative spectroscopy…

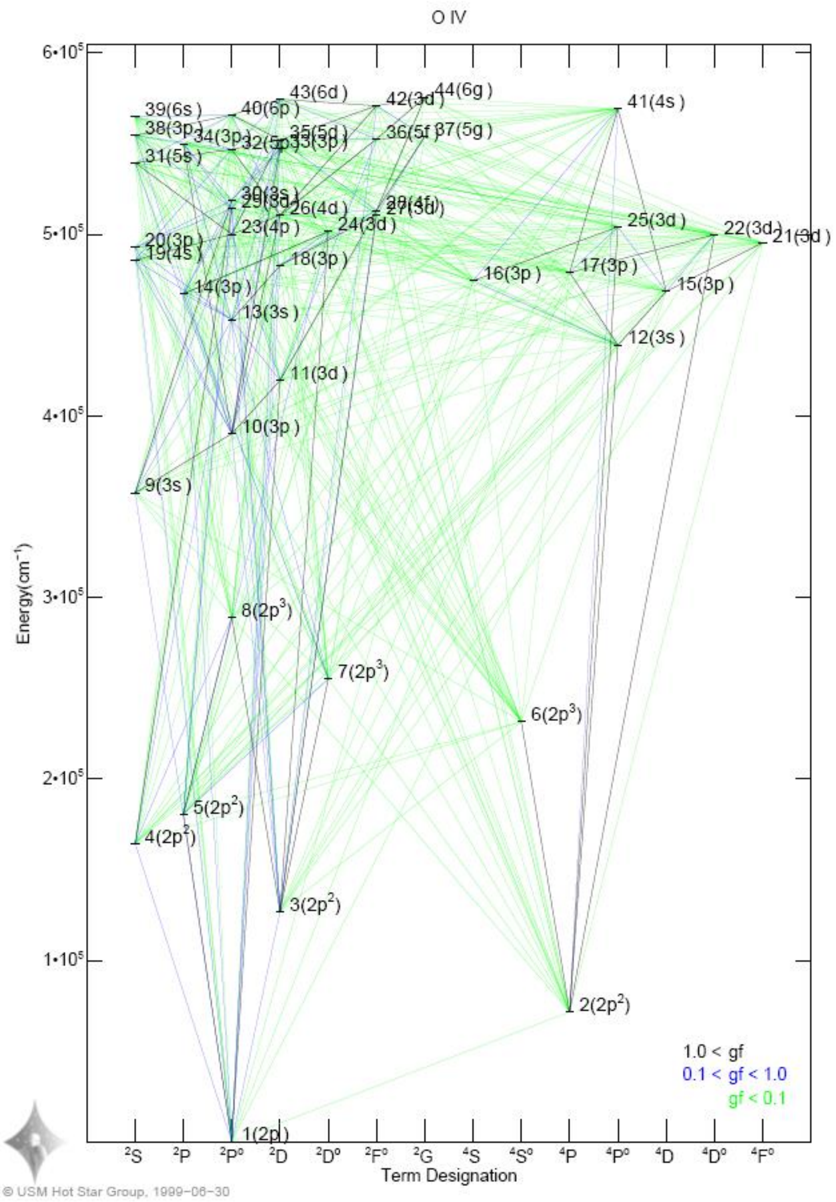
…gives insight into and understanding of our cosmos

- requires
  - plasma physics, plasma is "normal" state of atmospheres and interstellar matter (plasma diagnostics, line broadening, influence of magnetic fields,...)
  - atomic physics/quantum mechanics, interaction light/matter (micro quantities)
  - radiative transfer, interaction light/matter (macroscopic description)
  - thermodynamics, thermodynamic equilibria: TE, LTE (local), NLTE (non-local)
  - hydrodynamics, atmospheric structure, velocity fields, shockwaves,...

- provides
  - stellar properties, mass, radius, luminosity, energy production, chemical composition, properties of outflows
  - properties of (inter) stellar plasmas, temperature, density, excitation, chemical comp., magnetic fields

- INPUT for stellar, galactic and cosmologic evolution and for stellar and galactic structure atomic levels and allowed transitions ("Grotrian-diagram") in OIV

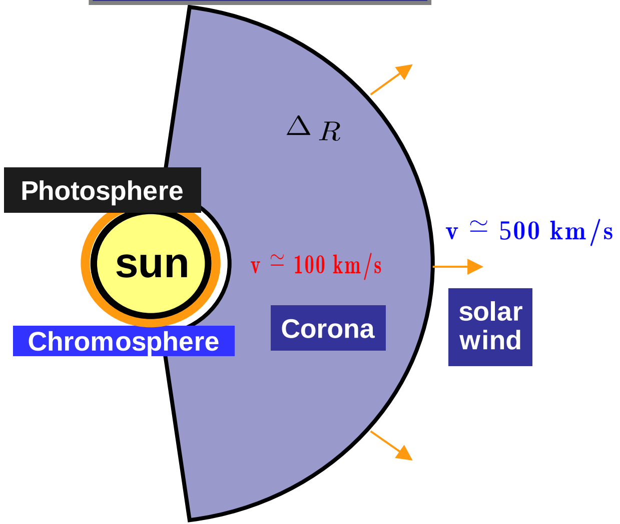
gf oscillator strength, measures "strength" of transition (cf. Chap. 7)



© USM Hot Star Group, 1999-06-30

Stellar atmospheres - an overview

Sun



Core:

$M_\odot = 2\ 10^{33}$ g

$R_\odot = 7\ 10^{10}$ cm

$L_\odot = 4\ 10^{33}$ erg s^{-1}

Photosphere:

$\Delta R \sim 200$ km

$T \sim 6000$ K

$n \sim 10^{15}$ cm^{-3}

Chromosphere:

$\Delta R \sim 1000$ km

$T \sim 20000$ K

$n \sim 10^{12}$ cm^{-3}

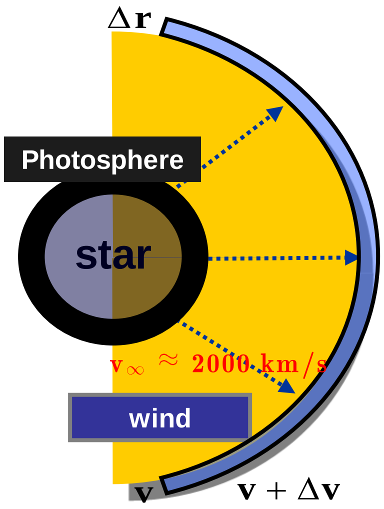
Corona:

$\Delta R \sim R_\odot$

$T \sim 10^{6}$ K

$n \sim 2 \cdot 10^{6}$ cm^{-3}

hot, massive star



Core:

$M \sim 50\ M_\odot$

$R \sim 20\ R_\odot$

$L \sim 10^{5} \mapsto 10^{6}\ L_\odot$

Photosphere:

$\Delta R / R \sim \Delta R_\odot / R_\odot$

$T \sim 20000 \mapsto 50000$ K

$n \sim 10^{14} \mapsto 10^{12}$ cm^{-3}

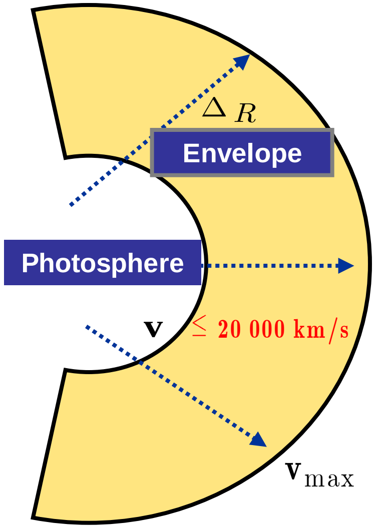
Wind:

$\Delta R \sim 100 R_*$

$T \sim 0.9 \ldots 0.3 \cdot T_{\mathrm{eff}}$

$n \sim 10^{12} \ldots 10^{8}$ cm^{-3}

Supernova Ia



Core:

$M \sim 1.44\ M_\odot$

exploded

$L_{\max} \sim 10^{11}\ L_\odot$

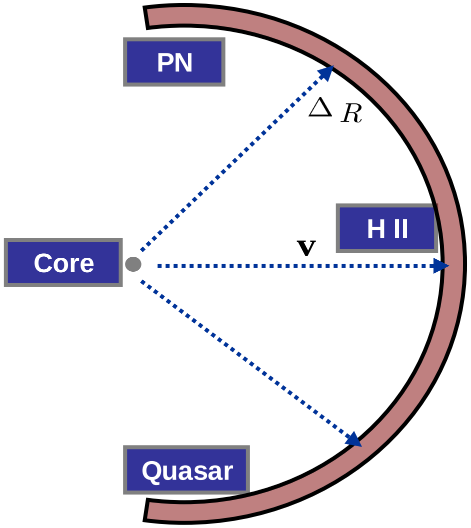
Envelope:

$\Delta R \sim 10^{5} R_\odot$

$T \sim 12000$ K

$n \sim 10^{6}$ cm^{-3}

Gaseous Nebula



Core:

$M_{PN} \sim 1\ M_\odot$

$M_{HII} \sim 50 \mapsto 10^{5}\ M_\odot$

$M_{Q} \sim 10^{9} \mapsto 10^{11}\ M_\odot$

$L_{PN} \sim 10^{4}\ L_\odot$

$L_{HII} \sim 10^{6}\ L_\odot$

$L_{Q} \sim 10^{12}\ L_\odot$

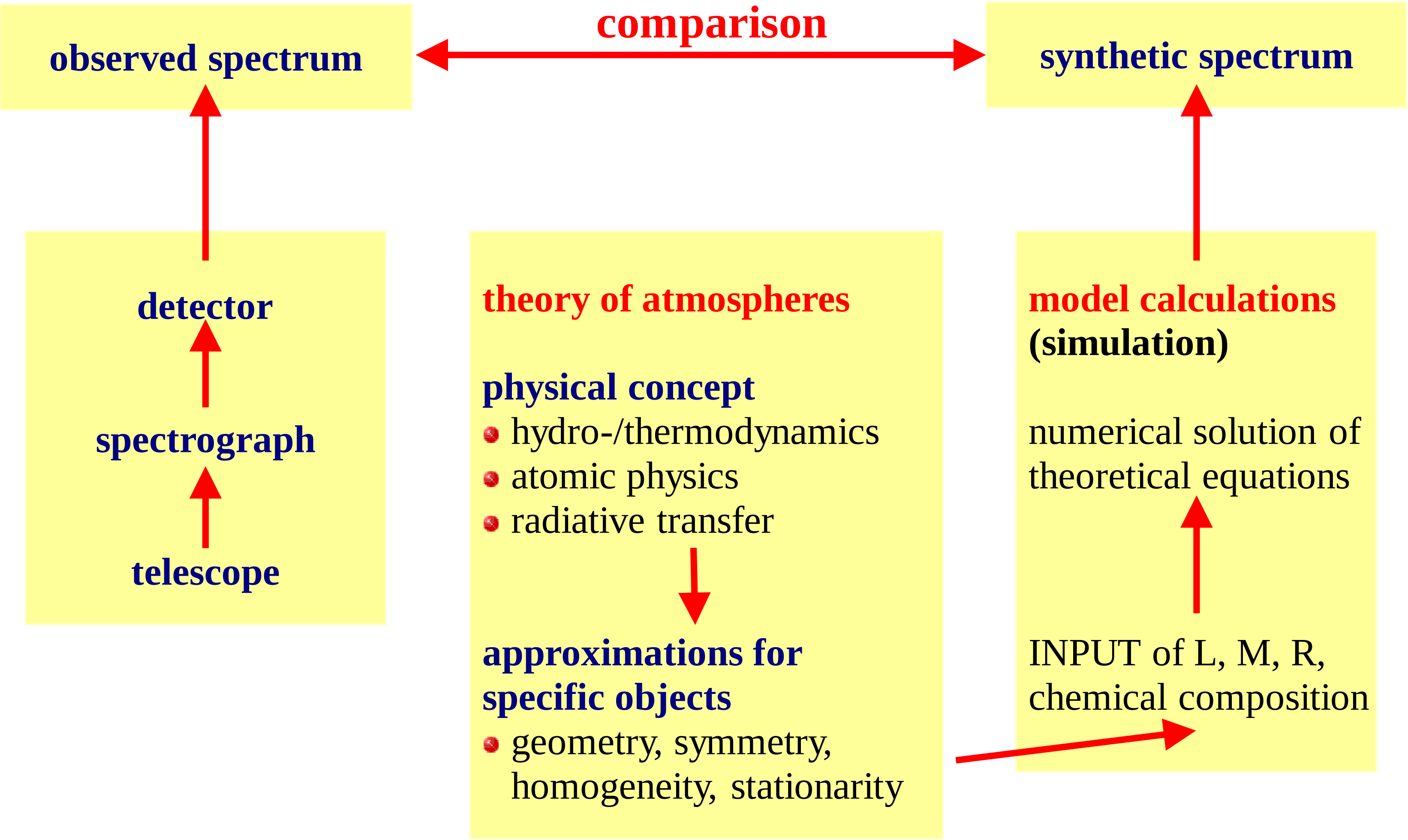
Envelope:

$\Delta R \sim 0.1$pc (PN)

$\Delta R \sim 10$ pc (H II)

$\Delta R \sim 10^{3}$ pc (Quasar)

Concept of spectral analysis



observed spectrum

comparison

synthetic spectrum

detector

spectrograph

telescope

theory of atmospheres

physical concept

- hydro-/thermodynamics
- atomic physics
- radiative transfer

approximations for specific objects

- geometry, symmetry, homogeneity, stationarity

model calculations
(simulation)

numerical solution of theoretical equations

INPUT of L, M, R, chemical composition

The VLT-FLAMES survey of massive stars (‘FLAMES I’)
The VLT-FLAMES Tarantula survey (‘FLAMES II’)

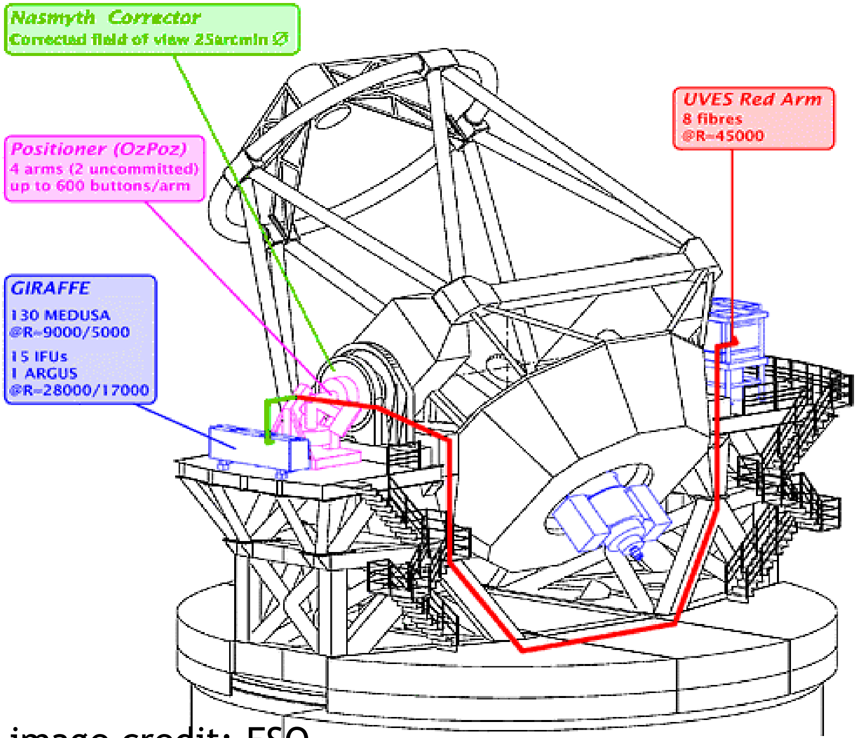
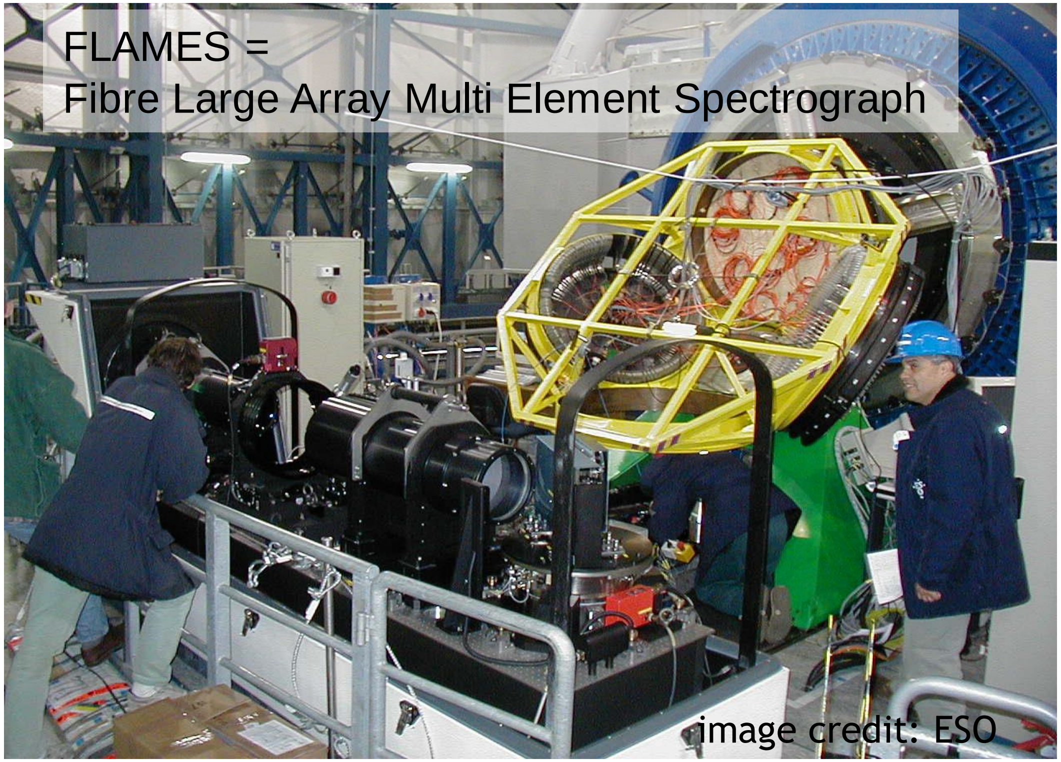


image credit: ESO



FLAMES =
Fibre Large Array Multi Element Spectrograph

image credit: ESO

- **FLAMES I:** high resolution spectroscopy of massive stars in 3 Galactic, 2 LMC and 2 SMC clusters (young and old)
  - total of 86 O- and 615 B-stars
- **FLAMES II**: high resolution spectroscopy of more than 1000 massive stars in Tarantula Nebula (incl. 300 O-type stars)

Major objectives

- rotation and abundances (test rotational mixing)
- stellar mass-loss as a function of metallicity
- binarity/multiplicity (fraction, impact)
- detailed investigation of the closest ‘proto-starburst’

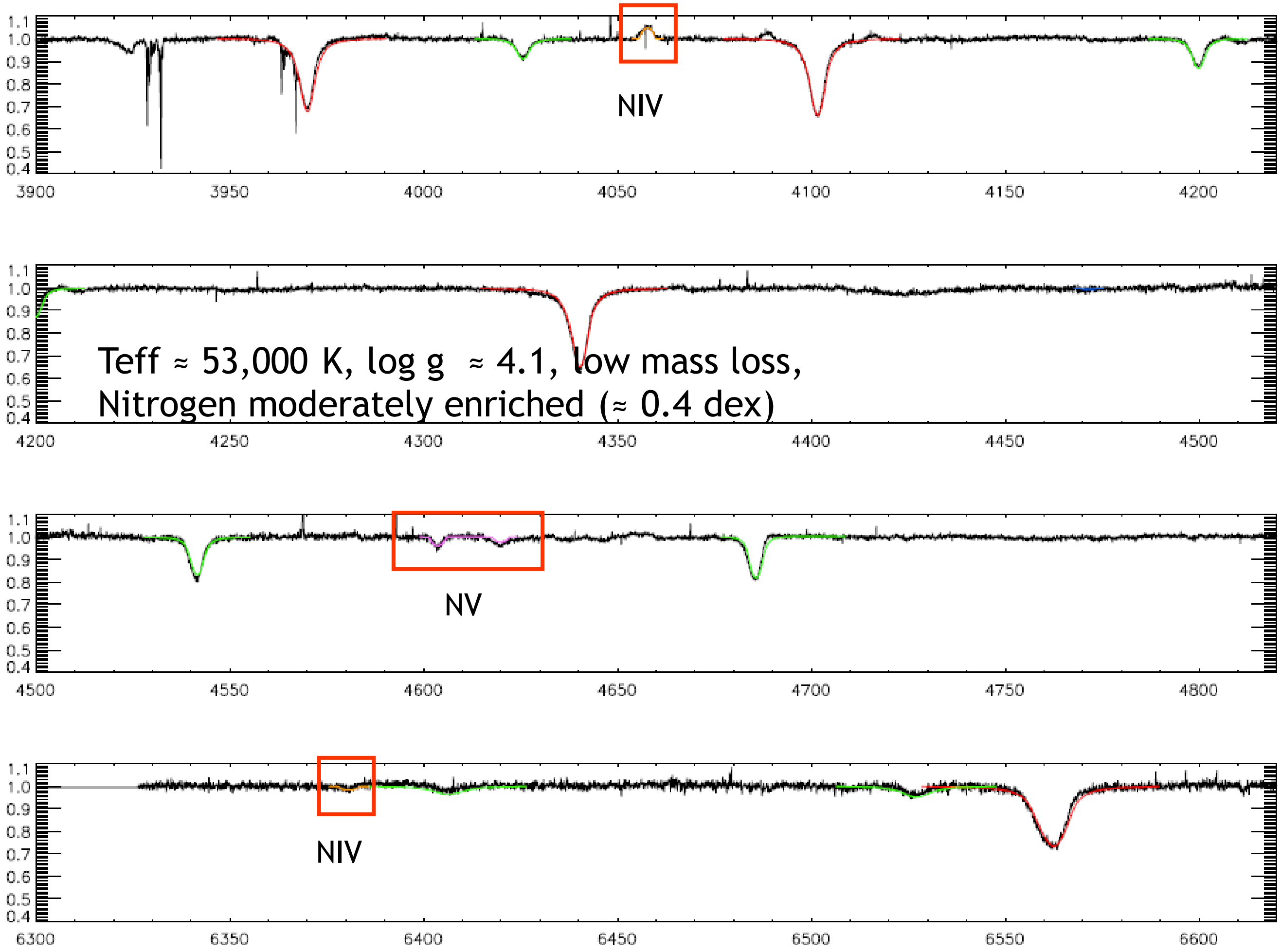
summary of FLAMES I results: Evans et al. (2008)

Optical spectrum of a very hot O-star

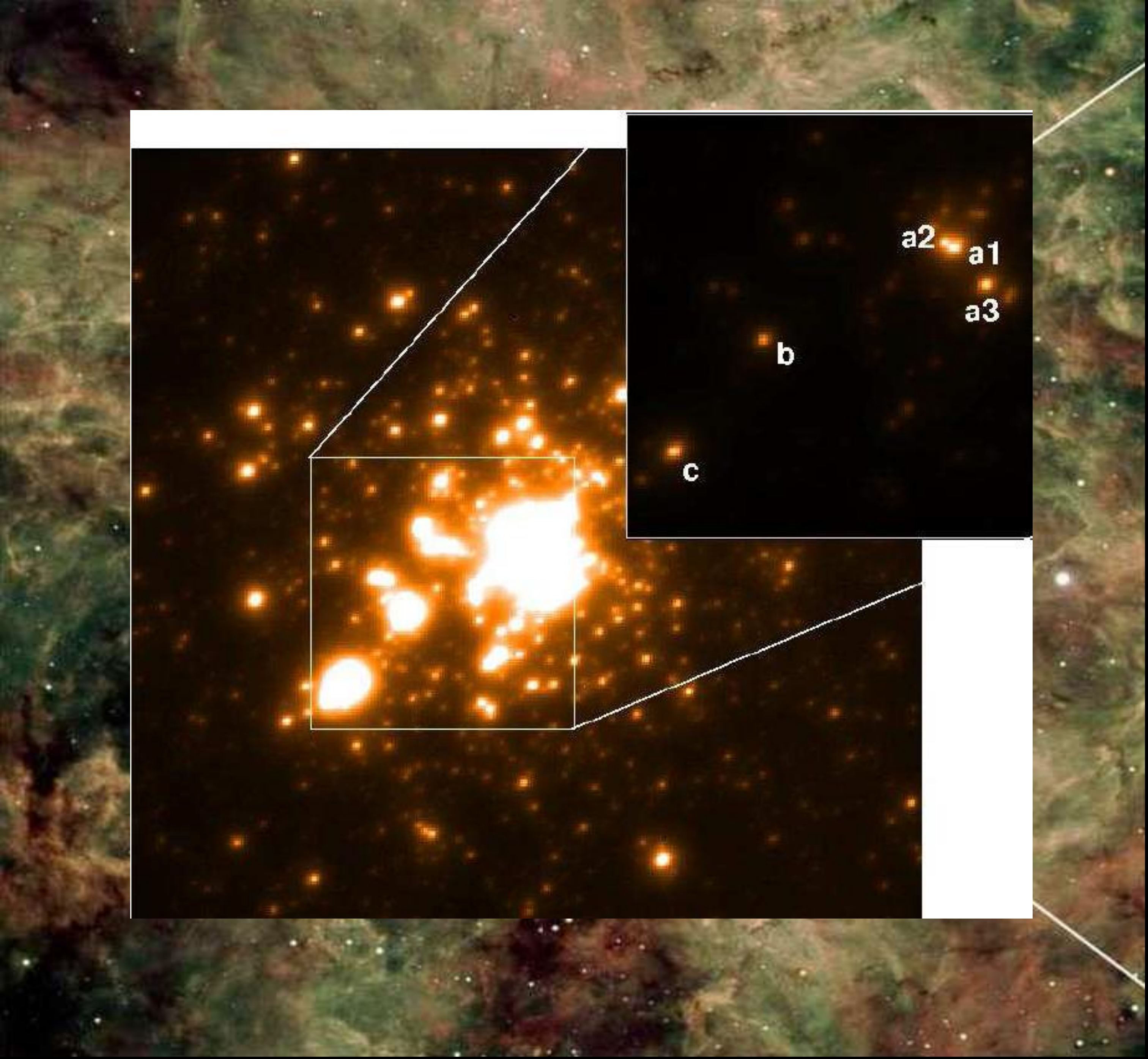
BI237 O2V (f*) (LMC) – vsini = 140 km/s

- Synthetic spectra from Rivero-Gonzalez et al. (2012)

red:	HI
blue:	HeI
green:	HeII
orange:	NIV
magenta:	NV



Teff ≈ 53,000 K, log g ≈ 4.1, low mass loss,
Nitrogen moderately enriched (≈ 0.4 dex)



- Tarantula Nebula (30 Dor) in the LMC
- Largest starburst region in Local Group
- Target of VLT-FLAMES Tarantula survey (‘FLAMES II’, PI: Chris Evans)
- Cluster R136 contains some of the *most massive, hottest, and brightest* stars known
- Crowther et al. (2010): 4 stars with initial masses from 165-320 (!!!) $M_\odot$
- problems with IR-photometry (background-correction), lead to overestimated luminosities → initial masses become reduced: 140 - 195 $M_\odot$ (Rubio-Diez et al., IAUS 329, 2016, and in prep. for A&A)

Spectral energy distribution of the most massive stars in our "neighbourhood"

from Crowther et al. 2010

lower masses from Rubio-Diez et al 2016

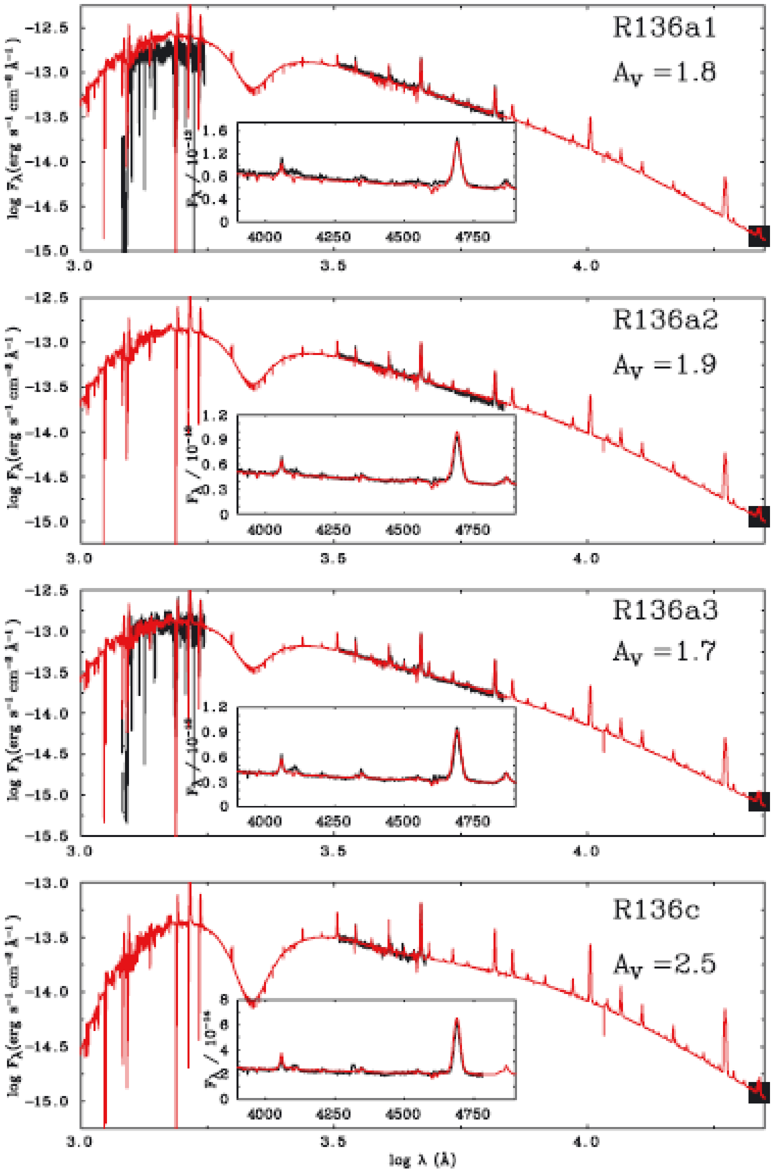


Figure 4. Spectral energy distributions of R136 WN 5h stars from *HST*/FOS together using K_s photometry from VLT/SINFONI calibrated with VLT/MAD imaging. Reddened theoretical spectral energy distributions are shown as red lines.

initial mass (Msun)	current mass (Msun)
320 → <194	265
240 → <140	195
165 → 141	135
220	175
typical uncertainty ± 40 Msun	

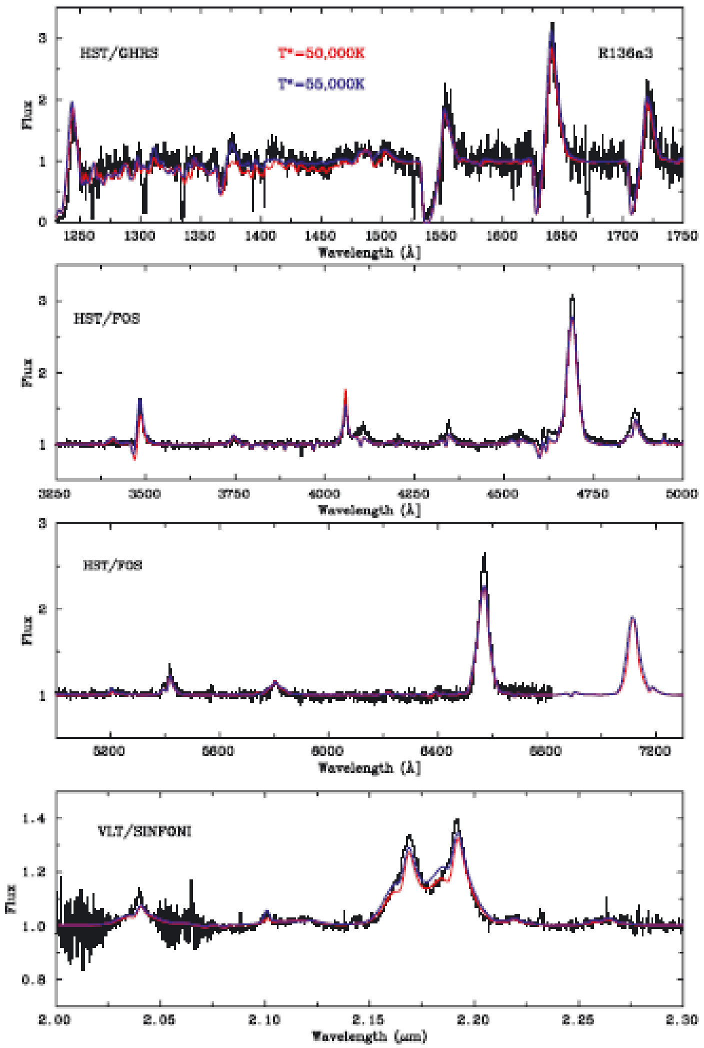


Figure 5. Rectified, ultraviolet (*HST*/GHRS), visual (*HST*/FOS) and near-IR (VLT/SINFONI) spectroscopy of the WN 5h star R136a3 together with synthetic UV, optical and near-IR spectra, for $T_* = 50\,000$ K (red) and $T_* = 55\,000$ K (blue). Instrumental broadening is accounted for, plus an additional rotational broadening of 200 km s^{-1}.

Chap. 3 – The radiation field

Number of particles in $(\mathbf{r},\mathbf{r}+d\mathbf{r})$ with momenta $(\mathbf{p},\mathbf{p}+d\mathbf{p})$ at time t

$$\delta N(\mathbf{r},\mathbf{p},t) = f(\mathbf{r},\mathbf{p},t)\, d^3\mathbf{r}\, d^3\mathbf{p}$$

distribution function f

For a detailed derivation and discussion, see, e.g., Cercignani, C., "The Boltzmann Equation and Its Applications", Appl. Math. Sciences 67, Springer, 1987

i) $f(\mathbf{r},\mathbf{p},t)$ is Lorentz-invariant

ii) $\delta N_0 = f(\mathbf{r}_0,\mathbf{p}_0,t_0)\, d^3\mathbf{r}_0\, d^3\mathbf{p}_0$

evolution

$$\delta N = f(\mathbf{r}_0 + d\mathbf{r}, \mathbf{p}_0 + d\mathbf{p}, t_0 + dt)\, d^3\mathbf{r}\, d^3\mathbf{p}$$

$$(\dot{p} = F) = f(\mathbf{r}_0 + \mathbf{v}dt, \mathbf{p}_0 + \mathbf{F}dt, t_0 + dt)\, d^3\mathbf{r}\, d^3\mathbf{p}$$

Theoretical mechanics: If no collisions, conservation of phase space volume:

$$d^3\mathbf{r}_0\, d^3\mathbf{p}_0 = d^3\mathbf{r}\, d^3\mathbf{p}$$

and

$\delta N_0 = \delta N$ (particles do not "vanish", again no collisions supposed)

$$\Rightarrow \boxed{f(\mathbf{r},\mathbf{p},t) = \text{const}, \quad \text{if no collisions}}$$

$$\Rightarrow \frac{\partial f}{\partial t} + \sum \frac{\partial f}{\partial r_i}\frac{\partial r_i}{\partial t} + \sum \frac{\partial f}{\partial p_i}\frac{\partial p_i}{\partial t} =$$

$$= \underbrace{\frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla) f}_{} + (\mathbf{F}\cdot\nabla_p) f = \begin{cases} 0 & \text{Vlasov} \\ \left(\dfrac{\delta f}{\delta t}\right)_{\text{coll}} & \text{Boltzmann if collisions} \end{cases}$$

D/Dt f, Lagrangian derivative

total derivative of f measured in fluid frame, at times t, t+Δt and positions **r**, **r** + **v** Δt

- implications for photon gas

$$\mathbf{p} = \frac{h\nu}{c}\mathbf{n}$$

$d^3\mathbf{p} = p^2 dp\, d\Omega$ ← solid angle with respect to $\mathbf{n}$

absolute value

$$= \left(\frac{h\nu}{c}\right)^2 \frac{h}{c}\, d\nu\, d\Omega = \frac{h^3}{c^3}\nu^2\, d\nu\, d\Omega$$

$$\Rightarrow f(\mathbf{r},\mathbf{p},t)\, d^3\mathbf{r}\, d^3\mathbf{p} = \frac{h^3}{c^3}\nu^2 f(\mathbf{r},\mathbf{n},\nu,t)\, d^3\mathbf{r}\, d\nu\, d\Omega =$$

$$\boxed{= \Psi(\mathbf{r},\mathbf{n},\nu,t)\, d^3\mathbf{r}\, d\nu\, d\Omega}$$

$$d^3\mathbf{p} = J(\mathbf{p},\mathbf{p}')\, d^3\mathbf{p}', \qquad \mathbf{p}' = (p,\theta,\phi)$$

cartesian Jacobi-det. spherical

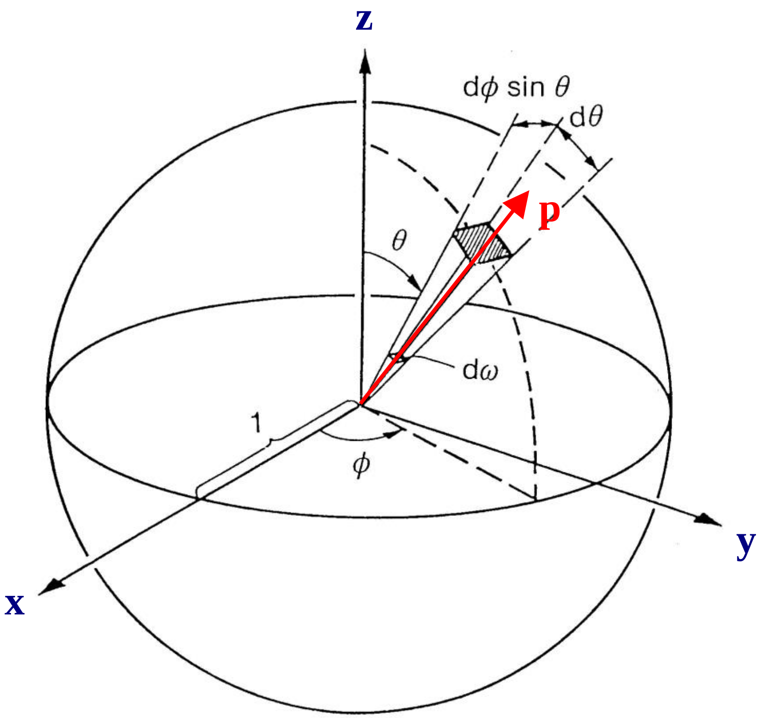
$$\mathrm{p}_x = p\sin\theta\cos\phi$$

$$\mathrm{p}_y = p\sin\theta\sin\phi$$

$$\mathrm{p}_z = p\cos\theta$$

$$J = \det\begin{pmatrix} \frac{\partial p_x}{\partial p} & \frac{\partial p_x}{\partial \theta} & \frac{\partial p_x}{\partial \phi} \\ \frac{\partial p_y}{\partial p} & \frac{\partial p_y}{\partial \theta} & \frac{\partial p_y}{\partial \phi} \\ \frac{\partial p_z}{\partial p} & \frac{\partial p_z}{\partial \theta} & \frac{\partial p_z}{\partial \phi} \end{pmatrix} = \det\begin{pmatrix} \sin\theta\cos\phi & \mathrm{p}\cos\theta\cos\phi & \text{-}\mathrm{p}\sin\theta\sin\phi \\ \sin\theta\sin\phi & \mathrm{p}\cos\theta\sin\phi & \mathrm{p}\sin\theta\cos\phi \\ \cos\theta & \text{-}\mathrm{p}\sin\theta & 0 \end{pmatrix}$$

$$= \text{(exercise)}\ \boxed{p^2\sin\theta}$$

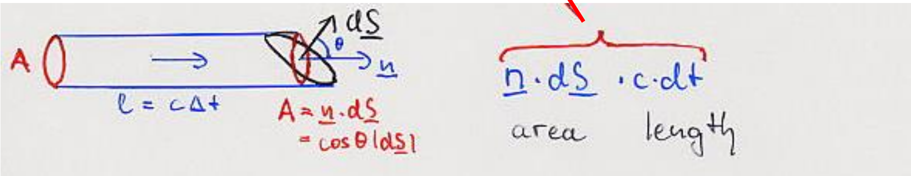


$$\Rightarrow d^3\mathbf{p} = dp_x dp_y dp_z = p^2 dp\, \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega}$$

The specific intensity

Number of photons with ν, $\nu+d\nu$ which propagate through surface element $d\mathbf{S}$ into direction $\mathbf{n}$ and solid angle $d\Omega$, at time t and with velocity c:

$$\delta N = \frac{h^3\nu^2}{c^3} f(\mathbf{r},\mathbf{n},\nu,t)\; d^3\mathbf{r}\; d\nu\, d\Omega$$



$$= \frac{h^3\nu^2}{c^3} f(\mathbf{r},\mathbf{n},\nu,t)\; \underset{\sphericalangle(\mathbf{n},\, d\mathbf{S})}{\cos\theta}\; c dt\; dS\; d\nu\, d\Omega$$

- corresponding energy transport

$$\delta E = h\nu\;\delta N = \underbrace{\frac{h^4\nu^3}{c^2} f(\mathbf{r},\mathbf{n},\nu,t)}_{I(\mathbf{r},\mathbf{n},\nu,t)} \cos\theta\; dS\; d\nu\; dt\; d\Omega$$

specific intensity
[erg cm^{-2} Hz^{-1} s^{-1}sr^{-1}]

summarized

$I = ch\nu\;\Psi = \frac{h^4\nu^3}{c^2} f$ function of $\mathbf{r},\mathbf{n},\nu,t$

specific intensity is radiation energy, which is transported into direction $\mathbf{n}$ through surface $d\mathbf{S}$, per frequency, time and solid angle.

basic quantity in theory of radiative transfer

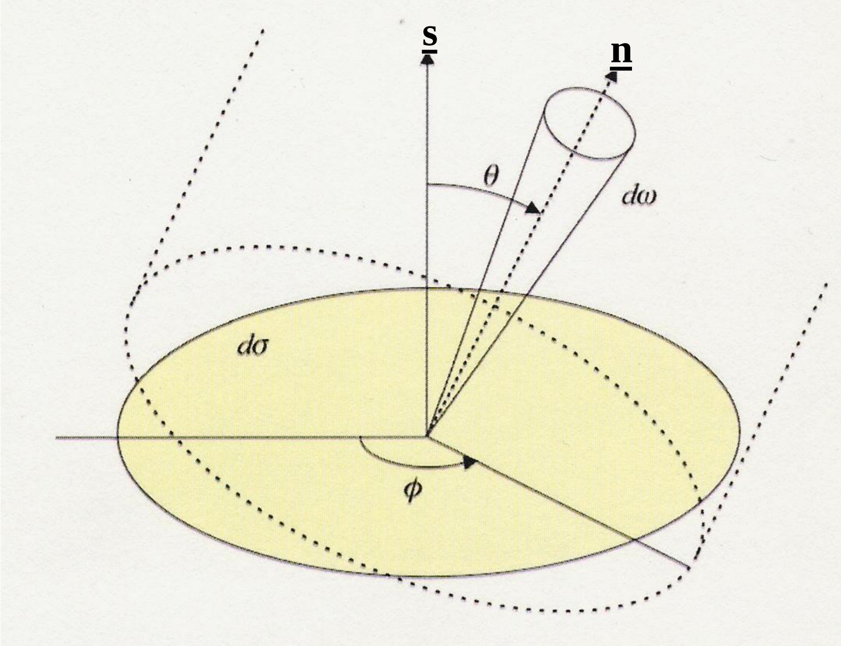
invariance of specific intensity

since $\frac{\mathrm{D}f}{\mathrm{D}t} = 0$ without collisions (Vlasov equation) and without GR (i.e., $\mathbf{F} \equiv \mathbf{0}$), we have

$I \sim f$

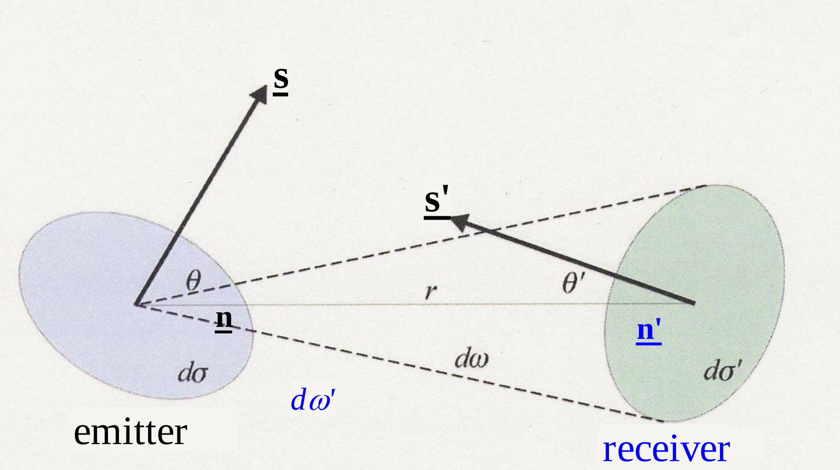
$\Rightarrow$ I = const in fluid frame, as long as no interaction with matter!

If stationary process, i.e. $\partial/\partial t = 0$, then $\underline{\mathbf{n}\nabla}\, I = d/ds\, I = 0$,
where ds is path element, i.e.
I = const also spatially!
(this is the major reason for working with specific intensities)



specific intensity is **radiation energy** with frequencies (ν, ν + dν), which is transported through *projected* area element dσcosθ into direction **n**, per time interval dt and solid angle dω.

$$\delta E = I(\vec{r},\vec{n},\nu,t)\cos\theta\, d\sigma\, d\nu\, dt\, d\omega$$



Invariance of specific intensity

Consider pencil of light rays which passes through both area elements δσ (emitter) and δσ' (receiver).

If no energy sinks and sources in between, the amount of energy which passes through both areas is given by

$$\delta E = I_\nu \cos\theta\, d\sigma\, dt\, d\omega =$$

$$\delta E' = I'_\nu \cos\theta' d\sigma' dt\, d\omega',$$ and, cf. figure,

$$d\omega = \frac{\text{projected area}}{\text{distance}^2} = \frac{\cos\theta' d\sigma'}{r^2}$$

$$d\omega' = \frac{\cos\theta\, d\sigma}{r^2}$$

$\Rightarrow I_\nu = I'_\nu$, independent of distance

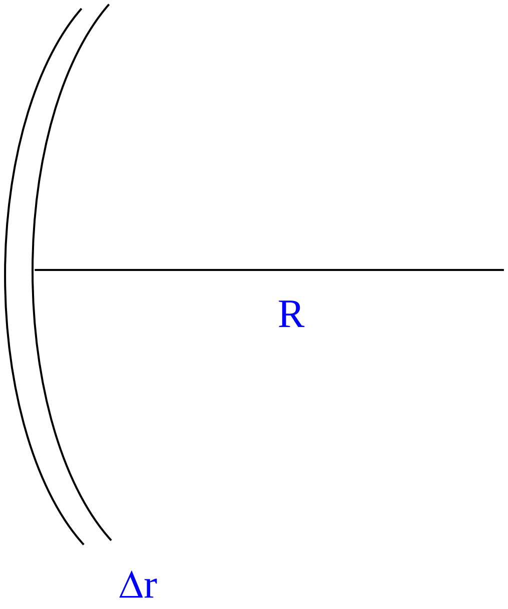
... but energy/unit area dilutes with r^{-2}!

Plane-parallel and spherical symmetries

stars = gaseous spheres => spherical symmetry

BUT rapidly rotating stars (e.g., Be-stars, $v_{rot} \approx 300 \ldots 400$ km/s) rotationally flattened, only axis-symmetry can be used

AND atmospheres usually very thin, i.e. $\Delta r / R \ll 1$



example: the sun

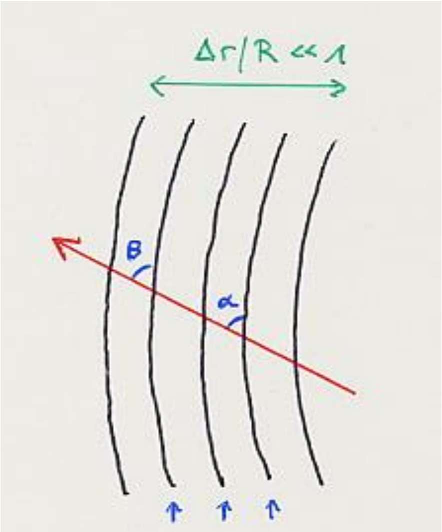
$R_{sun} \approx 700{,}000$ km
Δr (photo) ≈ 300 km

=> $\Delta r / R \approx 4\ 10^{-4}$

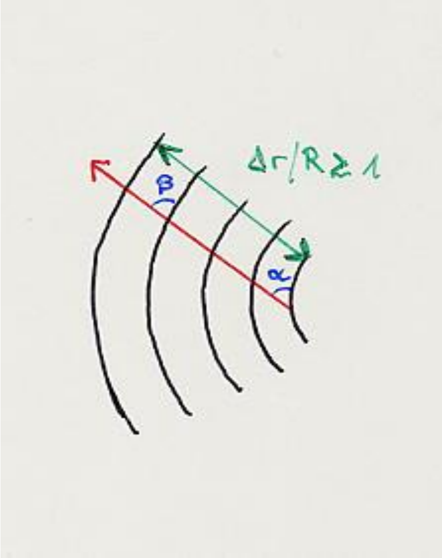
BUT corona
$\Delta r / R$ (corona) ≈ 3

as long as $\Delta r / R << 1$ => plane-parallel symmetry

light ray through atmosphere



lines of constant temperature
and density (isocontours)



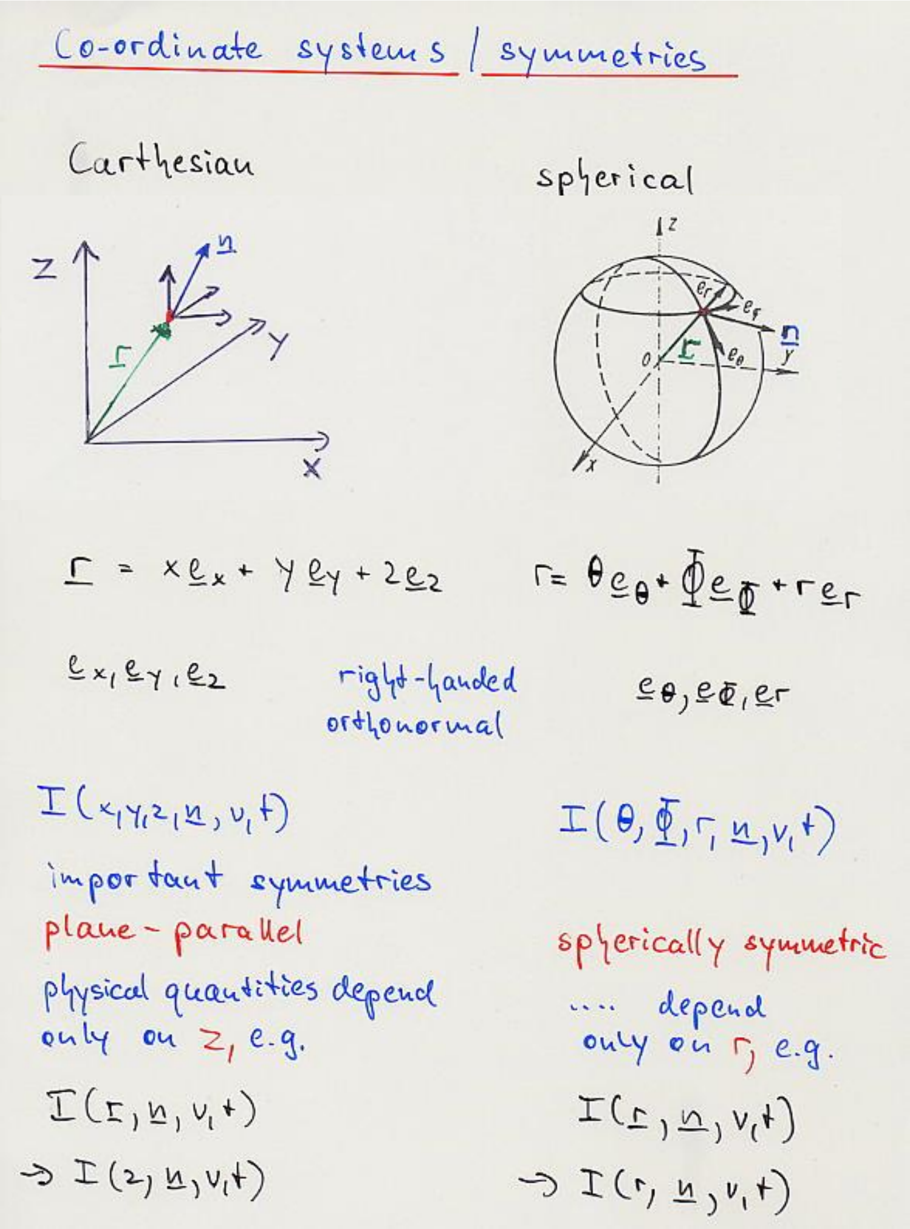
curvature of atmosphere insignificant for photons' path : $\alpha = \beta$

significant curvature : $\alpha \neq \beta$,
spherical symmetry

examples

solar photosphere / cromosphere
atmospheres of
- main sequence stars
- white dwarfs
- giants (partly)

solar corona
atmospheres of
- supergiants
- expanding envelopes (stellar winds)
- of OBA stars, M-giants and supergiants

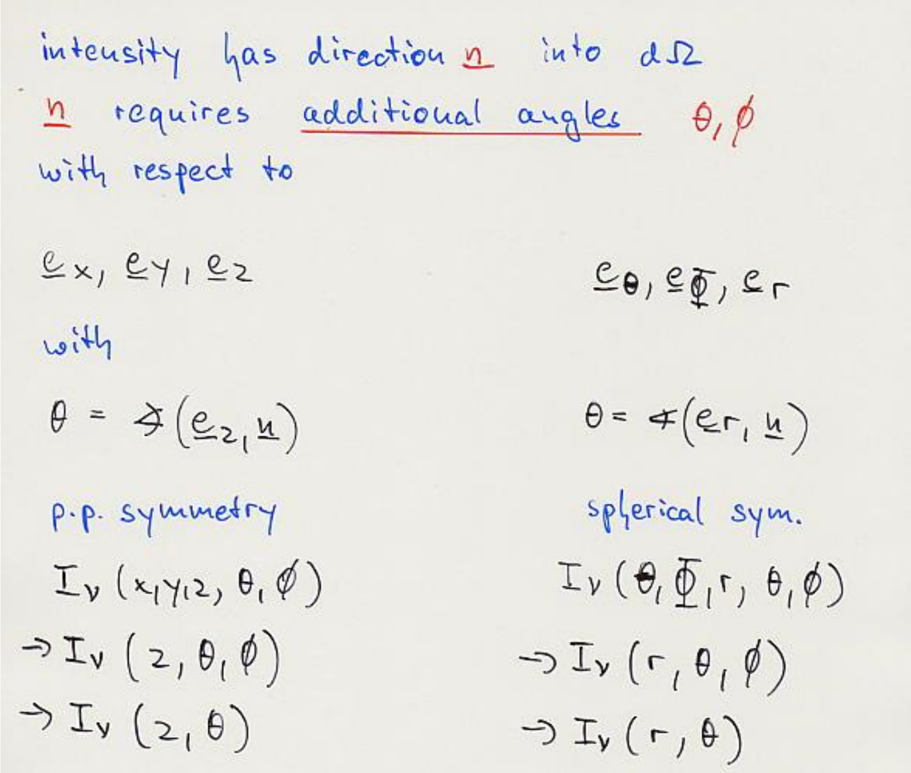


Co-ordinate systems / symmetries

Carthesian	spherical
$\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$	$r = \theta\underline{e}_\theta + \Phi\underline{e}_\Phi + r\underline{e}_r$
$\underline{e}_x, \underline{e}_y, \underline{e}_z$ right-handed orthonormal	$\underline{e}_\theta, \underline{e}_\Phi, \underline{e}_r$
$I(x,y,z,\underline{n},\nu,t)$	$I(\theta,\Phi,r,\underline{n},\nu,t)$

important symmetries

plane-parallel	spherically symmetric
physical quantities depend only on z, e.g.	.... depend only on r, e.g.
$I(\underline{r},\underline{n},\nu,t)$	$I(\underline{r},\underline{n},\nu,t)$
$\rightarrow I(z,\underline{n},\nu,t)$	$\rightarrow I(r,\underline{n},\nu,t)$

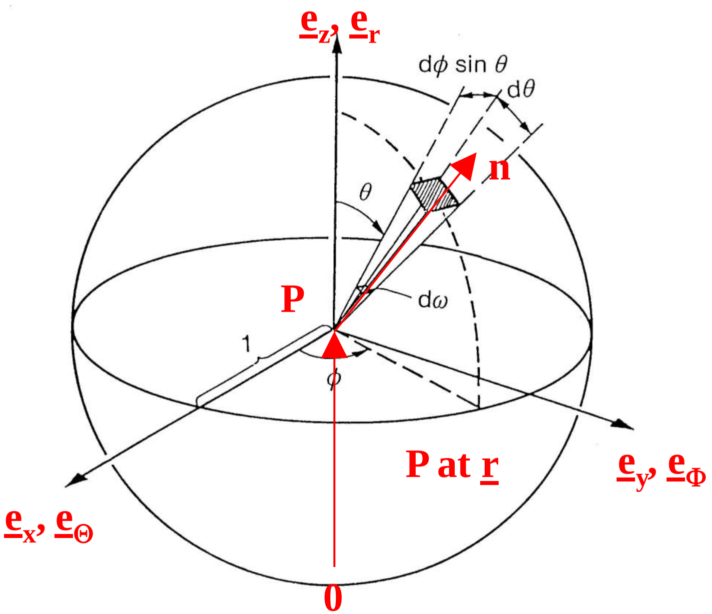


intensity has direction $\underline{n}$ into $d\Omega$

$\underline{n}$ requires additional angles θ, ϕ

with respect to

$\underline{e}_x, \underline{e}_y, \underline{e}_z$	$\underline{e}_\Theta, \underline{e}_\Phi, \underline{e}_r$
with	
$\theta = \measuredangle(\underline{e}_z, \underline{n})$	$\theta = \measuredangle(\underline{e}_r, \underline{n})$
p.p. symmetry	spherical sym.
$I_\nu(x,y,z,\theta,\phi)$	$I_\nu(\Theta,\Phi,r,\theta,\phi)$
$\rightarrow I_\nu(z,\theta,\phi)$	$\rightarrow I_\nu(r,\theta,\phi)$
$\rightarrow I_\nu(z,\theta)$	$\rightarrow I_\nu(r,\theta)$



Moments of the specific intensity

1. mean intensity

$$J(\underline{r},\nu,t) = \frac{1}{4\pi}\oint I(\underline{r},\underline{n},\nu,t)\,d\Omega$$

specific intensity, averaged over solid angle

def. of solid angle

solid angle = ratio of area of sphere to radius^2

total solid angle = $\frac{4\pi R^2}{R^2} = 4\pi$

$d\Omega$ with $r=1$ = dA

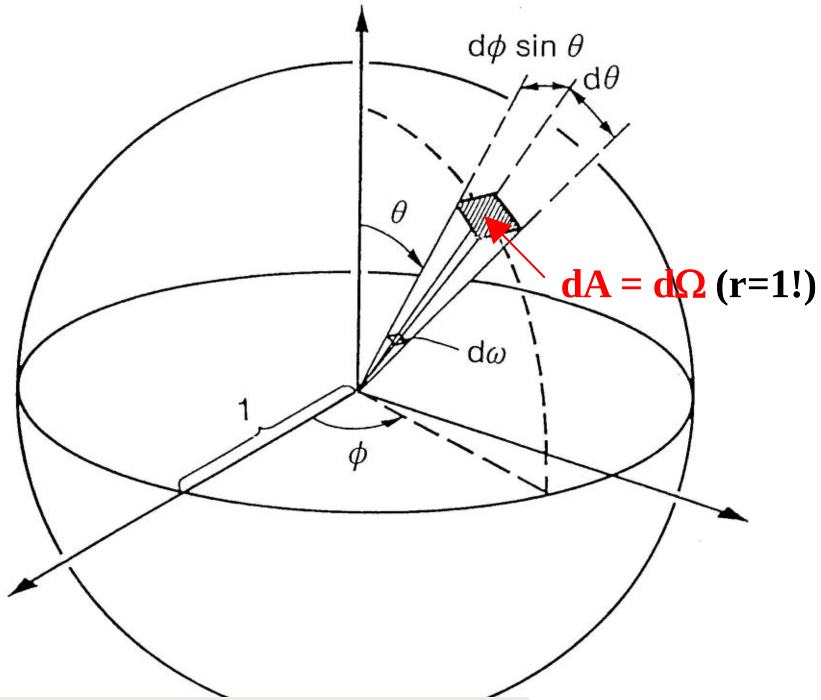
area = $d\theta \times \sin\theta\, d\phi$

def: $\mu =: \cos\theta$

$d\mu = -\sin\theta\, d\theta \Rightarrow d\Omega = -d\mu\, d\phi$

THUS

$$J(\underline{r},\nu,t) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{0\to+1}^{\pi\to-1} \underset{\text{usually } f(\theta,\phi)}{\underset{\downarrow}{I(\underline{r},\underline{n},\nu,t)}}\,\underbrace{\sin\theta\, d\theta}_{-d\mu}$$



In plane-parallel or spherical symmetry:

$$J_\nu(\underline{r},t) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{+1} I_\nu(\underline{r},\mu,t)\,d\mu =$$

$$= \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\,d\mu \qquad \text{"0th" moment}$$

The Planck function

... on the other hand

energy density (i.e., per Volume $d^3\underline{r}$) per $d\nu$ (i.e., spectral) $= h\nu \oint (\text{distr. function})\, d\Omega$

$$u_\nu(\underline{r},t) = h\nu \oint \psi_\nu(\underline{r},\mu,t)\, d\Omega$$

$$\overset{\text{def.}}{=} \frac{1}{c} \oint I_\nu(\underline{r},\mu,t)\, d\Omega = \frac{4\pi}{c} J_\nu(r,t)$$

$\dim[u_\nu] = \text{erg cm}^{-3}\,\text{Hz}^{-1}$

$\dim[J_\nu] = \text{erg cm}^{-2}\,\text{Hz}^{-1}\,\text{s}^{-1}$

- from thermodynamics, we know spectral energy density of a cavity or black body radiator (in thermodynamic equilibrium, "TE", with isotropic radiation, independent of material)

$$u_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT}-1}$$

$$\Rightarrow J_\nu = \frac{c}{4\pi} u_\nu \quad \text{and} \quad J_\nu = \frac{1}{2}\int_{-1}^{+1} \overset{\text{isotropic}}{I_\nu}\, d\mu = I_\nu$$

specific intensity of a cavity/black body radiator at temperature T

$$I_\nu^{*} \overset{\text{T.E.}}{=} B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT}-1} \quad \text{"Planck-Function"}$$

properties of Planck function

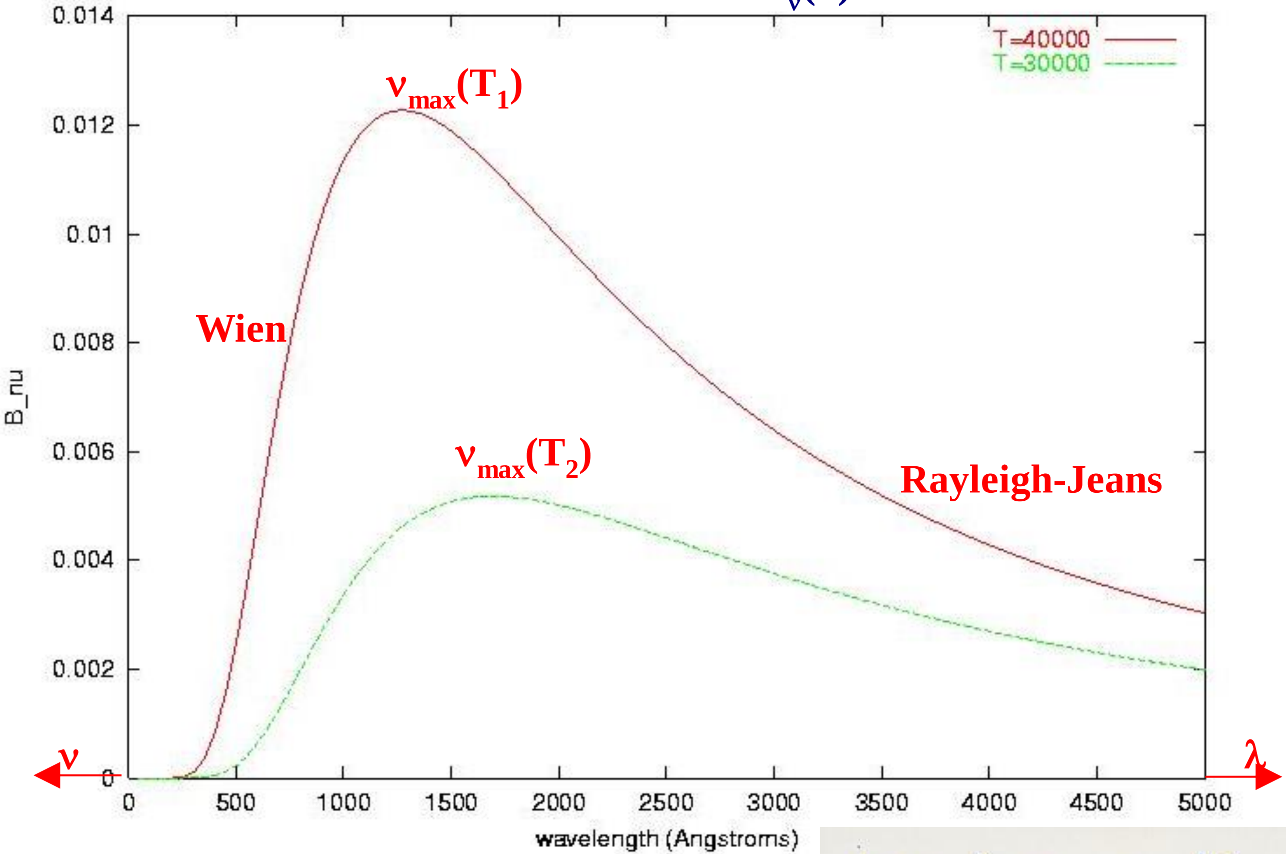
- $B_\nu(T_1) > B_\nu(T_2) \quad \forall \nu, \text{ if } T_1 > T_2$
  i.e., Planck functions do not cross each other!
- maximum is shifted towards higher wavelengths with decreasing temperature
  $\frac{\nu_{max}}{T} = \text{const}$, Wien's displacement law
- Wien regime $\frac{h\nu}{kT} \gg 1 \Rightarrow B_\nu \approx \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$
- Rayleigh Jeans regime $\frac{h\nu}{kT} \ll 1 \Rightarrow B_\nu \approx \frac{2h\nu^3}{c^2}\frac{kT}{h\nu} = \frac{2\nu^2}{c^2} kT$

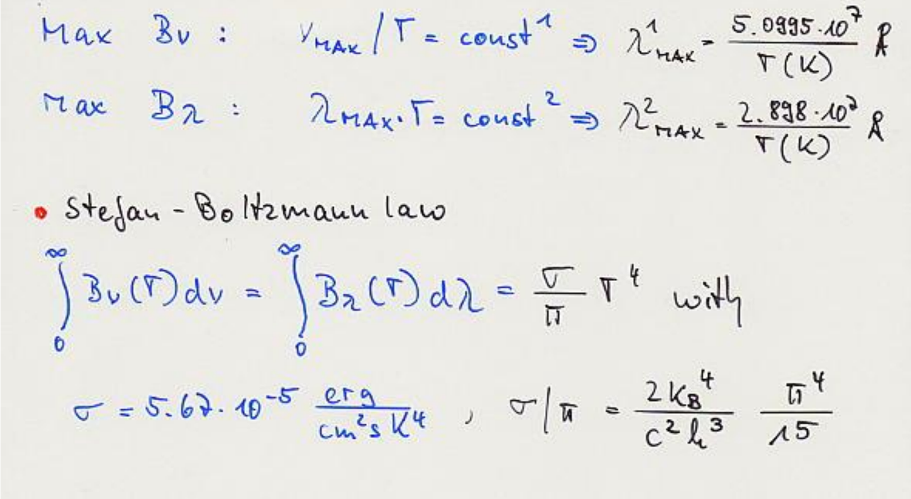
NOTE: in a number of cases one finds $B_\lambda \neq B_\nu$ since $B_\lambda d\lambda = B_\nu d\nu$

$$\Rightarrow B_\lambda = B_\nu \left|\frac{d\nu}{d\lambda}\right| = B_\nu \frac{c}{\lambda^2} = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/kT\lambda}-1}$$

$$\Rightarrow \text{Max}(B_\lambda) \neq \text{Max}(B_\nu)\,!$$

Planck function $B_\nu(\lambda)$





Max B_ν : $\nu_{MAX} / T = \text{const}^1 \Rightarrow \lambda^1_{MAX} = \frac{5.0995 \cdot 10^7}{T(K)}$ Å

Max B_λ : $\lambda_{MAX} \cdot T = \text{const}^2 \Rightarrow \lambda^2_{MAX} = \frac{2.898 \cdot 10^7}{T(K)}$ Å

- Stefan - Boltzmann law

$$\int_0^\infty B_\nu(T)\, d\nu = \int_0^\infty B_\lambda(T)\, d\lambda = \frac{\sigma}{\pi} T^4 \text{ with}$$

$$\sigma = 5.67 \cdot 10^{-5} \frac{erg}{cm^2 s K^4}, \quad \sigma / \pi = \frac{2 k_B^4}{c^2 h^3} \frac{\pi^4}{15}$$

1st moment: radiative flux

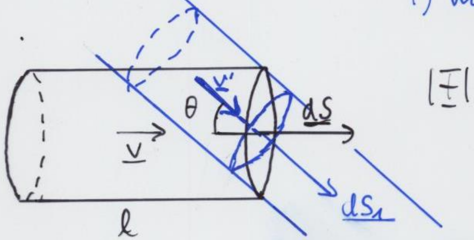
a) general definition

flux: rate of flow of a quantity across a given surface

flux-density: flux/unit area, also called flux

vector quantity

i) mass flux $\underline{v} \parallel d\underline{S}$



$$|\underline{\mathcal{F}}| = \frac{m}{\Delta t\, |d\underline{S}|}$$

$$= \frac{m}{\mathrm{Vol}} \frac{l}{\Delta t} = \rho |v|$$

mass flux = mass $\underline{\text{density}}$ • velocity

ii) $\underline{v}'$ arbitrarily oriented with respect to $d\underline{S}$

$$|\underline{\mathcal{F}}| = \frac{m}{\Delta t\, |dS|} = \frac{m}{\Delta t\, |dS_1|} \frac{|dS_1|}{|dS|} = \frac{m}{\mathrm{Vol}} |v'| \frac{|dS| \cos\theta}{|dS|}$$

$\uparrow$ $\mathrm{Vol} = |v'|\, \Delta t\, |dS_1|$

$$= \rho |v'| \cos\theta$$

⇒ mass flux through $\underline{dS}$ $= \underline{\mathcal{F}} \cdot d\underline{S} = \rho \cdot \underbrace{\underline{v}' \cdot d\underline{S}}_{|v'|\, |dS| \cdot \cos\theta}$

is reduced by factor $\cos\theta$,
since less material is transported
across smaller effective area ⊥ flow
(in same Δt)

iii) mass-loss rate for spherically sym. outflow

$$\dot{M} = \underbrace{(\rho v)(r)}_{\text{mass flux}} \cdot \underbrace{4\pi r^2}_{\text{surface area}}$$

transported mass/unit time across surface with radius r

$\cos\theta = 1$!

b) application to radiation field

- photon flux through surface $d\underline{S}$ into direction $\underline{n}$ and solid angle $d\Omega$ ("radiation pencil")

$$\frac{\delta N}{dt\, d\nu} = \Big(\underbrace{\Psi(\underline{r}, \underline{n}, \nu, t)\, d\Omega}_{\text{number DENSITY}} \cdot \underbrace{c \cdot \underline{n}}_{\text{velocity}} \Big) \cdot d\underline{S}$$

- net rate of $\underline{\text{total}}$ photon flow across $d\underline{S}$ (i.e., contribution of all pencils)

$$\frac{N}{dt\, d\nu} = \Big(c \oint \Psi(\underline{r}, \underline{n}, \nu, t)\, \underline{n}\, d\Omega \Big) \cdot d\underline{S}$$

- net rate of $\underline{\text{radiant energy flow}}$ across $d\underline{S}$

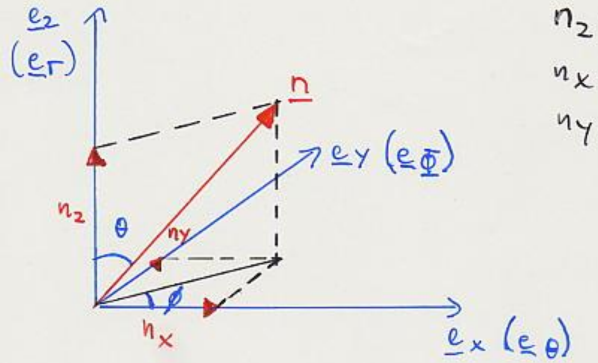
$$\frac{E}{dt\, d\nu} = \Big(c h\nu \oint \Psi(\underline{r}, \underline{n}, \nu, t)\, \underline{n}\, d\Omega \Big) d\underline{S} =$$

$$\overset{\text{def.}}{=} \Big(\oint I(\underline{r}, \underline{n}, \nu, t)\, \underline{n}\, d\Omega \Big) d\underline{S}$$

$$= \underline{\mathcal{F}}_\nu (\underline{r}, t) \cdot d\underline{S}$$

$$\underline{\mathcal{F}}_\nu (\underline{r}, t) = \oint I_\nu(\underline{r}, \underline{n}, t)\, \underline{n}\, d\Omega$$ radiative flux

$$\dim[\mathcal{F}_\nu] = \frac{\mathrm{erg}}{\mathrm{cm}^2\, \mathrm{s}\, \mathrm{Hz}} = \dim[J_\nu]$$



$n_z = \cos\theta = \mu$

$n_x = \sin\theta\cos\phi$

$n_y = \sin\theta\sin\phi$

Note: Carthesian / spherical co-ordinate system

$$\begin{pmatrix}\underline{e}_\theta\\ \underline{e}_\Phi\\ \underline{e}_r\end{pmatrix} \mathrel{\hat{=}} (\text{locally}) \begin{pmatrix}\underline{e}_x\\ \underline{e}_y\\ \underline{e}_z\end{pmatrix},$$

θ, ϕ defined similarly

$$\Rightarrow \underline{\mathcal{F}} = \begin{pmatrix}\mathcal{F}_{x,\theta}\\ \mathcal{F}_{y,\Phi}\\ \mathcal{F}_{z,r}\end{pmatrix} = \begin{pmatrix} & I n_x\, d\Omega\\ \oint & I n_y\, d\Omega\\ & I n_z\, d\Omega\end{pmatrix} = \int_0^{2\pi} d\phi \begin{matrix}\cos\phi\\ \sin\phi\\ 1\end{matrix} \int_{-1}^{1} d\mu \begin{matrix} I(1-\mu^2)^{\frac{1}{2}}\\ I(1-\mu^2)^{\frac{1}{2}}\\ I\mu\end{matrix}$$

p.p. / spherical symmetric

$I(\underline{r}, \underline{n}, \nu, t) \Rightarrow I(\genfrac{}{}{0pt}{}{r}{z}, \mu, \nu, t)$ independent of ϕ,

$x(\theta)$, $y(\Phi)$ comp. cancel each other

(math: $\cos\phi$, $\sin\phi$ integrals $= 0$)

$$\Rightarrow \underline{\mathcal{F}} = \left(0, 0, 2\pi\int_{-1}^{+1} I(\genfrac{}{}{0pt}{}{r}{z}, \mu, \nu, t)\,\mu\, d\mu\right)$$

- in analogy to mean intensity $J_\nu = \frac{1}{2}\int_{-1}^{+1} I(\mu)\, d\mu$

  we define the Eddington flux

$$H_\nu(\genfrac{}{}{0pt}{}{r}{z}, t) = \frac{1}{2}\int_{-1}^{+1} I_\nu(\genfrac{}{}{0pt}{}{r}{z}, \mu, t)\,\mu\, d\mu = \frac{1}{4\pi}\mathcal{F}_\nu(\genfrac{}{}{0pt}{}{r}{z}, t)$$

"first moment"

- "flux" from a cavity radiator

  small opening

$$\mathcal{F}_\nu = 2\pi\int_{-1}^{+1} I(\mu)\,\mu\, d\mu = 2\pi\int_0^1 I(\mu)\,\mu\, d\mu - 2\pi\int_0^1 I(-\mu)\,\mu\, d\mu$$

$$= \mathcal{F}^+ - \mathcal{F}^-$$

only photons escaping from radiatior

$I(\mu)$, $\mu = 0 \dots 1$ $= B_\nu(T)$ isotropic radiation

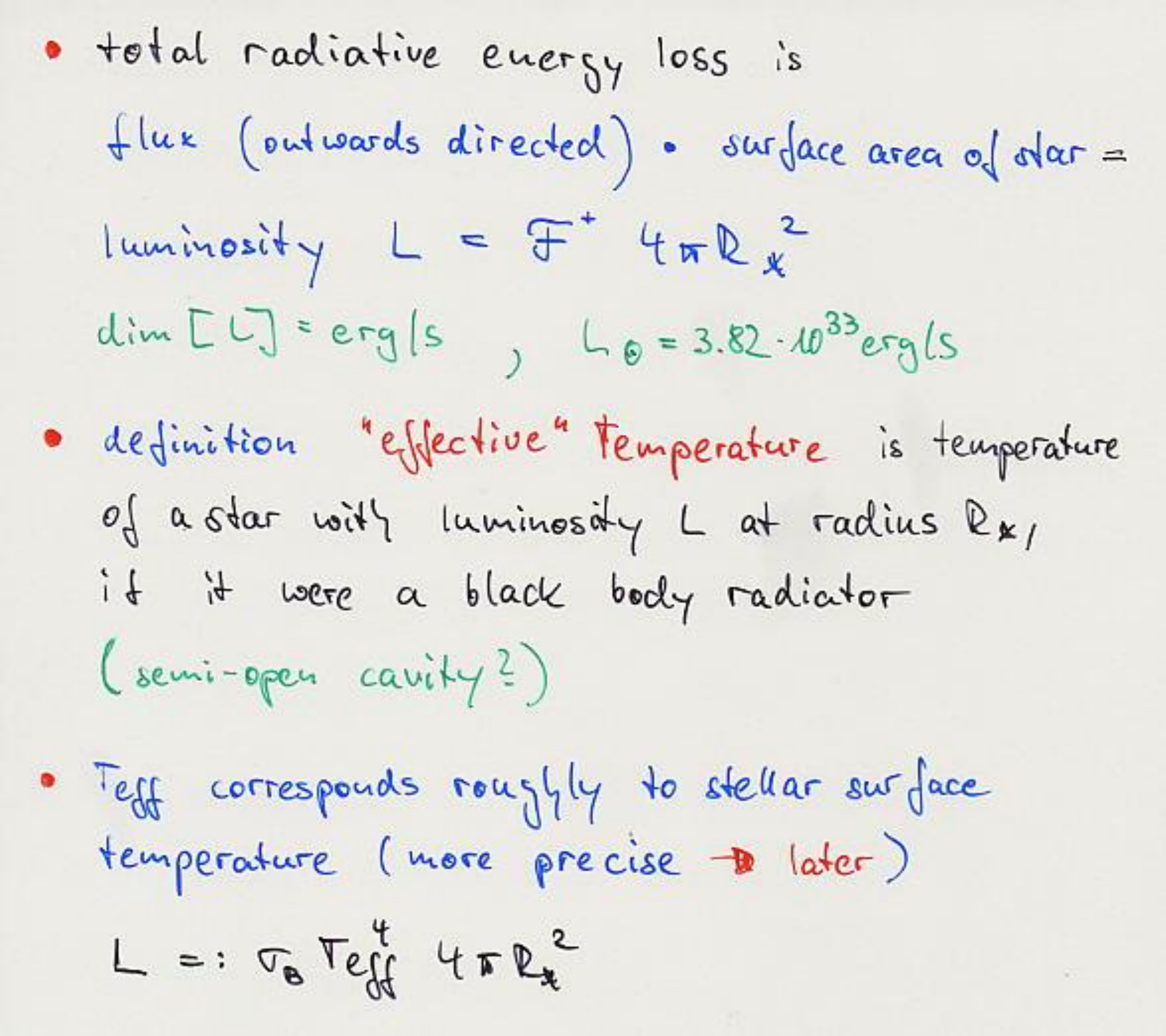
$I(-\mu)$ $= 0$

$$\Rightarrow \mathcal{F} = \int_0^\infty \pi B_\nu(T)\, d\nu = \pi \cdot \frac{\sigma_B}{\pi} T^4 = \sigma_B T^4$$

REMEMBER Black Body

frequ. integrated	specific and mean intensity	$\frac{\sigma_B}{\pi} T^4$
"	energy density	$\frac{4\sigma_B}{c} T^4$
"	flux	$\sigma_B T^4$

Effective temperature



- total radiative energy loss is

  flux (outwards directed) • surface area of star =

  luminosity $L = \mathcal{F}^{+} \, 4\pi R_*^2$

  dim $[L]$ = erg/s , $L_\odot = 3.82 \cdot 10^{33}$ erg/s

- definition "effective" Temperature is temperature of a star with luminosity L at radius R_*, if it were a black body radiator

  (semi-open cavity ?)

- T_{eff} corresponds roughly to stellar surface temperature (more precise → later)

  $$L =: \sigma_B T_{eff}^4 \, 4\pi R_*^2$$

Examples

i) isotropic radiation

see exercise

ii) extremely anisotropic radiation

see exercise

iii) $\mathcal{F}_\nu^+ = 2\pi \int_0^1 I(\mu)\mu d\mu$ is stellar radiation energy, emitted into ALL directions (per $dS, d\nu, dt$)

$= \frac{d^2}{R_*^2} f_\nu$, if f_ν is the energy received on earth (per $dS, d\nu, dt$), d is the distance and $d \gg R_*$ [no extinction!]

proof if no extinction, totally emitted stellar energy remains conserved

$$L = \text{const} = \mathcal{F}_\nu^+(R_*) \cdot 4\pi R_*^2 \overset{!}{=} f_\nu^{\text{obs}}(d)\, 4\pi d^2$$

$$\Rightarrow f_\nu^{\text{obs}}(d) = \mathcal{F}_\nu^+(R_*) \frac{R_*^2}{d^2} \qquad \text{q.e.d.}$$

("quadratic dilution")

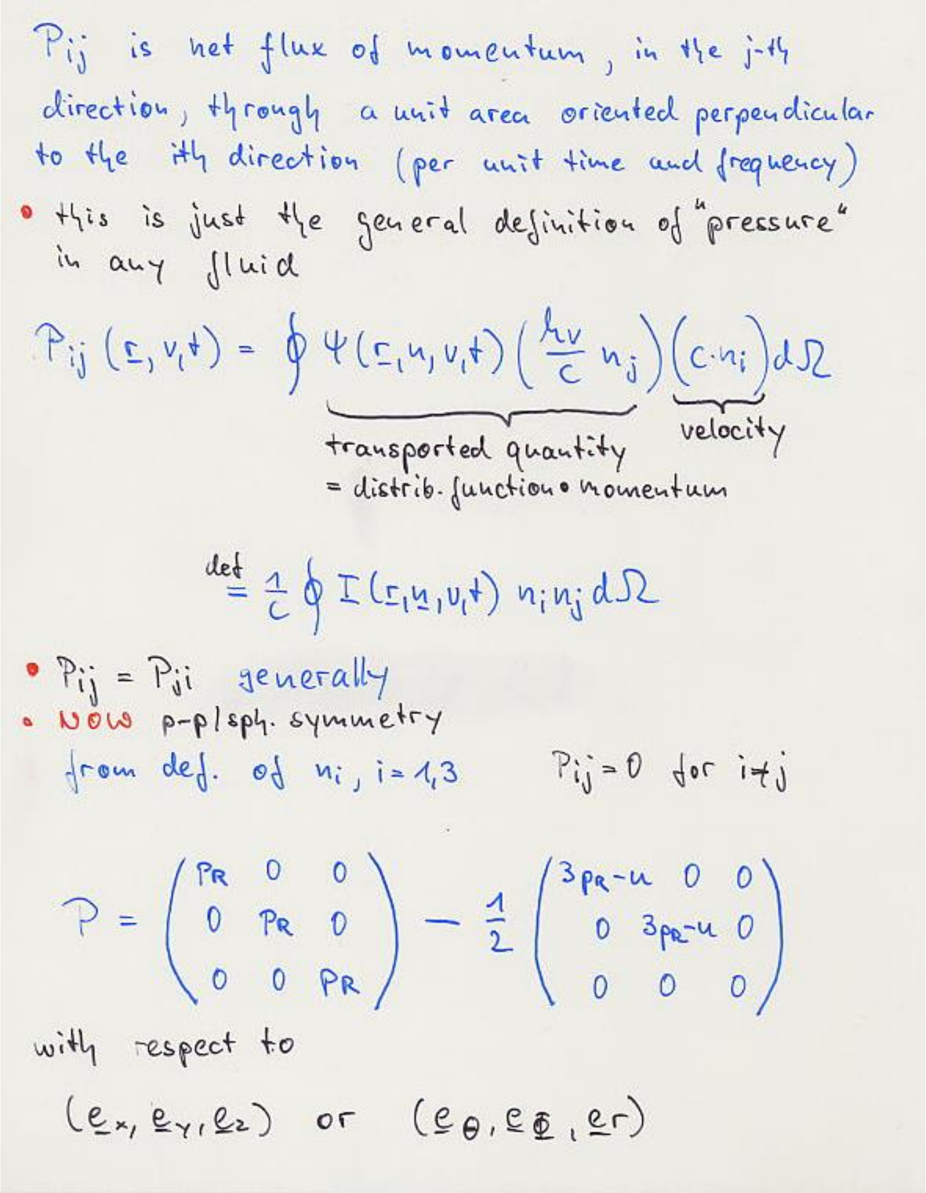
iv) solar constant

see exercise

v) exercise

How many $L_\odot$ is emitted by a typical O-supergiant with $T_{\text{eff}} = 40{,}000$ K and $R_* = 20 R_\odot$? Where is its spectral maximum?

2nd moment: radiation pressure (stress) tensor



P_{ij} is net flux of momentum, in the j-th direction, through a unit area oriented perpendicular to the ith direction (per unit time and frequency)

- this is just the general definition of "pressure" in any fluid

$$P_{ij}(\underline{r},\nu,t) = \oint \underbrace{\Psi(\underline{r},\underline{n},\nu,t)\left(\frac{h\nu}{c}n_j\right)}_{\substack{\text{transported quantity}\\ =\text{distrib. function}\cdot\text{momentum}}}\underbrace{\left(c\cdot n_i\right)}_{\text{velocity}}d\Omega$$

$$\overset{\text{def}}{=} \frac{1}{c}\oint I(\underline{r},\underline{n},\nu,t)\; n_i n_j \, d\Omega$$

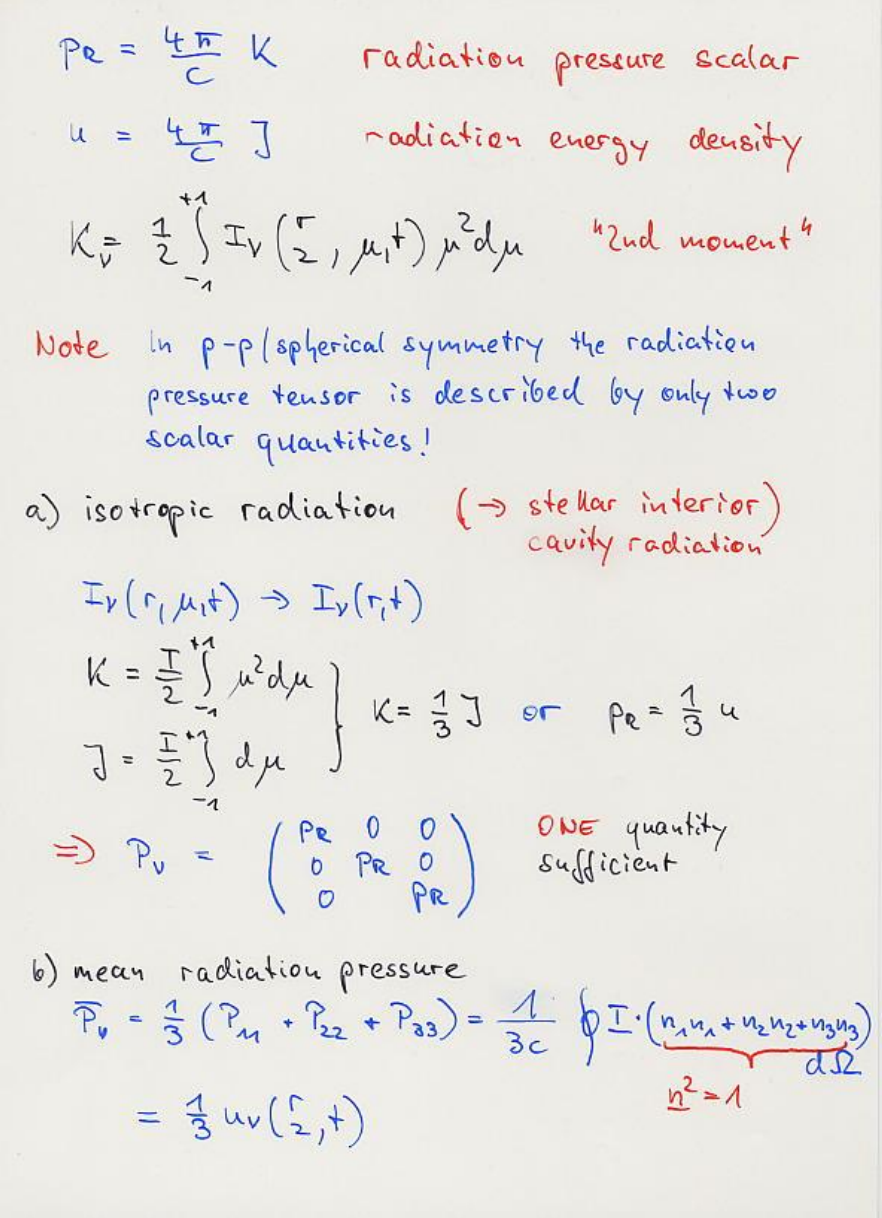
- $P_{ij} = P_{ji}$ generally
- NOW p-p/sph. symmetry

  from def. of n_i, i = 1,3 $\qquad P_{ij} = 0$ for $i \neq j$

$$P = \begin{pmatrix} P_R & 0 & 0 \\ 0 & P_R & 0 \\ 0 & 0 & P_R \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 3P_R - u & 0 & 0 \\ 0 & 3P_R-u & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to

$(\underline{e}_x, \underline{e}_y, \underline{e}_z)$ or $(\underline{e}_\Theta, \underline{e}_\Phi, \underline{e}_r)$



$P_R = \frac{4\pi}{c} K$ radiation pressure scalar

$u = \frac{4\pi}{c} J$ radiation energy density

$$K_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\underline{r},\mu,t)\,\mu^2 d\mu$$ "2nd moment"

Note In p-p/spherical symmetry the radiation pressure tensor is described by only two scalar quantities!

a) isotropic radiation (→ stellar interior) cavity radiation

$I_\nu(r,\mu,t) \rightarrow I_\nu(r,t)$

$$\left.\begin{aligned} K &= \frac{I}{2}\int_{-1}^{+1}\mu^2 d\mu \\ J &= \frac{I}{2}\int_{-1}^{+1} d\mu \end{aligned}\right\} K = \frac{1}{3}J \quad \text{or} \quad P_R = \frac{1}{3}u$$

$$\Rightarrow P_\nu = \begin{pmatrix} P_R & 0 & 0 \\ 0 & P_R & 0 \\ 0 & & P_R \end{pmatrix}$$ ONE quantity sufficient

b) mean radiation pressure

$$\overline{P}_\nu = \frac{1}{3}\left(P_{11}+P_{22}+P_{33}\right) = \frac{1}{3c}\oint I\cdot\Big(\underbrace{n_1n_1+n_2n_2+n_3n_3}_{\underline{n}^2=1}\Big)d\Omega$$

$$= \frac{1}{3}u_\nu(\underline{r},t)$$

c) divergence of radiation pressure tensor

gas pressure → pressure force $\sim -\underline{\nabla} p$

here: radiative acceleration = volume forces exerted by radiation field

$$(\underline{\nabla}\cdot\underline{\underline{P}})_i = \sum_j \frac{\partial}{\partial x_j} P_{ij}$$ ith component of divergence (Cartesian)

- p-p symmetry $p_R, u = f(z)$

  only $\frac{\partial}{\partial z} \neq 0 \Rightarrow$

  $$(\underline{\nabla}\cdot\underline{\underline{P}})_z = \frac{\partial p_R(z,\nu,t)}{\partial z}$$

- spherical symmetry

  only $(\underline{\nabla}\cdot\underline{\underline{P}})_r$ has non-vanishing component

  $$(\underline{\nabla}\cdot\underline{\underline{P}})_r = \frac{\partial p_R}{\partial r} + \frac{1}{r}(3p_R - u)$$

so far, this is the only expression which is different in p-p and spherical symmetry!

For symmetric tensors T^{ij} $(i, j = \Theta, \Phi, r)$ one can prove the following relations (e.g., Mihalas & Weibel Mihalas, "Foundations of Radiation Hydrodynamics", Appendix)

$$(\nabla\cdot T)_r = \frac{1}{r^2}\frac{\partial(r^2 T^{rr})}{\partial r} + f(T^{r\Theta}) + f(T^{r\Phi}) - \frac{1}{r}(T^{\Theta\Theta} + T^{\Phi\Phi})$$

$$(\nabla\cdot T)_\Theta = \frac{1}{r}\left\{ f(T^{r\Theta}) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, T^{\Theta\Theta})}{\partial\theta} + f(T^{\Theta\Phi}) + \frac{1}{r}(T^{r\Theta} - \cot\theta\, T^{\Phi\Phi})\right\}$$

$$(\nabla\cdot T)_\Phi = \frac{1}{r\sin\theta}\left\{ f(T^{r\Phi}) + f(T^{\Theta\Phi}) + \frac{1}{r\sin\theta}\frac{\partial T^{\Phi\Phi}}{\partial\phi} + f(\cot\theta\, T^{\Theta\Phi})\right\}$$

where f are (different) functions of the tensor-elements which are not relevant here.

Since in spherical symmetry the radiation pressure tensor P is diagonal (i.e., symmetric), and since p_R and u are functions of r alone, we have

$$(\nabla\cdot P)_r = \frac{1}{r^2}\left(2rP^{rr} + r^2\frac{\partial P^{rr}}{\partial r}\right) - \frac{1}{r}(P^{\Theta\Theta} + P^{\Phi\Phi}) = \frac{\partial P^{rr}}{\partial r} + \frac{1}{r}(2P^{rr} - P^{\Theta\Theta} - P^{\Phi\Phi})$$

(which in the isotropic case would yield $(\nabla\cdot P)_r = \frac{\partial P^{rr}}{\partial r} = \frac{\partial p_R}{\partial r}$)

$$(\nabla\cdot P)_\Theta = \frac{1}{r^2\sin\theta}\left(\cos\theta\, P^{\Theta\Theta} + \sin\theta\frac{\partial T^{\Theta\Theta}}{\partial\theta}\right) - \frac{1}{r^2}\cot\theta\, P^{\Phi\Phi} \to 0 \text{ (in spherical symmetry)}$$

$(\nabla\cdot P)_\Phi \to 0$ (in spherical symmetry).

Finally, we obtain

$$(\nabla\cdot P) \to (\nabla\cdot P)_r = \mathbf{e_r}\cdot\left\{\frac{\partial p_R}{\partial r} + \frac{1}{r}\left(2p_R - 2\left(p_R - \frac{1}{2}(3p_R - u)\right)\right)\right\} =$$

$$= \mathbf{e_r}\cdot\left(\frac{\partial p_R}{\partial r} + \frac{1}{r}(3p_R - u)\right), \text{ q.e.d.}$$

Chap. 4 – Coupling with matter

The equation of radiative transfer

- had Boltzmann eq. for particle distrib. function f

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla} + \underline{F}\cdot\underline{\nabla}_p\right) f = \left(\frac{\delta f}{\delta t}\right)_{\text{coll}}$$

for photons $v = c\cdot\underline{n}$, $\underline{F} \equiv 0$ without GR

$$\Rightarrow \left(\frac{\partial}{\partial t} + c\underline{n}\cdot\underline{\nabla}\right)\Psi_\nu = \left(\frac{\delta \Psi_\nu}{\delta t}\right)_{\text{coll}} \leftarrow \text{photon creation/destr. along path in phase space}$$

with

$$\Psi_\nu(\underline{r},\underline{n},t)\, d^3\underline{r}\, d\nu\, d\Omega = f(\underline{r},\underline{p},t)\, d^3\underline{r}\, d^3\underline{p}$$

and

$$\left(\frac{\partial}{\partial t} + c\cdot\underline{n}\cdot\underline{\nabla}\right)\frac{I_\nu}{ch\nu} = \frac{1}{ch\nu}\left(\frac{\delta I_\nu}{\delta t}\right)_{\text{"coll"}}$$

$$\Rightarrow \left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\underline{\nabla}\right) I_\nu = \left(\frac{\delta I_\nu}{ds}\right)_{\text{"coll"}} = \frac{\delta I_\nu^{\text{em}} - \delta I_\nu^{\text{abs}}}{ds}$$

with

$$I_\nu = ch\nu\,\Psi_\nu, \quad ds = c\cdot\delta t$$

gain/loss by interaction with matter

Equation of radiative transfer for specific intensity

Emissivity and opacity

a) vacuum

$\rightarrow$ no "collisions" $\rightarrow$ Vlasov equation

$$\Rightarrow \left[\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\underline{\nabla}\right] I = 0$$

stationary

$$(\underline{n}\cdot\underline{\nabla})\, I = \frac{d}{ds} I = 0 \Rightarrow I = \text{const} \quad (\text{cf. Chap } 3)$$

directional derivative

b) energy gain by emission

add energy to ray (matter in dV radiates) by emission / photon creation

$$\delta E_\nu^+ = \delta E_\nu^{\text{em}} \overset{\text{def}}{=} \eta_\nu(\underline{r},\underline{n},t)\, dV\, d\Omega\, d\nu\, dt$$

$$= \eta_\nu(\underline{r},\underline{n},t)\, \underbrace{\underbrace{\underline{n}\cdot d\underline{S}}_{\cos\theta\, dS} \cdot ds}_{dV}\, d\Omega\, d\nu\, dt$$

compare with def. of specific energy

$$\delta E_\nu = I_\nu(\underline{r},\underline{n},t)\cos\theta\, dS\, d\Omega\, d\nu\, dt$$

$$\Rightarrow \delta I_\nu^{\text{em}} = \eta_\nu\, ds \quad \text{macroscopic emission coefficient}$$

$$\dim[\eta_\nu] = \text{erg cm}^{-3}\, \text{sr}^{-1}\, \text{Hz}^{-1}\, \text{s}^{-1}$$

c) energy loss by absorption

remove energy from ray (matter in dV absorbs)
by absorption / photon distruction

NOTE i) energy gain/emission property of interacting matter

ii) BUT: energy loss must depend on properties of matter and radiation, since

no radiation field ⇒ no loss
no matter ⇒ no loss

THUS following definition

$$\delta E_\nu^{-} = \delta E_\nu^{abs} = (\chi_\nu I_\nu)(\underline{r},\underline{n},t)\cos\theta\, dS\, ds\, d\Omega\, d\nu\, dt$$

$$\delta I_\nu^{abs} = \chi_\nu I_\nu ds$$

χ_ν absorption coefficient or opacity

$\dim[\chi_\nu] = \mathrm{cm}^{-1}$

d) optical depth

define $d\tau_\nu = \chi_\nu ds \rightarrow \tau_\nu(s) = \int_0^s \chi_\nu(s) ds$

$\delta I_\nu^{abs} = I_\nu d\tau_\nu$ the higher τ, the more is absorbed

$\dim[\tau_\nu]$ dimensionless

interpretation later

e) emission and absorption in parallel

$$\left(\frac{\delta I_\nu}{ds}\right)_{con} = \frac{\delta I_\nu^{em} - \delta I_\nu^{abs}}{ds} = \eta_\nu - \chi_\nu I_\nu$$

⇒ finally

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\underline{\nabla}\right) I_\nu = \eta_\nu - \chi_\nu I_\nu$$

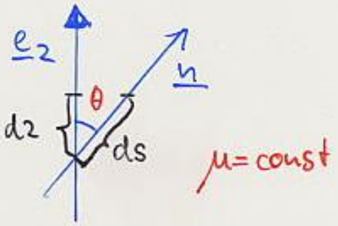
η_ν, χ_ν depend on microphysics of interacting matter

NOTE
- in static media η_ν, χ_ν (mostly) isotropic
- in moving media: Dopplereffect

  matter "sees" light at frequencies different than the observer ⇒ dependency on angle

The equation of transfer for specific geometries

a) plane-parallel symmetry



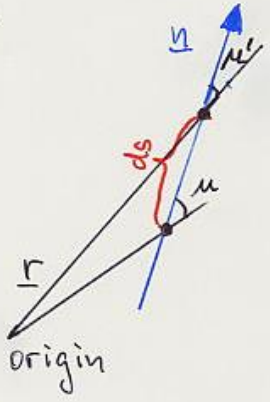
$$dz = \mu\, ds$$

$$\rightarrow (\underline{n}\cdot\underline{\nabla}) = \frac{d}{ds} = \mu \frac{d}{dz}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial z}\right) I_\nu(z,\mu,t) = \eta_\nu - \chi_\nu I_\nu$$

b) spherical symmetry

along ds, $\mu \neq \text{const}$

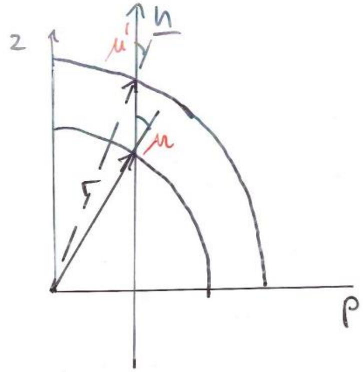


$$(\underline{n}\cdot\underline{\nabla}) = \frac{d}{ds} = \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\right) I_\nu(r,\mu,t) = \eta_\nu - \chi_\nu I_\nu$$

c) in general

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \Phi}, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \phi}\right] I_\nu(\theta,\Phi,r,\mu,\phi,t)$$



so-called p-z geometry

$$\frac{d}{ds} = \frac{d}{dz}\Big|_p$$

mit $r^2 = z^2 + p^2$

$\mu = \frac{z}{r}$

$$\Rightarrow \frac{d}{ds} = \frac{d}{dz}\Big|_p = \frac{\partial r}{\partial z}\Big|_p \frac{\partial}{\partial r} + \frac{\partial \mu}{\partial z}\Big|_p \frac{\partial}{\partial \mu}$$

$$r^2 = z^2 + p^2 \rightarrow \frac{\partial r}{\partial z}\Big|_p = \frac{z}{r} = \mu$$

$$\mu = \frac{z}{(z^2+p^2)^{\frac{1}{2}}} \rightarrow \frac{\partial \mu}{\partial z}\Big|_p = \frac{1}{r} - \frac{z^2}{r^3} = \frac{1}{r}(1-\mu^2)$$

$$\Rightarrow \underline{n}\cdot\underline{\nabla} = \frac{d}{ds} = \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}$$

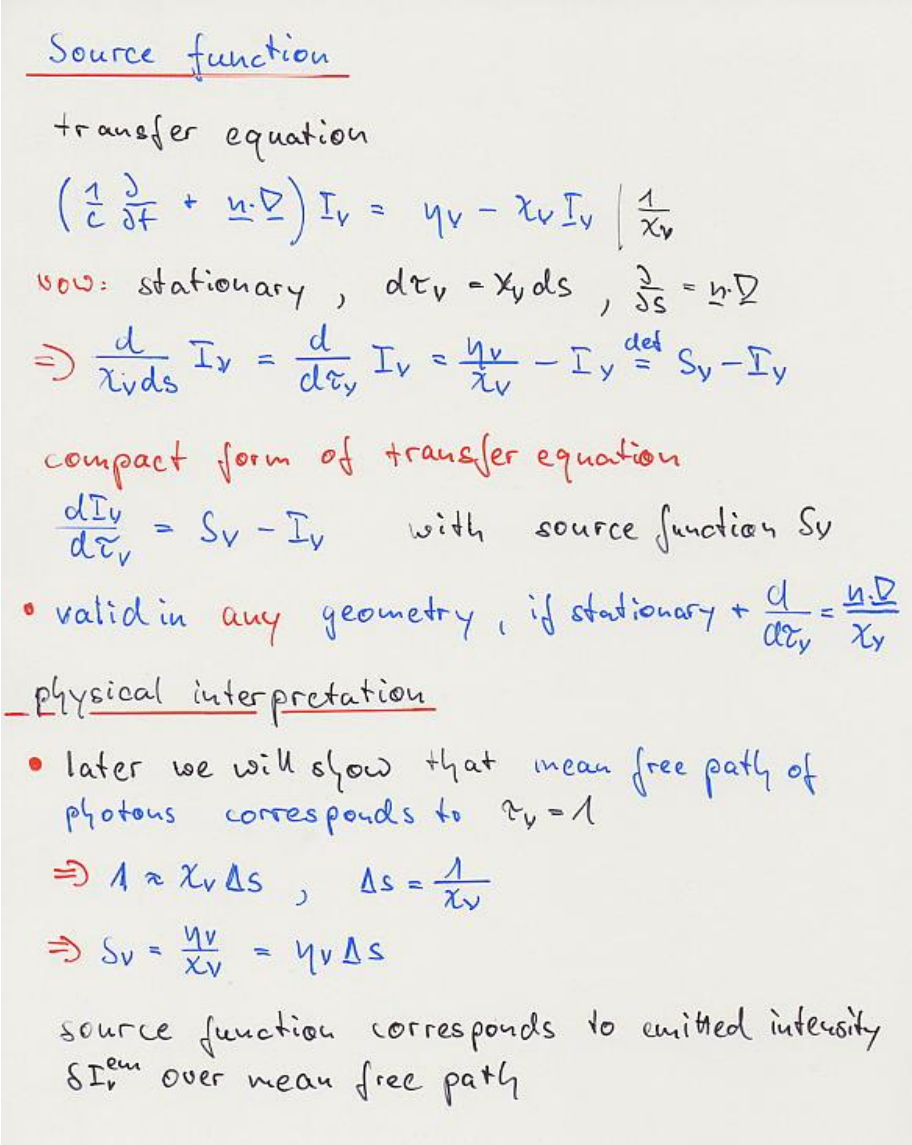
$$\left\| \left(\frac{1}{c}\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\right) I_\nu(r,\mu,t) = \eta_\nu - \chi_\nu I_\nu \right\|$$

General (without proof) for θ, Φ, r

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial r} + \frac{\gamma}{r}\frac{\partial}{\partial \theta} + \frac{\sigma}{r\sin\theta}\frac{\partial}{\partial \Phi} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu} - \frac{\sigma\cot\theta}{r}\frac{\partial}{\partial \phi}\right) I_\nu(\theta,\Phi,r,\mu,\phi,\nu,t) = \eta_\nu - \chi_\nu I_\nu$$

mit $\gamma = \cos\phi \sin\theta$
$\sigma = \sin\phi \sin\theta$

Source function and Kirchhoff-Planck law



Source function

transfer equation

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\underline{\nabla}\right) I_\nu = \eta_\nu - \chi_\nu I_\nu \quad \Big| \; \frac{1}{\chi_\nu}$$

now: stationary, $d\tau_\nu = \chi_\nu ds$, $\frac{\partial}{\partial s} = \underline{n}\cdot\underline{\nabla}$

$$\Rightarrow \frac{d}{\chi_\nu ds} I_\nu = \frac{d}{d\tau_\nu} I_\nu = \frac{\eta_\nu}{\chi_\nu} - I_\nu \overset{\text{def}}{=} S_\nu - I_\nu$$

compact form of transfer equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$ with source function S_ν

- valid in any geometry, if stationary + $\frac{d}{d\tau_\nu} = \frac{\underline{n}\cdot\underline{\nabla}}{\chi_\nu}$

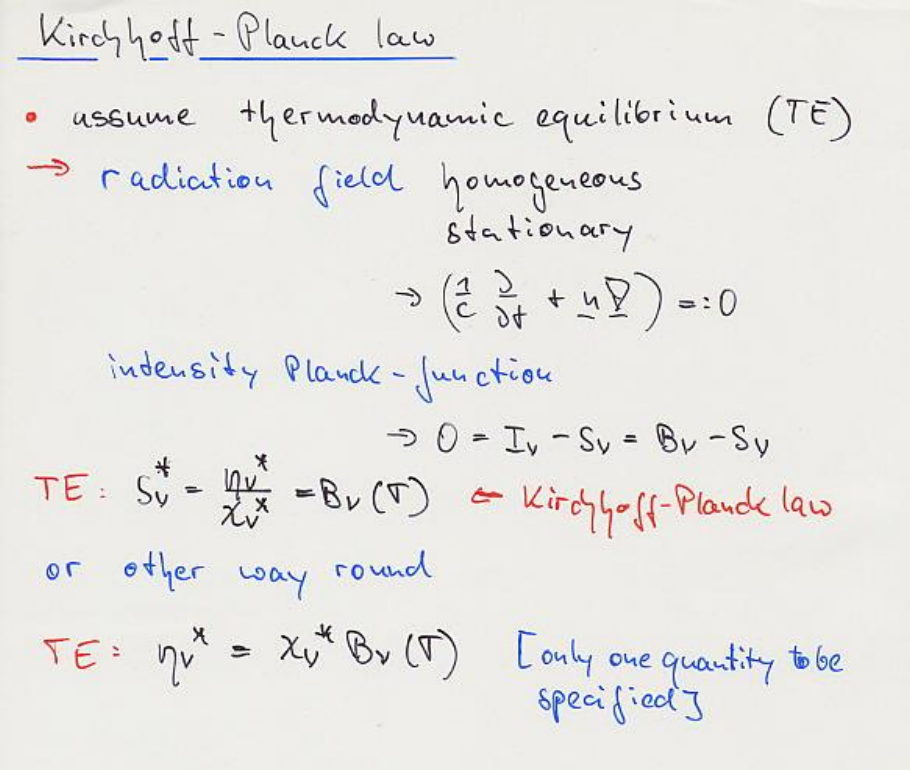
physical interpretation

- later we will show that mean free path of photons corresponds to $\tau_\nu = 1$

$$\Rightarrow 1 \approx \chi_\nu \Delta s, \quad \Delta s = \frac{1}{\chi_\nu}$$

$$\Rightarrow S_\nu = \frac{\eta_\nu}{\chi_\nu} = \eta_\nu \Delta s$$

source function corresponds to emitted intensity δI_ν^{em} over mean free path



Kirchhoff-Planck law

- assume thermodynamic equilibrium (TE)

$\rightarrow$ radiation field homogeneous
stationary

$$\rightarrow \left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\underline{\nabla}\right) =: 0$$

intensity Planck-function

$$\rightarrow 0 = I_\nu - S_\nu = B_\nu - S_\nu$$

TE: $S_\nu^* = \frac{\eta_\nu^*}{\chi_\nu^*} = B_\nu(T)$ ⟵ Kirchhoff-Planck law

or other way round

TE: $\eta_\nu^* = \chi_\nu^* B_\nu(T)$ [only one quantity to be specified]

True absorption and scattering

"true" absorption processes:

radiation energy => thermal pool
if not TE, temperature T(r) is changed
examples: photo-ionization
bound-bound absorption with subsequent collisional de-excitation

scattering:

no interaction with thermal pool
absorbed photon energy is directly reemitted (as photon)
no influence on T(r)
But direction $\underline{n}$ -> $\underline{n}'$ is changed (change in frequency mostly small)

examples: Thomson scattering at free electrons
Rayleigh scattering at atoms and molecules
resonance line scattering

ESSENTIAL POINT

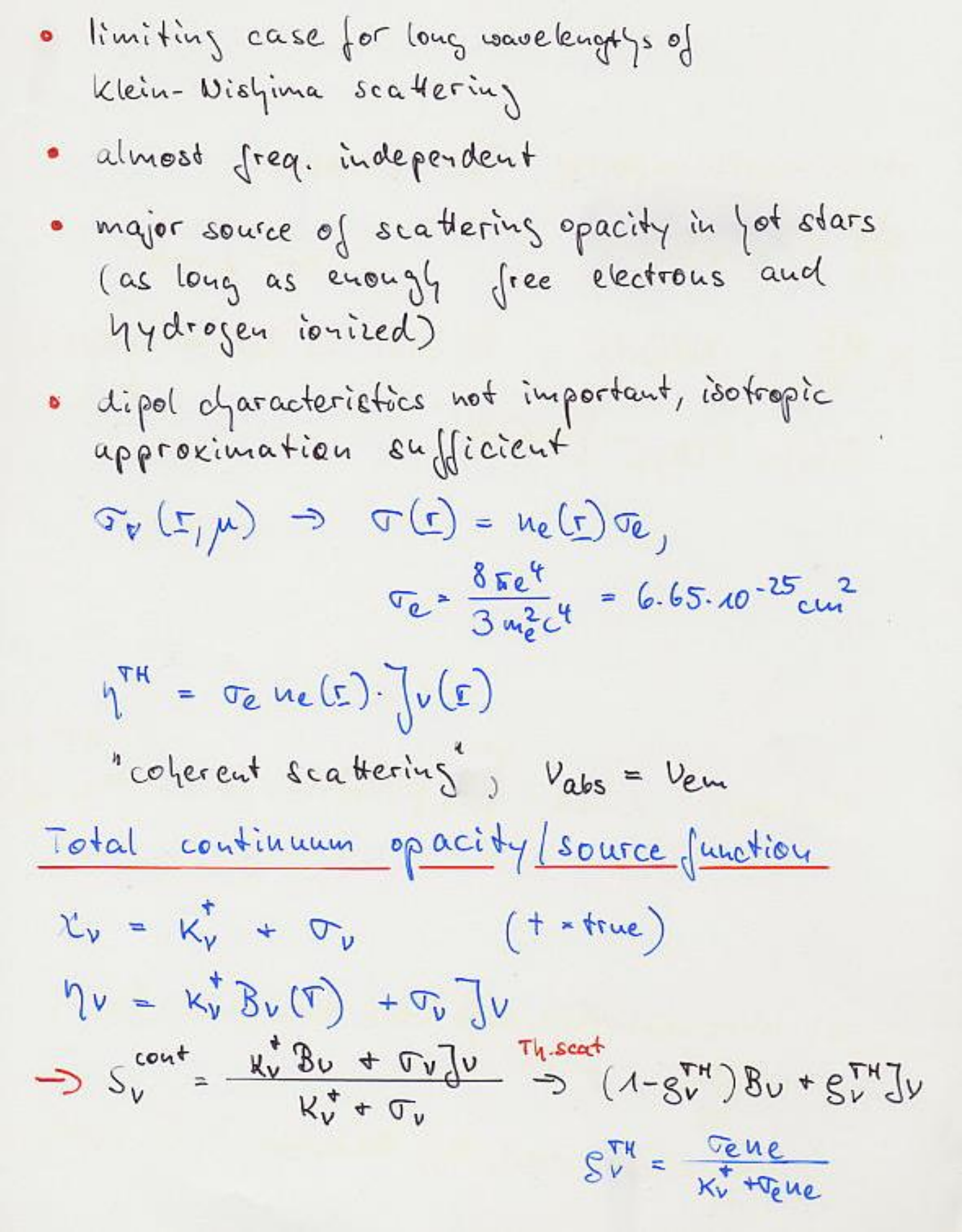
true processes:

localized interaction with thermal pool,
drive physical conditions into local equilibrium
often (e.g., in LTE – page 107): $\eta_\nu(\text{true}) = \kappa_\nu B_\nu(T)$

scattering processes:

(almost) no influence on local thermodynamic properties of plasma
propagate information of radiation field (sometimes over large distances)
η_ν (Thomson) = $\sigma_{TH}\, J_\nu$ (-> next page)

Thomson scattering



- limiting case for long wavelengths of Klein-Nishima scattering
- almost freq. independent
- major source of scattering opacity in hot stars (as long as enough free electrons and hydrogen ionized)
- dipol characteristics not important, isotropic approximation sufficient

$$\sigma_\nu(\underline{r},\mu) \rightarrow \sigma(\underline{r}) = n_e(\underline{r})\,\sigma_e,$$

$$\sigma_e = \frac{8\pi e^4}{3 m_e^2 c^4} = 6.65\cdot 10^{-25}\,\mathrm{cm}^2$$

$$\eta^{TH} = \sigma_e\, n_e(\underline{r})\cdot J_\nu(\underline{r})$$

"coherent scattering", $\nu_{abs} = \nu_{em}$

Total continuum opacity/source function

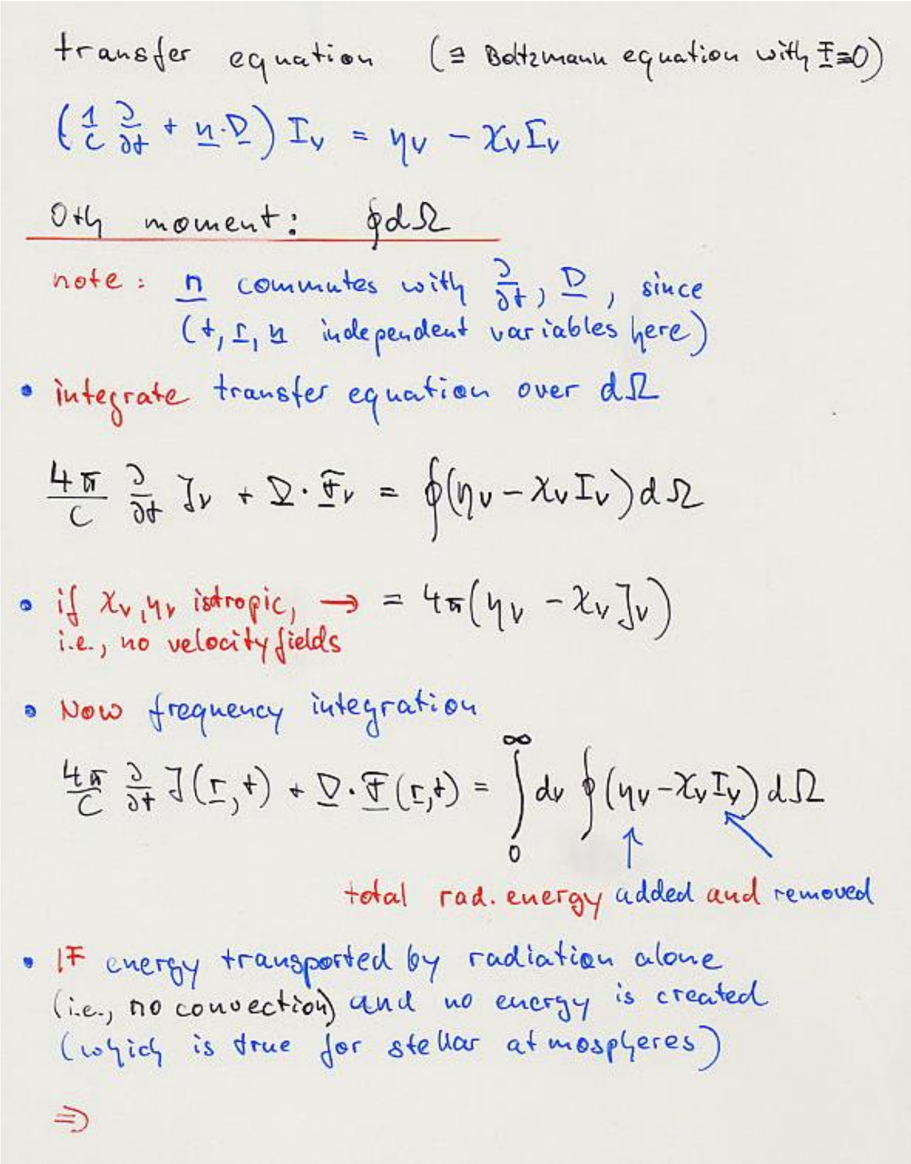
$$\chi_\nu = \kappa_\nu^+ + \sigma_\nu \qquad (+ = \text{true})$$

$$\eta_\nu = \kappa_\nu^+ B_\nu(T) + \sigma_\nu J_\nu$$

$$\rightarrow S_\nu^{cont} = \frac{\kappa_\nu^+ B_\nu + \sigma_\nu J_\nu}{\kappa_\nu^+ + \sigma_\nu} \xrightarrow{Th.scat} (1-\xi_\nu^{TH}) B_\nu + \xi_\nu^{TH} J_\nu$$

$$\xi_\nu^{TH} = \frac{\sigma_e n_e}{\kappa_\nu^+ + \sigma_e n_e}$$

Moments of the transfer equation



transfer equation (≙ Boltzmann equation with $\underline{F} \equiv 0$)

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\underline{\nabla}\right) I_\nu = \eta_\nu - \chi_\nu I_\nu$$

0th moment: $\oint d\Omega$

note: $\underline{n}$ commutes with $\frac{\partial}{\partial t}$, $\underline{\nabla}$, since (t, $\underline{r}$, $\underline{n}$ independent variables here)

- integrate transfer equation over $d\Omega$

$$\frac{4\pi}{c}\frac{\partial}{\partial t} J_\nu + \underline{\nabla}\cdot\underline{F}_\nu = \oint(\eta_\nu - \chi_\nu I_\nu)\,d\Omega$$

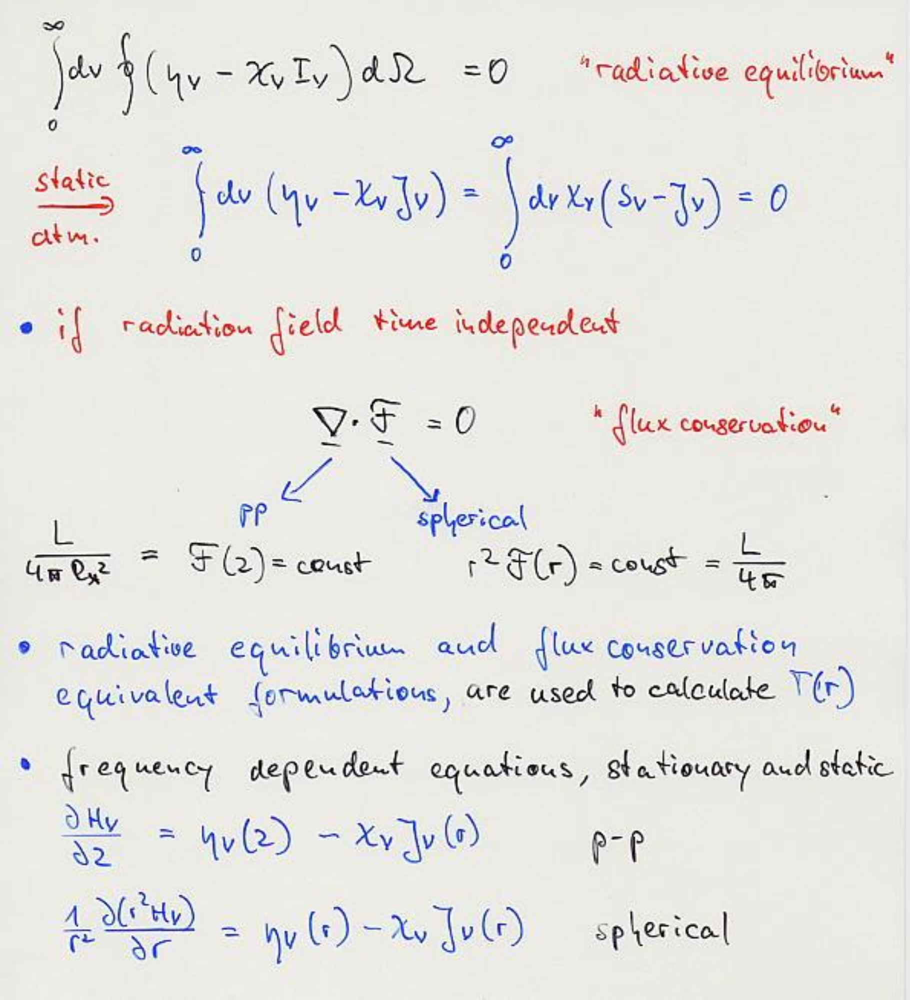
- if χ_ν, η_ν isotropic, → $= 4\pi(\eta_\nu - \chi_\nu J_\nu)$
  i.e., no velocity fields
- Now frequency integration

$$\frac{4\pi}{c}\frac{\partial}{\partial t} J(\underline{r},t) + \underline{\nabla}\cdot\underline{F}(\underline{r},t) = \int_0^\infty d\nu \oint(\eta_\nu - \chi_\nu I_\nu)\,d\Omega$$

total rad. energy added and removed

- IF energy transported by radiation alone (i.e., no convection) and no energy is created (which is true for stellar atmospheres)

⇒



$$\int_0^\infty d\nu \oint(\eta_\nu - \chi_\nu I_\nu)\,d\Omega = 0$$

"radiative equilibrium"

$$\xrightarrow[\text{atm.}]{\text{static}} \int_0^\infty d\nu\,(\eta_\nu - \chi_\nu J_\nu) = \int_0^\infty d\nu\,\chi_\nu(S_\nu - J_\nu) = 0$$

- if radiation field time independent

$$\underline{\nabla}\cdot\underline{F} = 0$$

"flux conservation"

pp ↙ ↘ spherical

$$\frac{L}{4\pi R_*^2} = F(z) = \text{const} \qquad r^2 F(r) = \text{const} = \frac{L}{4\pi}$$

- radiative equilibrium and flux conservation equivalent formulations, are used to calculate $T(r)$
- frequency dependent equations, stationary and static

$$\frac{\partial H_\nu}{\partial z} = \eta_\nu(z) - \chi_\nu J_\nu(z) \qquad \text{p-p}$$

$$\frac{1}{r^2}\frac{\partial(r^2 H_\nu)}{\partial r} = \eta_\nu(r) - \chi_\nu J_\nu(r) \qquad \text{spherical}$$

1st moment : $\oint \underline{n}\, d\Omega / c$

$$\oint \frac{d\Omega}{c}\left(\underline{n}\,\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\;\underline{n}\cdot\underline{\nabla}\right) I_\nu = \frac{1}{c}\oint(\eta_\nu - \chi_\nu I_\nu)\,\underline{n}\,d\Omega$$

$\Rightarrow$

$$\frac{1}{c^2}\frac{\partial}{\partial t}\underline{\mathcal{F}}_\nu + \underline{\nabla}\cdot\mathbb{P}_\nu = \frac{1}{c}\oint(\eta_\nu - \chi_\nu I_\nu)\,\underline{n}\,d\Omega$$

$\nearrow$ Tensor, cf. Chap. 3

frequency integrated analogous

- can be shown

$$\frac{1}{c}\int_0^\infty d\nu \oint \chi_\nu I_\nu \underline{n}\, d\Omega$$ is force/Volume, by radiation on matter (momentum transfer photons -> matter via absorption)

$$= \underline{f}_{\mathrm{rad}}(\underline{r})$$

"radiation force"

$$\frac{\text{force}}{\text{volume}}\cdot\frac{1}{\rho} = \frac{\text{force}}{\text{mass}} = g_{\mathrm{rad}}$$ "radiative acceleration"

and

$$\int d\nu \oint \eta_\nu \underline{n}\, d\Omega = 0$$ because of fore/aft symmetry of emission process (even in v-fields)

- in total

$$\frac{1}{c^2}\frac{\partial}{\partial t}\underline{\mathcal{F}}(\underline{r},t) + \underline{\nabla}\cdot\mathbb{P}(\underline{r},t) = -\frac{1}{c}\int d\nu \oint \chi_\nu I_\nu \underline{n}\, d\Omega$$

$$= -\rho\, \underline{g}_{\mathrm{rad}}(\underline{r})$$

- stationary

$$\underline{\nabla}\cdot\mathbb{P}(\underline{r}) = -\rho(\underline{r})\,\underline{g}_{\mathrm{rad}}(\underline{r}) = -\frac{1}{c}\int_0^\infty d\nu \oint d\Omega\,(\chi_\nu I_\nu)\,\underline{n}$$

- static $\rightarrow -\frac{1}{c}\int_0^\infty d\nu\, \chi_\nu\, \underline{\mathcal{F}}_\nu(\underline{r})$

1-D $\rightarrow$ $$g_{\mathrm{rad}}(z) = \frac{4\pi}{c\rho(z)}\int_0^\infty d\nu\, \chi_\nu(z) H_\nu(z)$$

- frequency dependent equations, stationary and static

$$\underline{\nabla}\cdot\mathbb{P}_\nu = -\frac{1}{c}\chi_\nu \underline{\mathcal{F}}_\nu \quad (= -\rho(\underline{r})\, g_{\mathrm{rad}})$$

p-p $\rightarrow$ $$\frac{\partial p_R(z,\nu)}{\partial z} = -\frac{1}{c}\chi_\nu(z)\mathcal{F}_\nu(z) \text{ or } \frac{\partial K_\nu(z)}{\partial z} = -\chi_\nu(z)H_\nu(z)$$

sph. $\rightarrow$ $$\frac{\partial p_R(r,\nu)}{\partial r} + \frac{3p_R - u_\nu}{r} = -\frac{1}{c}\chi_\nu(r)\mathcal{F}_\nu(r) \text{ or}$$

$$\frac{\partial K_\nu(r)}{\partial r} + \frac{3K_\nu(r) - J_\nu(r)}{r} = -\chi_\nu(r)H_\nu(r)$$

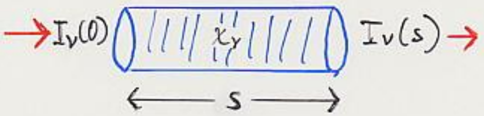
The change in radiative pressure drives the flux!

Chap. 5 – Radiative transfer: simple solutions

Pure absorption and optical depth

- from here on, stationary description
  (→ stellar atmospheres)
- radiative transfer without emission

$$\frac{dI_\nu}{ds} = -\chi_\nu I_\nu$$



$$\frac{dI_\nu}{I_\nu} = -\chi_\nu(s)\,ds$$

$$\ln I_\nu(s) - \ln I_\nu(0) = -\int_0^s \chi_\nu(s')\,ds'$$

$$I_\nu(s) = I_\nu(0)\, e^{-\int_0^s \chi_\nu(s')ds'} = I_\nu(0)\, e^{-\tau_\nu(s)}$$

↗ optical depth, central quantity (more precisely: optical thickness)

or

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu}$$

- since $I_\nu \sim e^{-\tau_\nu}$, we look only until $\tau_\nu = 1$
  (freq. dep.!)
- Question: What is the average distance over which photons travel?

Answer:
$$\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu\, p(\tau_\nu)\, d\tau_\nu$$

↗ expectation value ↑ probability density function

$p(\tau_\nu)\,d\tau$ gives probability, that photon is absorbed in interval $\tau_\nu, \tau_\nu + d\tau_\nu$

- is probability, that photon is NOT absorbed between $0, \tau_\nu$ and then absorbed between $\tau_\nu, \tau_\nu + d\tau_\nu$

a) prob., that photon is absorbed

$$P(0,\tau_\nu) = \frac{\Delta I(\tau)}{I_0} = \frac{I_0 - I(\tau_\nu)}{I_0} = 1 - \frac{I(\tau_\nu)}{I_0}$$

b) prob, that photon is not absorbed

$$1 - P(0,\tau_\nu) = \frac{I(\tau_\nu)}{I_0} = e^{-\tau_\nu}$$

c) prob., that photon is absorbed in $\tau_\nu, \tau_\nu + d\tau_\nu$

$$P(\tau_\nu, \tau_\nu + d\tau_\nu) = \left|\frac{dI(\tau_\nu)}{I(\tau_\nu)}\right| = d\tau_\nu$$

d) total probability is $e^{-\tau_\nu} d\tau_\nu$

THUS

$$\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = \underline{1}$$

mean free path $\bar{s}$ corresponds to $\langle\tau_\nu\rangle = 1$

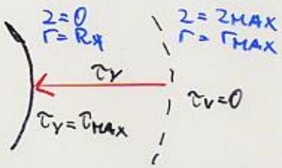
$$\Delta\tau_\nu = \chi_\nu \Delta s \;\to\; \underset{=\bar{s}}{\Delta s} = \frac{1}{\chi_\nu}\text{, q.e.d.}$$

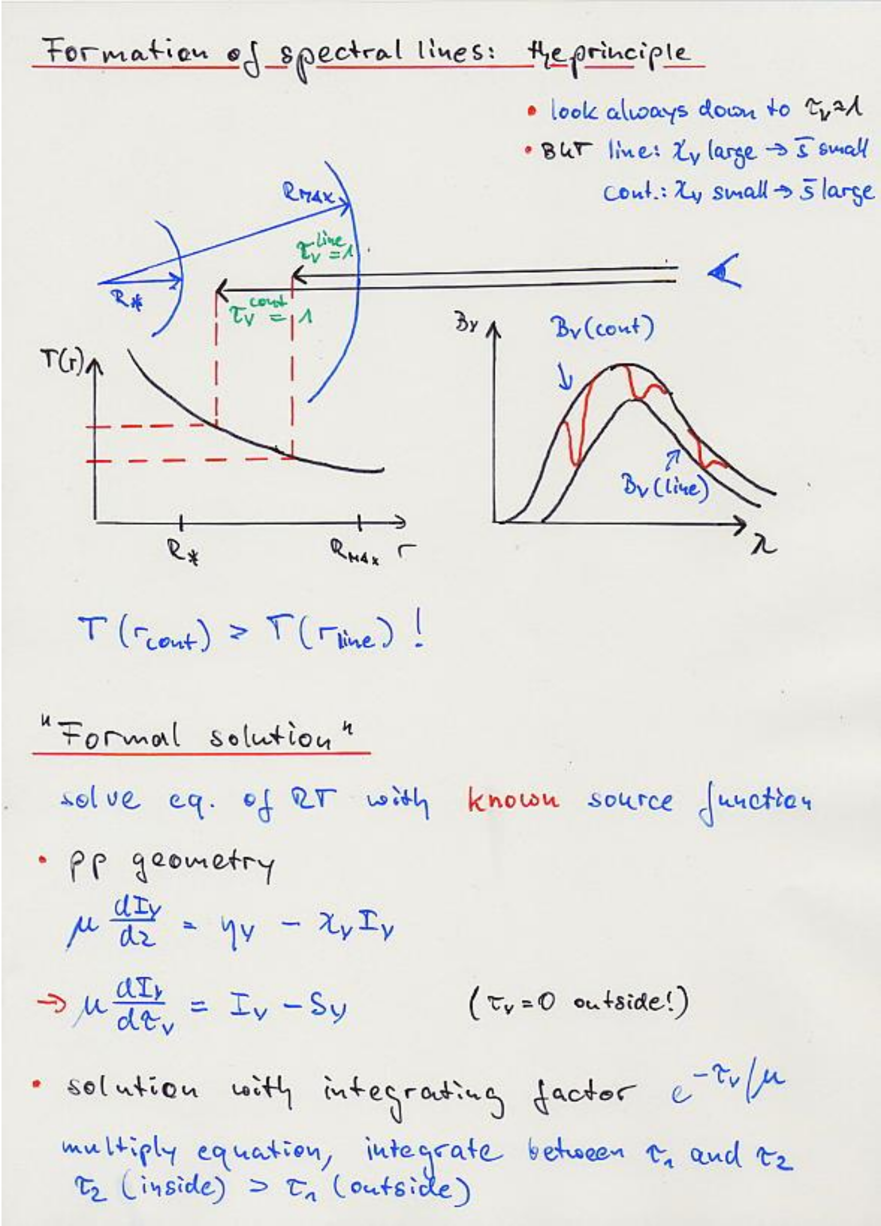
USUAL convention

- Since we "measure" from outside to inside, $\tau_\nu = 0$ is defined at outer "edge" of atmosphere

$$\Rightarrow ds = -\,dz \;(\text{or } -dr)$$

$$\Rightarrow \underbrace{d\tau_\nu}_{>0!} = -\chi_\nu \underbrace{\begin{pmatrix} dz \\ dr \end{pmatrix}}_{<0!}$$





Formation of spectral lines: the principle

- look always down to $\tau_\nu \approx 1$
- BUT line: χ_ν large → $\bar{s}$ small

  cont.: χ_ν small → $\bar{s}$ large

$T(r_{cont}) > T(r_{line})$!

"Formal solution"

solve eq. of RT with **known** source function

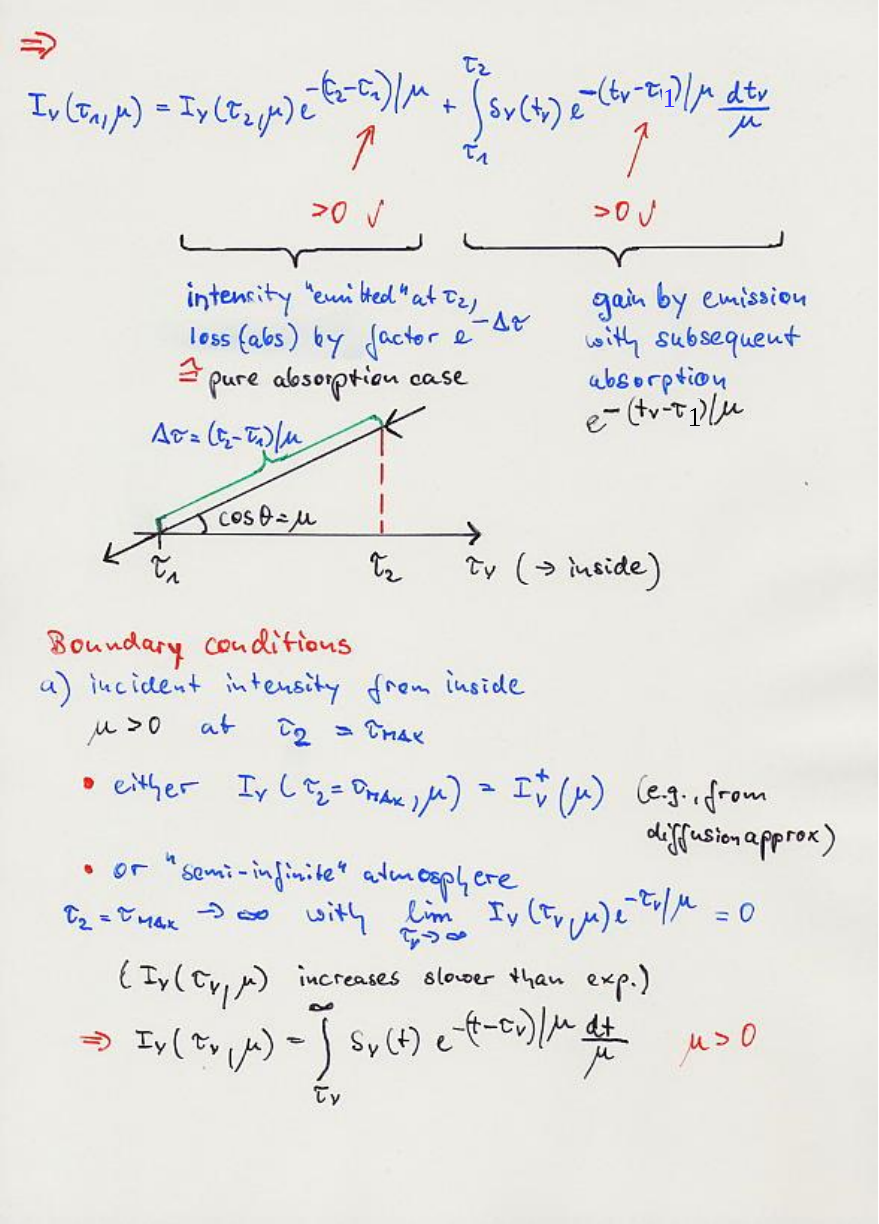
- pp geometry

$$\mu \frac{dI_\nu}{dz} = \eta_\nu - \chi_\nu I_\nu$$

$$\rightarrow \mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \qquad (\tau_\nu = 0 \text{ outside!})$$

- solution with integrating factor $e^{-\tau_\nu/\mu}$

multiply equation, integrate between τ_1 and τ_2
τ_2 (inside) $>$ τ_1 (outside)



$$\Rightarrow$$

$$I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)\, e^{-(\tau_2-\tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S_\nu(t_\nu)\, e^{-(t_\nu-\tau_1)/\mu}\, \frac{dt_\nu}{\mu}$$

>0 ✓ $\qquad$ >0 ✓

intensity "emitted" at τ_2, loss (abs) by factor $e^{-\Delta\tau}$
$\hat{=}$ pure absorption case

gain by emission with subsequent absorption $e^{-(t_\nu-\tau_1)/\mu}$

Boundary conditions

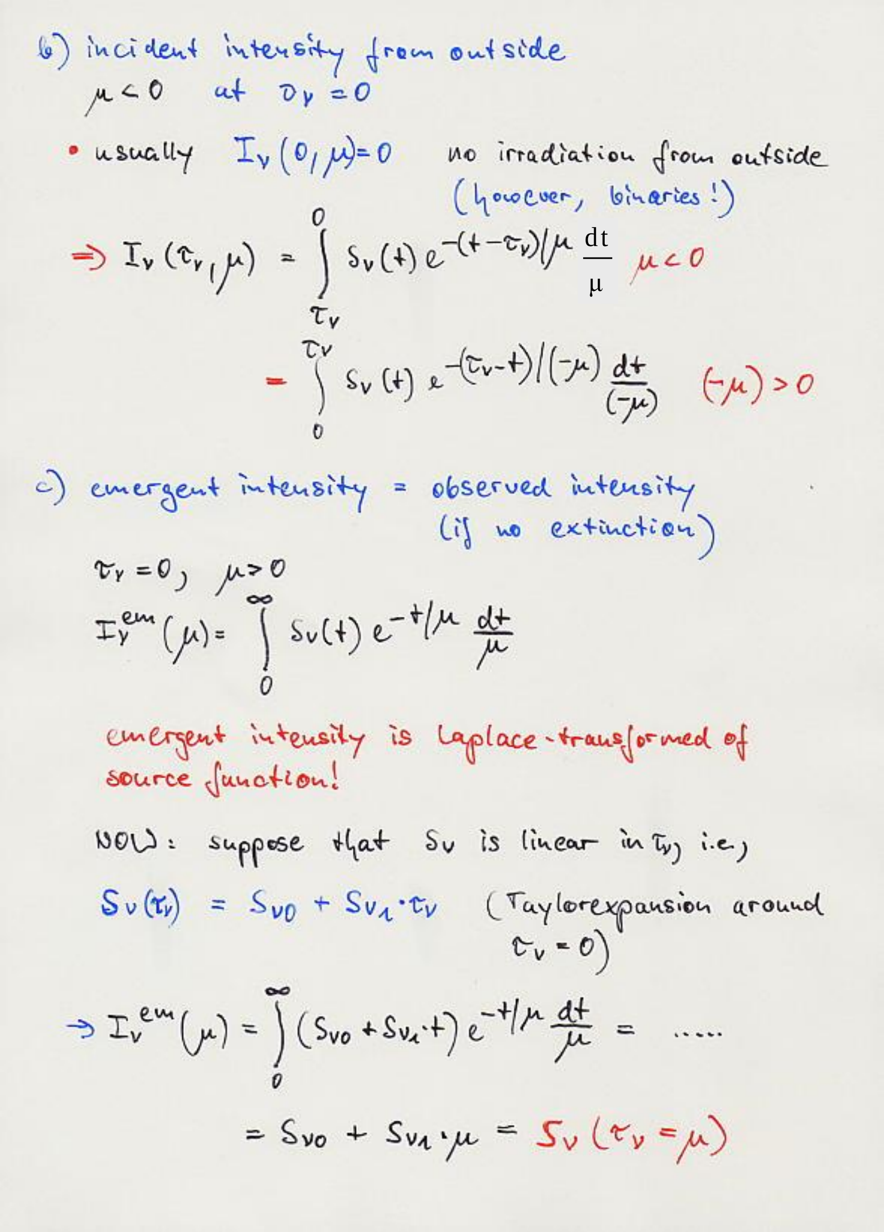
a) incident intensity from inside
$\mu > 0$ at $\tau_2 = \tau_{max}$

- either $I_\nu(\tau_2 = \tau_{max}, \mu) = I_\nu^+(\mu)$ (e.g., from diffusion approx)
- or "semi-infinite" atmosphere

$\tau_2 = \tau_{max} \rightarrow \infty$ with $\lim\limits_{\tau_\nu \to \infty} I_\nu(\tau_\nu,\mu)\, e^{-\tau_\nu/\mu} = 0$

($I_\nu(\tau_\nu,\mu)$ increases slower than exp.)

$$\Rightarrow I_\nu(\tau_\nu,\mu) = \int_{\tau_\nu}^{\infty} S_\nu(t)\, e^{-(t-\tau_\nu)/\mu}\, \frac{dt}{\mu} \qquad \mu > 0$$



b) incident intensity from outside

$\mu < 0$ at $\tau_\nu = 0$

- usually $I_\nu(0,\mu) = 0$ no irradiation from outside (however, binaries!)

$$\Rightarrow I_\nu(\tau_\nu,\mu) = \int_{\tau_\nu}^{0} S_\nu(t)\, e^{-(t-\tau_\nu)/\mu} \frac{dt}{\mu} \quad \mu < 0$$

$$= \int_{0}^{\tau_\nu} S_\nu(t)\, e^{-(\tau_\nu-t)/(-\mu)} \frac{dt}{(-\mu)} \quad (-\mu) > 0$$

c) emergent intensity = observed intensity (if no extinction)

$\tau_\nu = 0$, $\mu > 0$

$$I_\nu^{em}(\mu) = \int_0^\infty S_\nu(t)\, e^{-t/\mu} \frac{dt}{\mu}$$

emergent intensity is Laplace-transformed of source function!

NOW: suppose that S_ν is linear in τ_ν, i.e.,

$S_\nu(\tau_\nu) = S_{\nu 0} + S_{\nu 1} \cdot \tau_\nu$ (Taylorexpansion around $\tau_\nu = 0$)

$$\rightarrow I_\nu^{em}(\mu) = \int_0^\infty (S_{\nu 0} + S_{\nu 1} \cdot t)\, e^{-t/\mu} \frac{dt}{\mu} = \ldots..$$

$$= S_{\nu 0} + S_{\nu 1} \cdot \mu = S_\nu(\tau_\nu = \mu)$$

Eddington-Barbier-relation

$I_\nu^{em}(\mu) \approx S_\nu(\tau_\nu = \mu)$

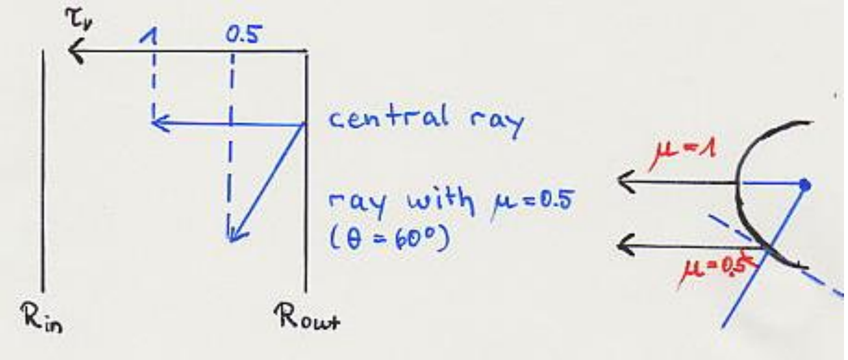
We “see” source function at location $\tau_\nu = \mu$ (remember: τ_ν radial quantity)
(corresponds to optical depth along path $\tau_\nu / \mu = 1!$)

Generalization of principle that we can see only until $\Delta\tau_\nu = 1$

i) spectral lines (as before)

for fixed μ, $\tau_\nu / \mu = 1$ is reached further out in lines (compared to continuum)

=> $S_\nu^{line}(\tau_\nu^{line} / \mu = 1) < S_\nu^{cont}(\tau_\nu^{cont} / \mu = 1)$ => "dip" is created

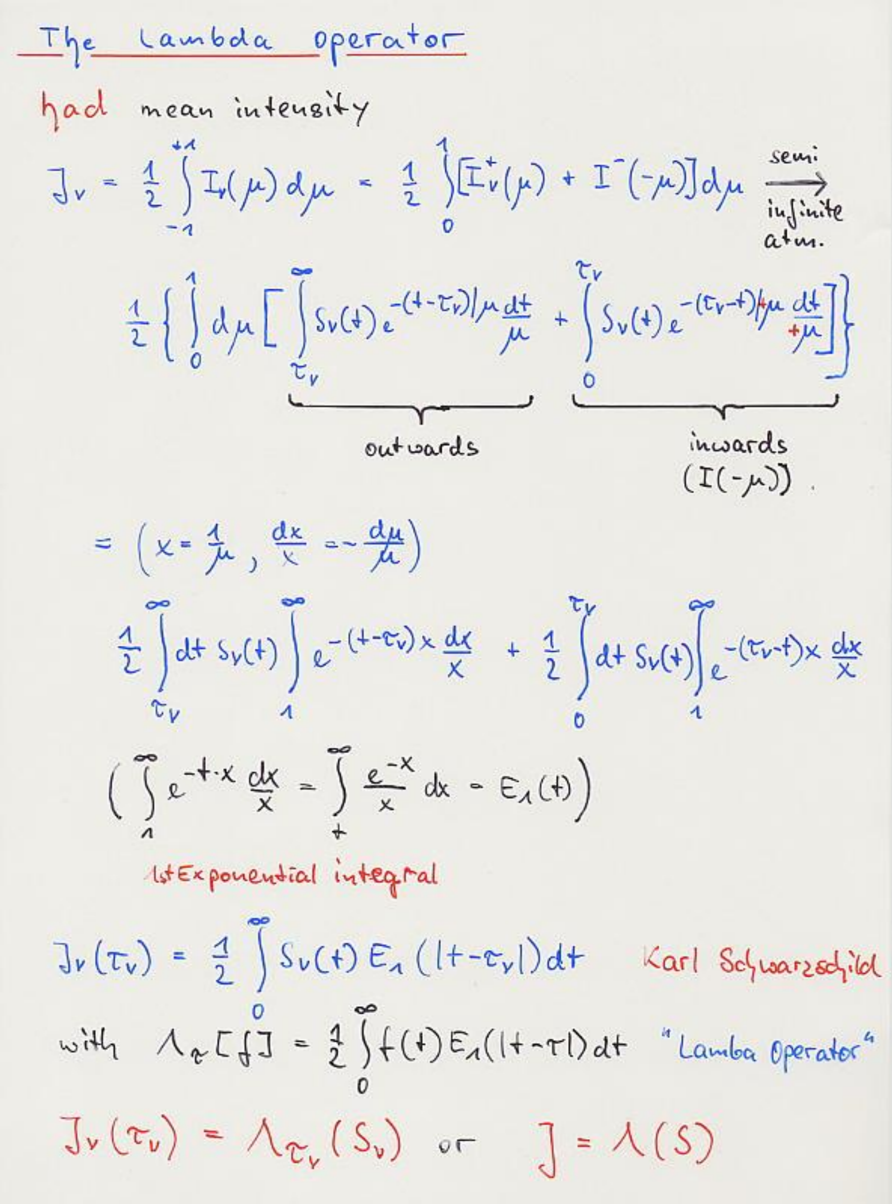


ii) limb darkening

for $\mu = 1$ (central ray), we reach maximum in depth (geometrical)
temperature / source function rises with τ
=> central ray: largest source function, limb darkening

iii) "observable" information only from layers with $\tau_\nu \le 1$
deepest atmospheric layers can be analyzed only indirectly

Lambda operator and diffusion approximation



The lambda operator

had mean intensity

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\,d\mu = \frac{1}{2}\int_0^1 [I^+_\nu(\mu) + I^-(-\mu)]\,d\mu \xrightarrow[\text{infinite atm.}]{\text{semi}}$$

$$\frac{1}{2}\left\{\int_0^1 d\mu\left[\underbrace{\int_{\tau_\nu}^{\infty} S_\nu(t)\,e^{-(t-\tau_\nu)/\mu}\frac{dt}{\mu}}_{\text{outwards}} + \underbrace{\int_0^{\tau_\nu} S_\nu(t)\,e^{-(\tau_\nu-t)/+\mu}\frac{dt}{+\mu}}_{\substack{\text{inwards}\\(I(-\mu))}}\right]\right\}$$

$$= \left(x = \frac{1}{\mu},\ \frac{dx}{x} = -\frac{d\mu}{\mu}\right)$$

$$\frac{1}{2}\int_{\tau_\nu}^{\infty} dt\,S_\nu(t)\int_1^{\infty} e^{-(t-\tau_\nu)x}\frac{dx}{x} + \frac{1}{2}\int_0^{\tau_\nu} dt\,S_\nu(t)\int_1^{\infty} e^{-(\tau_\nu-t)x}\frac{dx}{x}$$

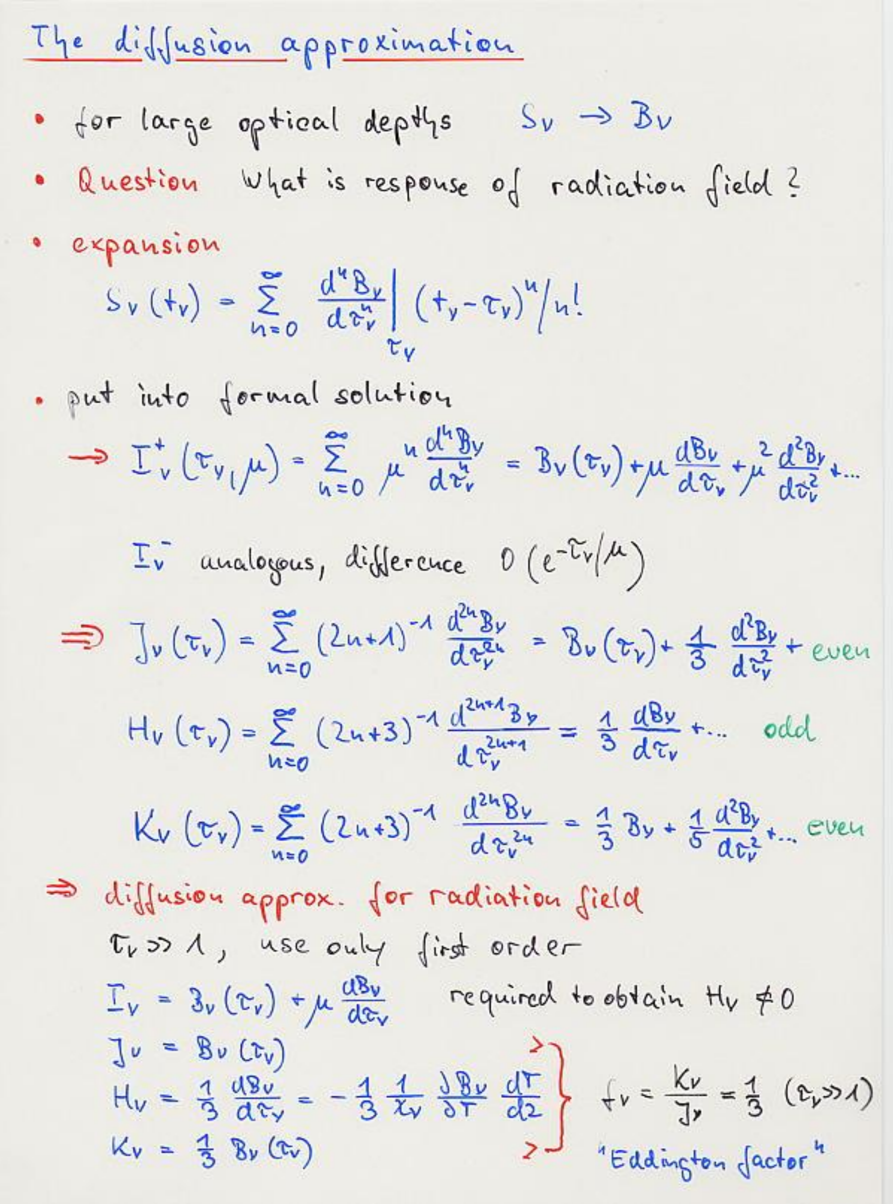
$$\left(\int_1^{\infty} e^{-t\cdot x}\frac{dx}{x} = \int_t^{\infty}\frac{e^{-x}}{x}\,dx = E_1(t)\right)$$

1st Exponential integral

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^{\infty} S_\nu(t)\,E_1(|t-\tau_\nu|)\,dt$$ Karl Schwarzschild

with $\Lambda_\tau[f] = \frac{1}{2}\int_0^{\infty} f(t)\,E_1(|t-\tau|)\,dt$ "Lamba Operator"

$J_\nu(\tau_\nu) = \Lambda_{\tau_\nu}(S_\nu)$ or $J = \Lambda(S)$



The diffusion approximation

- for large optical depths $S_\nu \rightarrow B_\nu$
- Question What is response of radiation field?
- expansion

$$S_\nu(t_\nu) = \sum_{n=0}^{\infty} \left.\frac{d^n B_\nu}{d\tau_\nu^n}\right|_{\tau_\nu} (t_\nu - \tau_\nu)^n / n!$$

- put into formal solution

$$\rightarrow I^+_\nu(\tau_\nu,\mu) = \sum_{n=0}^{\infty} \mu^n \frac{d^n B_\nu}{d\tau_\nu^n} = B_\nu(\tau_\nu) + \mu\frac{dB_\nu}{d\tau_\nu} + \mu^2\frac{d^2B_\nu}{d\tau_\nu^2} + \dots$$

I^-_ν analogous, difference $O(e^{-\tau_\nu/\mu})$

$$\Rightarrow J_\nu(\tau_\nu) = \sum_{n=0}^{\infty}(2n+1)^{-1}\frac{d^{2n}B_\nu}{d\tau_\nu^{2n}} = B_\nu(\tau_\nu) + \frac{1}{3}\frac{d^2B_\nu}{d\tau_\nu^2} + \text{even}$$

$$H_\nu(\tau_\nu) = \sum_{n=0}^{\infty}(2n+3)^{-1}\frac{d^{2n+1}B_\nu}{d\tau_\nu^{2n+1}} = \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} + \dots \quad \text{odd}$$

$$K_\nu(\tau_\nu) = \sum_{n=0}^{\infty}(2n+3)^{-1}\frac{d^{2n}B_\nu}{d\tau_\nu^{2n}} = \frac{1}{3}B_\nu + \frac{1}{5}\frac{d^2B_\nu}{d\tau_\nu^2} + \dots \quad \text{even}$$

⇒ diffusion approx. for radiation field

$\tau_\nu \gg 1$, use only first order

$I_\nu = B_\nu(\tau_\nu) + \mu\frac{dB_\nu}{d\tau_\nu}$ required to obtain $H_\nu \neq 0$

$$\left.\begin{aligned} J_\nu &= B_\nu(\tau_\nu) \\ H_\nu &= \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} = -\frac{1}{3}\frac{1}{\chi_\nu}\frac{\partial B_\nu}{\partial T}\frac{dT}{dz} \\ K_\nu &= \frac{1}{3}B_\nu(\tau_\nu) \end{aligned}\right\} \quad f_\nu = \frac{K_\nu}{J_\nu} = \frac{1}{3} \quad (\tau_\nu \gg 1)$$

"Eddington factor"

Solar limb-darkening
Empirical temperature stratification

- $H_\nu = -\frac{1}{3}\frac{1}{\chi_\nu}\underbrace{\frac{\partial B_\nu}{\partial T}}_{>0}\frac{dT}{dz}$

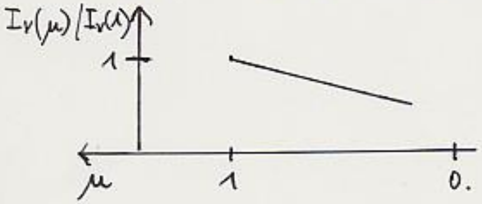
⇒ in order to transport flux $H_\nu > 0$, $\frac{dT}{dz} < 0$,
i.e., temperature must decrease!

Application: solar limb-darkening

Had $I_\nu^{em}(\mu) = S_{\nu 0} + \mu S_{\nu 1}$

→ LTE $S_\nu = B_\nu$, $I_\nu^{em} = B_\nu(0) + \mu \frac{dB_\nu}{d\tau_\nu}\Big|_0$

→ $\frac{I_\nu(\mu)}{I_\nu(1)} = \frac{B_\nu(0) + \mu dB_\nu/d\tau_\nu}{B_\nu(0) + dB_\nu/d\tau_\nu}$



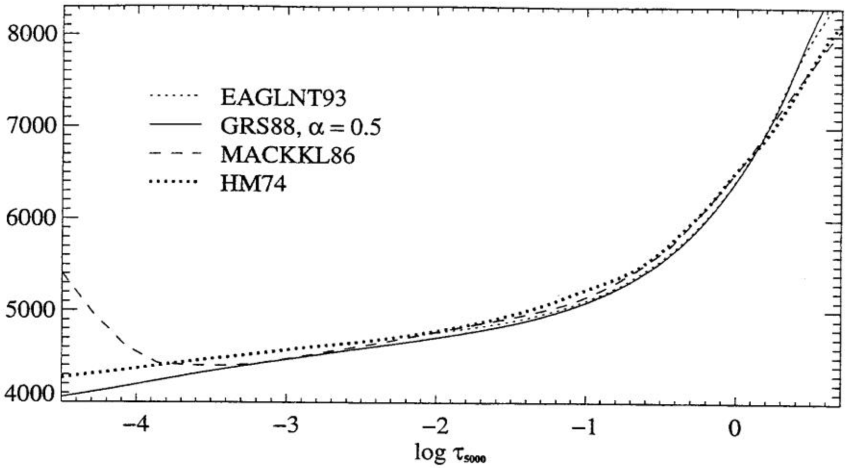
measurement
⇒ $B_\nu(0), \frac{dB_\nu}{d\tau_\nu}\Big|_0$

(one absolute measurement required, e.g., $B_\nu(0)$)

⇒ $B_\nu(\tau) = B_\nu(0) + \frac{dB_\nu}{d\tau_\nu}\Big|_0 \cdot \tau =: \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT(\tau)}-1}$

⇒ $T(\tau)$, empirical temperature stratification of solar photosphere

empirical temperature structure of solar photosphere by Holweger & Müller (1974)



The Milne-Eddington model

The Milne-Eddington model for continua with scattering

- allows understanding of emergent (continuum) fluxes from stellar atmospheres
- can be extended to include lines
- required for curve of growth method (➔ Chap. 7)

assume source function (→ page 71)

$$S_\nu = (1-\rho_\nu)B_\nu + \rho_\nu J_\nu \quad \text{with} \quad \rho_\nu = \frac{\sigma_e n_e}{\overset{+}{k}_\nu + \sigma_e n_e}$$

$$=: \varepsilon_\nu B_\nu + (1-\varepsilon_\nu) J_\nu, \quad \varepsilon_\nu = 1-\rho_\nu$$

and

$$B_\nu = a_\nu + b_\nu \cdot \tau_\nu$$ + plane-parallel symmetry

- 0th moment

$$\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu, \qquad d\tau_\nu = -(\overset{+}{k}_\nu + n_e\sigma_e)dz$$

$$= J_\nu - (\varepsilon_\nu B_\nu + (1-\varepsilon_\nu)J_\nu) = \varepsilon_\nu (J_\nu - B_\nu)$$

- 1st moment

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu$$

in diffusion approximation, we had

$$K_\nu = \frac{1}{3} J_\nu \quad (\tau_\nu \to \infty)$$

- Eddington's approximation (1929, 'The formation of absorption lines')

use $K_\nu / J_\nu = \frac{1}{3}$ everywhere ... not so wrong

$$\Rightarrow \frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu \rightarrow \frac{1}{3}\left(\frac{\partial J_\nu}{\partial \tau_\nu}\right) = H_\nu$$

⇒ (with 0th moment)

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial \tau_\nu^2} = \varepsilon_\nu (J_\nu - B_\nu) = \frac{1}{3}\frac{\partial^2 (J_\nu - B_\nu)}{\partial \tau_\nu^2},$$

since B_ν linear in τ_ν!

assume ε_ν = const (otherwise similar solution)

$$J_\nu - B_\nu = \text{const}' \cdot \exp\left(-(3\varepsilon_\nu)^{\frac{1}{2}}\tau_\nu\right)$$ [with lower b.c. $J_\nu \to B_\nu$ for $\tau \to \infty$]

- Eddington's approximation implies also

a) $J_\nu(0) = \sqrt{3}\, H_\nu(0)$ (see problem 6.2.c)

b) $\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu \rightarrow \frac{1}{3}\frac{\partial J_\nu}{\partial \tau_\nu}\Big|_0 = H_\nu(0)$

Thus $\frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau_\nu}\Big|_0 = J_\nu(0)$

⇒ insert in above equation

$$\text{const}' = \frac{b_\nu/\sqrt{3} - a_\nu}{(1+\varepsilon_\nu^{\frac{1}{2}})}$$

$$\Rightarrow J_\nu = a_\nu + b_\nu\tau_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{1+\varepsilon_\nu^{\frac{1}{2}}}\, e^{-(3\varepsilon_\nu)^{\frac{1}{2}}\tau_\nu}$$

$$J_\nu = a_\nu + b\tau_\nu + \frac{b/\sqrt{3} - a_\nu}{1+\varepsilon_\nu^{\frac{1}{2}}} e^{-(3\varepsilon_\nu)^{\frac{1}{2}}\tau_\nu}$$

$$J_\nu(0) = a_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{1+\varepsilon_\nu^{\frac{1}{2}}}$$

$$H_\nu(0) = \frac{1}{\sqrt{3}} J_\nu(0)$$

- assume isothermal atmosphere, $b_\nu = 0$ (possible, if gradient not too strong)

$$\Rightarrow J_\nu(0) = \frac{\varepsilon_\nu^{\frac{1}{2}}}{1+\varepsilon_\nu^{\frac{1}{2}}} a_\nu \begin{cases} \nearrow B_\nu/2 \text{ for } \varepsilon_\nu = 1, \text{ i.e. } \sigma = 0 \\ \searrow \varepsilon_\nu^{\frac{1}{2}} B_\nu \ll B_\nu \text{ for } \varepsilon_\nu \ll 1 \end{cases}$$

→ $J_\nu(0) < B_\nu(0)$!!!

- Thermalization

only for large arguments of the exponent, we have $J_\nu \approx B_\nu$

$\Rightarrow \tau_\nu \gtrsim \frac{1}{\varepsilon_\nu^{\frac{1}{2}}}$ thermalisation depth

a) $\sigma \ll k^+ \Rightarrow J_\nu(\tau_\nu \gtrsim 1) \Rightarrow B_\nu$

b) SN remnants: scattering dominated, very large thermalization depth

- pure scattering (test case)

$\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu = 0$ for $\varepsilon_\nu = 0$ Flux conservation

$+ H_\nu \approx \frac{1}{3} \frac{\partial B_\nu}{\partial \tau_\nu}$ from diffusion limit

in Milne Eddington model

$$H_\nu(0) = \frac{1}{\sqrt{3}}\left(a_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{1+\varepsilon_\nu^{\frac{1}{3}}}\right) \overset{\varepsilon_\nu \to 0}{\Rightarrow} \frac{b_\nu}{3} \triangleq \frac{1}{3}\frac{\partial B_\nu}{\partial \tau_\nu}$$

consistent result

- Question: why $J_\nu(0) \ll B_\nu(0)$?

- remember: $J_\nu(0)$ determined by $S_\nu(\tau_\nu = 1)$
- $J_\nu(1)$ might fall significantly below $B_\nu(1)$, since many photons can escape from photosphere (into interstellar medium)
- minimum value is given by incident flux, if no thermal emission

- interesting possibility

if ε_ν small, $H_\nu(0)$ can become larger than $H_\nu(0)$ $(\varepsilon_\nu = 1)$, if

$$\underbrace{a_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{2}}_{J_\nu(0, \varepsilon_\nu = 1)} < \underbrace{\frac{b_\nu}{\sqrt{3}}}_{J_\nu(0, \varepsilon_\nu \ll 1)}, \text{ i.e. } \frac{b_\nu}{a_\nu} > \sqrt{3}$$

i.e. for large temperature gradients

(information is transported from hotter regions to outer boundary by scattering dominated stratifications)

- further consequences later

Chap. 6 – Stellar atmospheres

Basic assumptions

1. Geometry
plane-parallel or spherically symmetric (-> Chap. 3)

2. Homogeneity
atmospheres assumed to be homogenous (both vertical and horizontal)

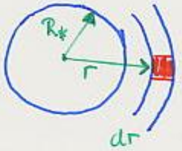
BUT: **sun** with spots, granulation, non-radial pulsations ...
white dwarfs with depth dependent abundances (diffusion)
stellar winds of hot stars (partly) with clumping $(\langle\rho^2\rangle \neq \langle\rho\rangle^2)$

HOPE: "mean" = homogenous model describes non-resolvable phenomena in a reasonable way
[attention for (magnetic) Ap-stars: ***very*** strong inhomogeneities!]

3. Stationarity
vast majority of spectra time-independent => $\partial/\partial t = 0$

BUT: explosive phenomena (supernovae)
pulsations
close binaries with mass transfer ...

Density stratification



$dm = \rho A dr$

mass element dm
in (spherically sym.) atmosphere

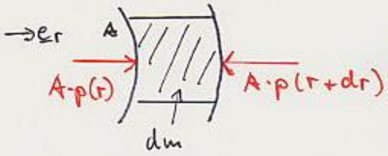
assume (at first) no velocity-fields, i.e. hydrostatic stratification

$\sum_i df_i = 0$, if f_i are forces acting on dm

- $df_{grav} = -G \frac{M_r dm}{r^2} = -g(r) dm$ with grav. accel.

  $g(r) = \frac{G M_r}{r^2}$ and M_r mass within r

- df_p pressure forces



gaspressure causes forces on surfaces $\perp e_r$. Forces on surfaces $\parallel e_r$ compensate each other in spherical (or p-p) symmetry

$$df_p = A \cdot p(r) - A\, p(r+dr) = -A \frac{dp}{dr} dr$$

- df_{Rad} (radiation force) $= g_{rad}(r) dm$

$$\sum dk_i = -g(r) dm + g_{rad}(r) dm - A \frac{dp}{dr} dr = 0$$

$$dm = A \cdot \rho(r) dr$$

$$\Rightarrow \frac{1}{\rho} \frac{dp}{dr} = -g(r) + g_{rad}(r)$$ or

$$\frac{dp}{dr} = -\rho(r) \left[g(r) - g_{rad}(r) \right]$$

Hydrostatic equilibrium

Approximation $\left\| g(r) = \frac{G M_r}{r^2} \rightarrow \frac{G M_*}{r^2} \right\|$

since mass within atmosph: $M(r) - M(R_*) \ll M(R_*)$

example: The sun

$$\Delta M_{phot} = \bar{\rho} \frac{4\pi}{3} \left((R+\Delta r)^3 - R^3 \right) \approx \bar{\rho}\, 4\pi R^2 \Delta r$$

$R \approx 7 \cdot 10^{10}$ cm, $\Delta r \approx 3 \cdot 10^7$ cm (later), $\bar{\rho} \approx m_H \bar{N}$,
with $\bar{N} = 10^{15}$ cm^{-3} and $m_H \approx 1.7 \cdot 10^{-24}$ g

$\Rightarrow \Delta M_{phot} \approx 3 \cdot 10^{21}$ g $\ll M_\odot \approx 2 \cdot 10^{33}$ g

(same argument holds also if atmosphere is extended)

in plane-parallel geometry, we have additionally

$\Delta r \ll R_*$, thus $\left\| g(r) = g_* = \frac{G M_*}{R_*^2} \right\|$

examples	log g [cgs] ≈
main seq. stars (O → A)	4
supergiants	3.5 … 0.8
white dwarfs	8 !
Sun	4.44
earth	3.0

- if stellar wind present, hydrodynamic description

$\dot{M} = 4\pi r^2 \rho(r)\, v(r)$ equation of continuity

$\rightarrow v(r) = \frac{\dot{M}}{4\pi} \frac{1}{r^2 \rho(r)} \neq 0$ (everywhere)

Question When are velocity fields important, i.e. induce significant deviations from hydrostatic equilibrium?

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Equation of momentum

("Euler equation")

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \underbrace{\nabla \cdot (\rho \mathbf{v}\mathbf{v})}_{\mathbf{v}[\nabla \cdot (\rho \mathbf{v})] + [\rho \mathbf{v} \cdot \nabla]\mathbf{v}} = -\nabla p + \rho \mathbf{g}^{\text{ext}}$$

$\Rightarrow$

stationarity, i.e., $\frac{\partial}{\partial t} = 0$

and spherical symmetry,

i.e., $\nabla \cdot \mathbf{u} \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r)$

$$r^2 \rho v = \text{const} = \frac{\dot{M}}{4\pi} \quad \text{(I)}$$

with $\nabla \cdot (\rho \mathbf{v}) = 0$

$$\underline{\rho v \frac{\partial v}{\partial r}} = -\frac{\partial p}{\partial r} + \rho g_r^{\text{ext}} \quad \text{(II)}$$

"advection term",
(from inertia)

I: Conservation of mass-flux (Chap. 3)

II: "Equation of motion"

$$\Rightarrow \frac{\partial p}{\partial r} = \rho(r)\left(-\frac{GM}{r^2} + g_{\text{Rad}}(r)\right) - \rho(r)v(r)\frac{\partial v}{\partial r}$$

Exercise:

Show, by using the cont. eq., that the Euler eq. can be alternatively written as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g}^{\text{ext}}$$

$p = \frac{k_B}{\mu m_H} \rho T$ equation of state for ideal gas, μ mean molecular weight

$$\rightarrow \frac{dp}{dr} = \frac{k_B}{\mu m_H}\left(T \frac{d\rho}{dr} + \rho \frac{dT}{dr}\right)$$

$$= v_{\text{sound}}^2 \left(\frac{d\rho}{dr} + \rho \frac{d\ln T}{dr}\right)$$

$\frac{k_B T}{\mu m_H} = v_{\text{sound}}^2$, isothermal speed of sound

- with $\dot{M} = 4\pi r^2 \rho v = \text{const}$

$$r^2 \rho \frac{dv}{dr} = -2r\rho v - r^2 v \frac{d\rho}{dr}$$

$$\rho v \frac{dv}{dr} = -\frac{2\rho v^2}{r} - v^2 \frac{d\rho}{dr}$$

$\Rightarrow$

alternative formulation of equation of motion

$$(v_s^2 - v^2)\frac{d\rho}{dr} = -\rho\left(g - g_{\text{rad}} + \frac{dv_s^2}{dr} - \frac{2v^2}{r}\right) \quad \text{hydrodyn.}$$

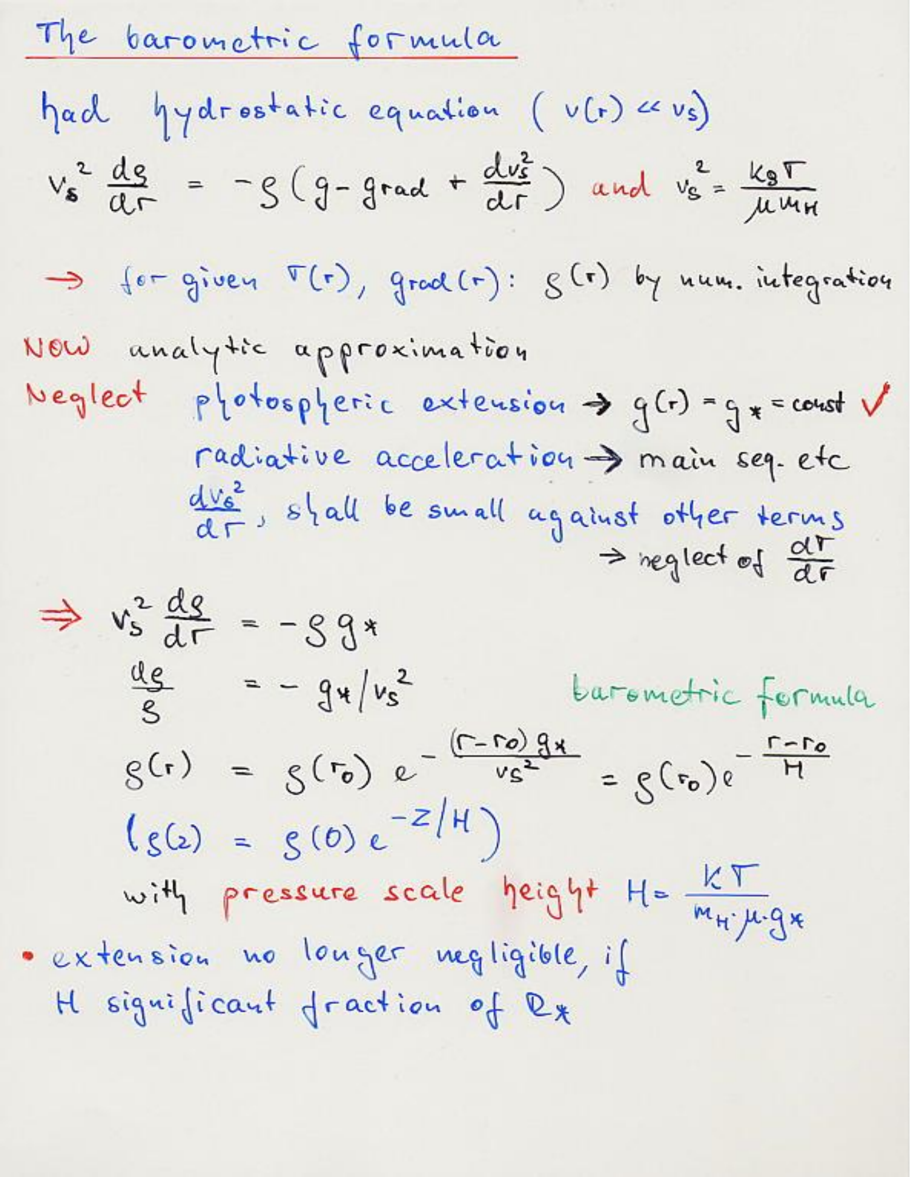
$$v_s^2 \frac{d\rho}{dr} = -\rho\left(g - g_{\text{rad}} + \frac{dv_s^2}{dr}\right) \quad \text{hydrostatic}$$

- conclusion: For $\underline{v \ll v_{\text{sound}}}$, density stratification becomes ('quasi'-) hydrostatic
- is reached in deeper photospheric layers (well below "sonic point", defined by $v(r_s) = v_s$)

example v_{sound} (solar photosphere) $\approx$ 6 km/s
" (O-stars) 20 km/s

- density stratification for stars with wind
  - a) deep layers $\rho(r)$ hydrostatic (→ next sect.)
    $\rightarrow v(r) = \frac{\dot{M}}{4\pi}\frac{1}{r^2\rho}$ ($v \ll v_{\text{sound}}$)
  - c) outer layers $v(r)$, from obs. or theory
    $\Rightarrow \rho(r) = \frac{\dot{M}}{4\pi}\frac{1}{r^2 v}$
  - b) intermediate (transonic) regions
    (smooth) transition from a) to c)

Barometric formula



The barometric formula

had hydrostatic equation ($v(r) \ll v_s$)

$$v_s^2 \frac{d\rho}{dr} = -\rho\left(g - g_{rad} + \frac{dv_s^2}{dr}\right) \text{ and } v_s^2 = \frac{k_B T}{\mu m_H}$$

→ for given $T(r)$, $g_{rad}(r)$: $\rho(r)$ by num. integration

NOW analytic approximation

Neglect photospheric extension → $g(r) = g_* = \text{const}$ ✓

radiative acceleration → main seq. etc

$\frac{dv_s^2}{dr}$, shall be small against other terms

→ neglect of $\frac{dT}{dr}$

$$\Rightarrow v_s^2 \frac{d\rho}{dr} = -\rho g_*$$

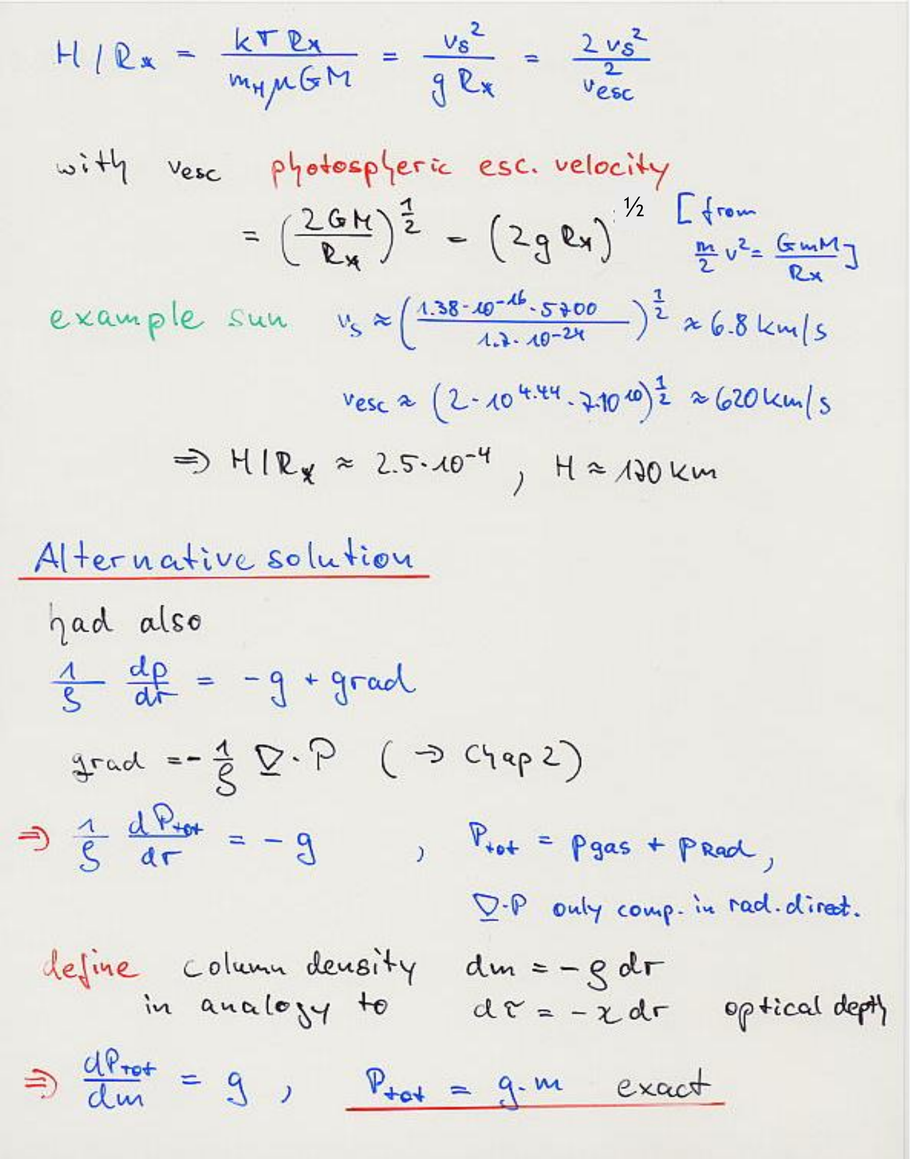
$$\frac{d\rho}{\rho} = -g_*/v_s^2 \qquad \text{barometric formula}$$

$$\rho(r) = \rho(r_0)\, e^{-\frac{(r-r_0) g_*}{v_s^2}} = \rho(r_0)\, e^{-\frac{r-r_0}{H}}$$

$$\left(\rho(z) = \rho(0)\, e^{-z/H}\right)$$

with pressure scale height $H = \frac{kT}{m_H \cdot \mu \cdot g_*}$

- extension no longer negligible, if H significant fraction of R_*



$$H/R_* = \frac{kT R_*}{m_H \mu G M} = \frac{v_s^2}{g R_*} = \frac{2 v_s^2}{v_{esc}^2}$$

with v_{esc} photospheric esc. velocity

$$= \left(\frac{2GM}{R_*}\right)^{\frac{1}{2}} - (2 g R_*)^{½} \quad \left[\text{from } \frac{m}{2} v^2 = \frac{GmM}{R_*}\right]$$

example sun $v_s \approx \left(\frac{1.38 \cdot 10^{-16} \cdot 5700}{1.7 \cdot 10^{-24}}\right)^{\frac{1}{2}} \approx 6.8$ km/s

$$v_{esc} \approx \left(2 \cdot 10^{4.44} \cdot 7 \cdot 10^{10}\right)^{\frac{1}{2}} \approx 620 \text{ km/s}$$

$$\Rightarrow H/R_* \approx 2.5 \cdot 10^{-4}, \quad H \approx 170 \text{ km}$$

Alternative solution

had also

$$\frac{1}{\rho}\frac{dp}{dr} = -g + g_{rad}$$

$$g_{rad} = -\frac{1}{\rho} \underline{\nabla} \cdot P \quad (\to \text{Chap } 2)$$

$$\Rightarrow \frac{1}{\rho}\frac{dP_{tot}}{dr} = -g, \quad P_{tot} = p_{gas} + p_{Rad},$$

$\underline{\nabla} \cdot P$ only comp. in rad. direct.

define column density $dm = -\rho\, dr$

in analogy to $d\tau = -\chi\, dr$ optical depth

$$\Rightarrow \frac{dP_{tot}}{dm} = g, \quad \underline{P_{tot} = g \cdot m \quad \text{exact}}$$

Eddington limit

or

$$\frac{dp_{gas}}{dm} = g - g_{rad} = g - \frac{4\pi}{c\rho}\int_0^\infty \chi_\nu H_\nu d\nu$$

- solution by numerical integration
- analytic approx: neglect … as before

$$\rightarrow \quad p_{gas} = g_* \cdot m$$

$$\rho = \frac{g_* \mu m_H}{k \cdot T} \cdot m = \frac{1}{H} \cdot m$$

or $\log \rho = \log m - \log H$

Example

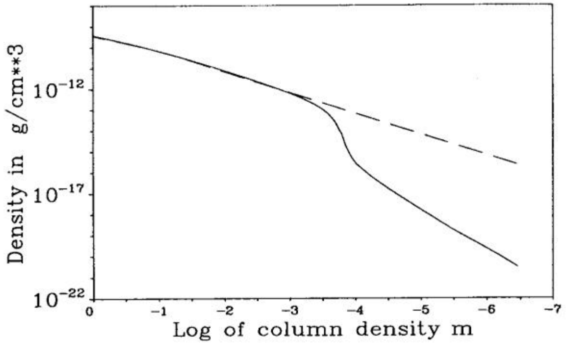


Fig. 16. Mass density ρ as function of logarithm of atmospheric column density m for a typical unified model (solid) and a hydrostatic model (dashed) with similar $T_{\rm eff}$ and log g

Exercise: derive H directly from above figure
compare with result from
calculation of H (T$_{\rm eff}$ = 40,000 K, log g = 3.6)

The Eddington limit

$$\frac{dp_{gas}}{dm} = g - g_{rad} = g_{eff} \quad \text{(without rotation)}$$

inwards outwards

- $g_{rad} = \frac{4\pi}{c\rho}\int_0^\infty \chi_\nu H_\nu d\nu$ in static atmospheres (χ_ν isotropic)
- minimum value (≙ main part of total continuum rad. acceleration in outer atmospheres of hot stars)

Thomson scattering

$$g_{rad}^{\text{Thomson}} = \frac{4\pi}{c\rho}\int_0^\infty \underset{\substack{n_e\sigma_e,\ \text{freq.} \\ \text{independent}}}{\sigma^{TH}} H_\nu d\nu = \frac{4\pi}{c} \underbrace{\frac{n_e \sigma_e}{\rho}}_{s_e} H(\tau)$$

s_e: 0.34 H/He fully ionized; 0.4 pure H, fully ionized

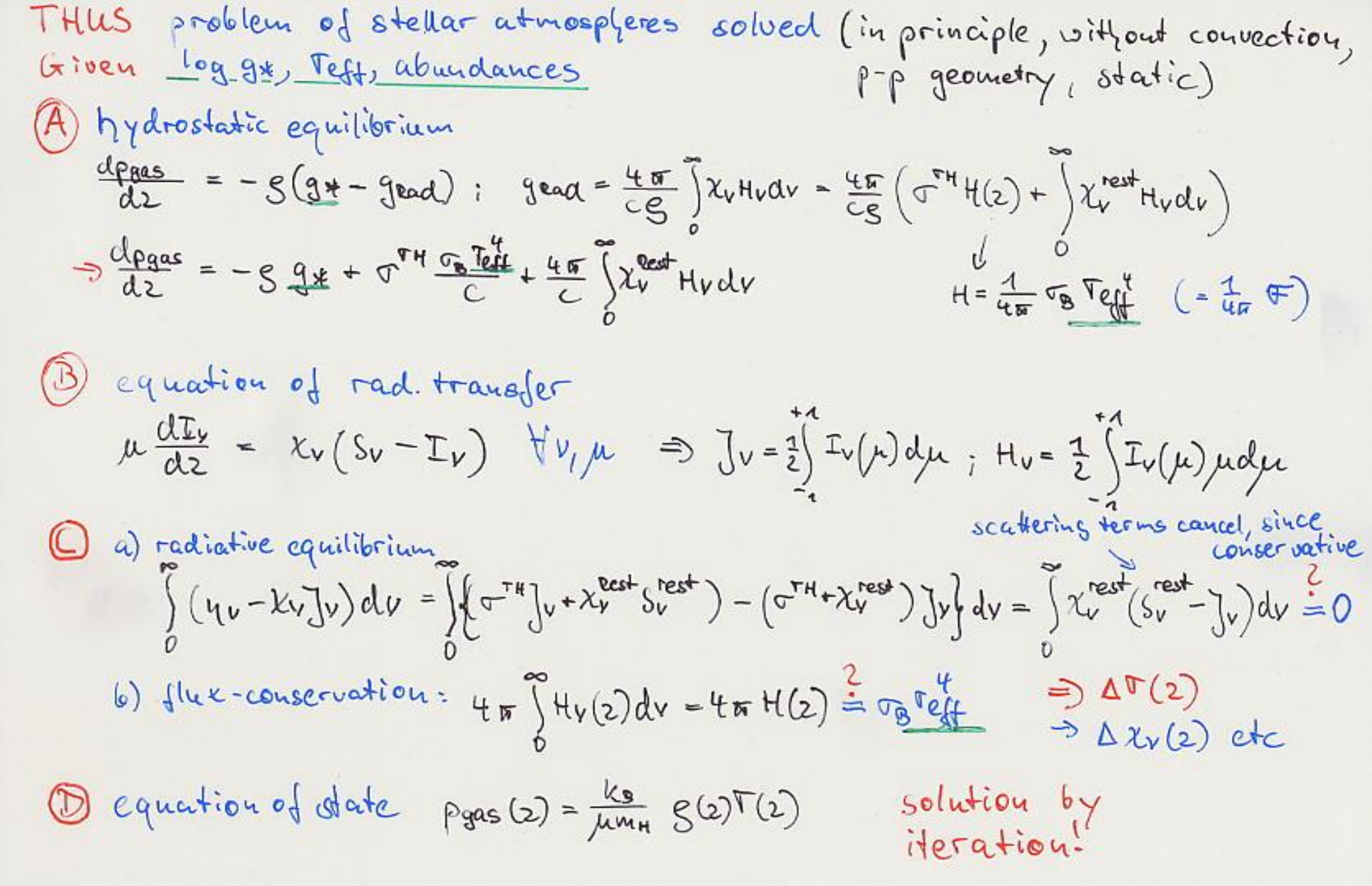
Define $\Gamma_e = \frac{g_{rad}^{TH}}{g_{grav}} = \frac{\frac{4\pi}{c} s_e \frac{L}{16\pi^2 r^2}}{\frac{GM}{r^2}} = \text{const (for } s_e = \text{const)}$

$$= \frac{L}{4\pi c G M} s_e = 7.64 \cdot 10^{-5} \cdot s_e \cdot \frac{L/L_\odot}{M/M_\odot}$$

- $\Gamma_e = 1$ defines "Eddington limit": unstable atmosphere
- $g_{eff} = g - g_{rad} = g(1-\Gamma_e) \quad (-g_{rad}^{rest})$
  defines "effective" gravity

NOTE
- bound-free + free-free absorption has similar contribution (in intermediate layers)
- bound-bound absorption dominates the radiative acceleration in hot, luminous stars ⇒ "line driven winds"

Summary: stellar atmospheres – the solution principle



THUS problem of stellar atmospheres solved (in principle, without convection, p-p geometry, static)

Given $\underline{\log g_*, T_{\rm eff}, \text{abundances}}$

Ⓐ hydrostatic equilibrium

$$\frac{dp_{\rm gas}}{dz} = -\rho(\underline{g_*} - g_{\rm rad}) \; ; \quad g_{\rm rad} = \frac{4\pi}{c\rho}\int_0^\infty \chi_\nu H_\nu d\nu = \frac{4\pi}{c\rho}\Big(\sigma^{\rm TH} \underset{\displaystyle\downarrow \atop \displaystyle H = \frac{1}{4\pi}\sigma_B \underline{T_{\rm eff}^4} \quad \left(= \frac{1}{4\pi}\mathcal{F}\right)}{H(z)} + \int_0^\infty \chi_\nu^{\rm rest} H_\nu d\nu\Big)$$

$$\Rightarrow \frac{dp_{\rm gas}}{dz} = -\rho\, \underline{g_*} + \sigma^{\rm TH}\frac{\sigma_B \underline{T_{\rm eff}^4}}{c} + \frac{4\pi}{c}\int_0^\infty \chi_\nu^{\rm rest} H_\nu d\nu$$

Ⓑ equation of rad. transfer

$$\mu \frac{dI_\nu}{dz} = \chi_\nu(S_\nu - I_\nu) \quad \forall \nu, \mu \quad \Rightarrow J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)d\mu \; ; \; H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\mu d\mu$$

Ⓒ a) radiative equilibrium

scattering terms cancel, since conservative

$$\int_0^\infty (\eta_\nu - \chi_\nu J_\nu)d\nu = \int_0^\infty \left\{(\sigma^{\rm TH} J_\nu + \chi_\nu^{\rm rest} S_\nu^{\rm rest}) - (\sigma^{\rm TH} + \chi_\nu^{\rm rest})J_\nu\right\} d\nu = \int_0^\infty \chi_\nu^{\rm rest}(S_\nu^{\rm rest} - J_\nu)d\nu \overset{?}{=} 0$$

b) flux-conservation: $4\pi\int_0^\infty H_\nu(z)d\nu = 4\pi H(z) \overset{?}{=} \sigma_B \underline{T_{\rm eff}^4}$ $\Rightarrow \Delta T(z)$

$\rightarrow \Delta \chi_\nu(z)$ etc

Ⓓ equation of state $p_{\rm gas}(z) = \frac{k_B}{\mu m_H}\rho(z)T(z)$

solution by iteration!

Solution of differential equations A and B by discretization
differential operators => finite differences
all quantities have to be evaluated on suitable grid

Eq. of radiative transfer (B)
usually solved by the so-called
Feautrier and/or Rybicki scheme

Grey temperature stratification

- for iteration, we need initial values
- analytic understanding

⇒ "grey" approximation

assume $\chi_\nu = \chi$, freq. independent opacities (corresponds to suitable averages)

$$\Rightarrow \mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu \qquad \overset{\wedge}{=} \text{radiative eq.}$$

$$\left.\begin{aligned} \frac{dH_\nu}{d\tau} &= J_\nu - S_\nu \\ \frac{dK_\nu}{d\tau} &= H_\nu \end{aligned}\right\} \text{(freq. integr.)} \quad J = \int_0^\infty J_\nu d\nu \text{ etc} \qquad \begin{aligned} \frac{dH}{d\tau} &= J - S \ (=0) \\ \frac{dK}{d\tau} &= H \end{aligned}$$

$$\Rightarrow \frac{dK}{d\tau} = H, \quad \text{i.e.} \quad K = H \cdot \tau + C$$

For large $\tau \gg 1$, we know from diff. approx. that $K_\nu / J_\nu = \frac{1}{3}$

Eddington's approx. $K/J = \frac{1}{3}$ everywhere

$$\Rightarrow J = 3H(\tau + c)$$

- From rad. equilibrium

$$J = S, \qquad S = 3H(\tau + c)$$

- remember Λ-operator

$$J = \Lambda_\tau(S)$$

- analogous

$H = \phi_\tau(S)$, in particular

$$H(0) = \frac{1}{2}\int_0^\infty S(t) E_2(t) dt \qquad E_2 \text{ 2nd Exp. integral}$$

$$\Rightarrow H(0) = \frac{1}{2}\int_0^\infty (3H(t+c)) E_2(t) dt = \ldots\ldots$$

$$\ldots\ldots H\left(\frac{1}{2} + c\frac{3}{4}\right)$$

But $H(0) = H$, i.e., $\left(\frac{1}{2} + c\frac{3}{4}\right) = 1$

$c = \frac{2}{3}$ in Eddington approx

Exact sol. $c = q(\tau)$, "Hopf function", $0.51 < q(\tau) < 0.71$

- $J = 3H(\tau + 2/3)$

$$H = \frac{\sigma T_{\text{eff}}^4}{4\pi}; \qquad J \xrightarrow{\text{LTE}} B = \frac{\sigma_B T^4}{\pi}$$

Finally

$$\left\| \; T^4 = \frac{3}{4} T_{\text{eff}}^4 (\tau + 2/3) \; \right\|$$

grey temp. in Eddington approx!

Consequences

- $T = T_{\text{eff}}$ at $\tau = 2/3$
- $T(0)/T_{\text{eff}} = \left(\frac{1}{2}\right)^{1/4} = 0.841$

grey temp. in spherical symmetry

basic difference

$J, H \sim \frac{1}{r^2}$ for $r \gg R_*$ quadratic dilution

$J/K = 1$ for $r \gg R_*$

result

$$T^4(r) = T_{eff}^4 \left(W + \frac{3}{4}\tau' \right)$$

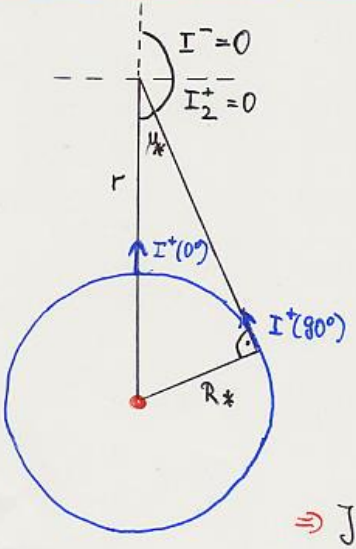
W dilution factor, $\frac{1}{2}\left[1-\left(1-\left(\frac{R_*}{r}\right)^2\right)^{\frac{1}{2}}\right]$

$$\tau' = \int_r^\infty \chi(r) \left(\frac{R_*}{r}\right)^2 dr$$

NOTE

$$T^{sph}(r) \xrightarrow{r \to R_*} T^{pp}(\tau)$$

Radiation field in optically thin envelopes



assume

- envelope optically thin
  $\Rightarrow I = \text{const}$
- radiation field leaving photosphere isotropic
  $\Rightarrow I^+_{phot}(\mu) = \text{const}$

$$\Rightarrow J_\nu(r) = \frac{1}{2}\int_{-1}^{+1} I_\nu(r)\, d\mu \longrightarrow$$

$$= \frac{1}{2}\int_{\mu_*}^{1} I_\nu^+(R_*)\, d\mu + \frac{1}{2}\int_{0}^{\mu^*} \underset{0}{\underset{\uparrow}{I_2^+}}\, d\mu + \frac{1}{2}\int_{-1}^{0} \underset{0}{\underset{\uparrow}{I^-}}\, d\mu$$

$$= \frac{1}{2} I_\nu^+(R_*)(1-\mu_*)$$

$\sin\theta = \frac{R_*}{r} \Rightarrow \mu_* = \cos\theta = \sqrt{1-\left(\frac{R_*}{r}\right)^2}$

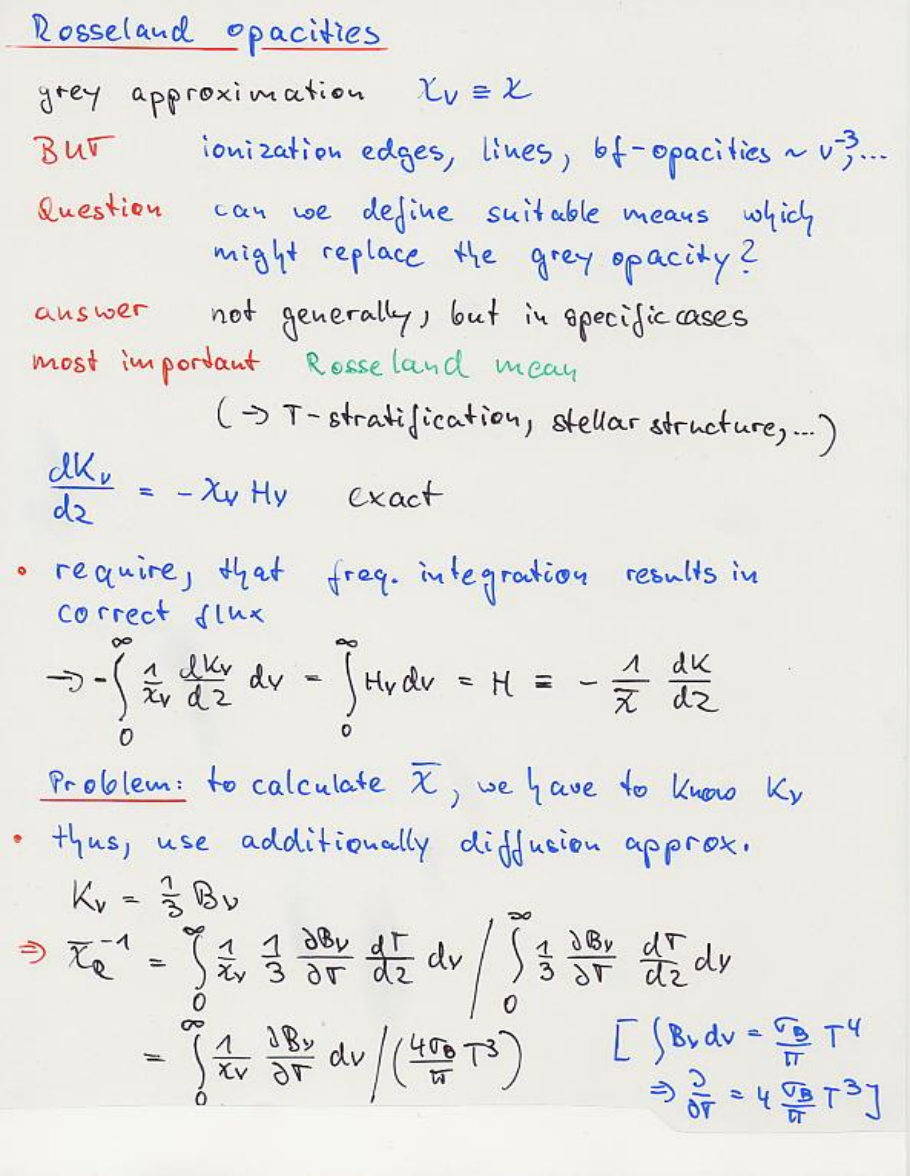
$J_\nu(r) = W \cdot I_\nu^+(R_*)$, $W = \frac{1}{2}\left(1-\left(1-\frac{R_*^2}{r^2}\right)^{\frac{1}{2}}\right)$

"Dilution factor"

exercise: show that for $r \gg R_*$,

$$J_\nu(r) \approx H_\nu(r) \approx K_\nu(r)$$

Rosseland opacities



Rosseland opacities

grey approximation $\chi_\nu \equiv \chi$

BUT ionization edges, lines, bf-opacities $\sim \nu^{-3}$, …

Question: can we define suitable means which might replace the grey opacity?

answer: not generally, but in specific cases

most important: Rosseland mean

(→ T-stratification, stellar structure, …)

$$\frac{dK_\nu}{dz} = -\chi_\nu H_\nu \quad \text{exact}$$

- require, that freq. integration results in correct flux

$$\Rightarrow -\int_0^\infty \frac{1}{\chi_\nu}\frac{dK_\nu}{dz}\,d\nu = \int_0^\infty H_\nu\,d\nu = H = -\frac{1}{\bar\chi}\frac{dK}{dz}$$

Problem: to calculate $\bar\chi$, we have to know K_ν

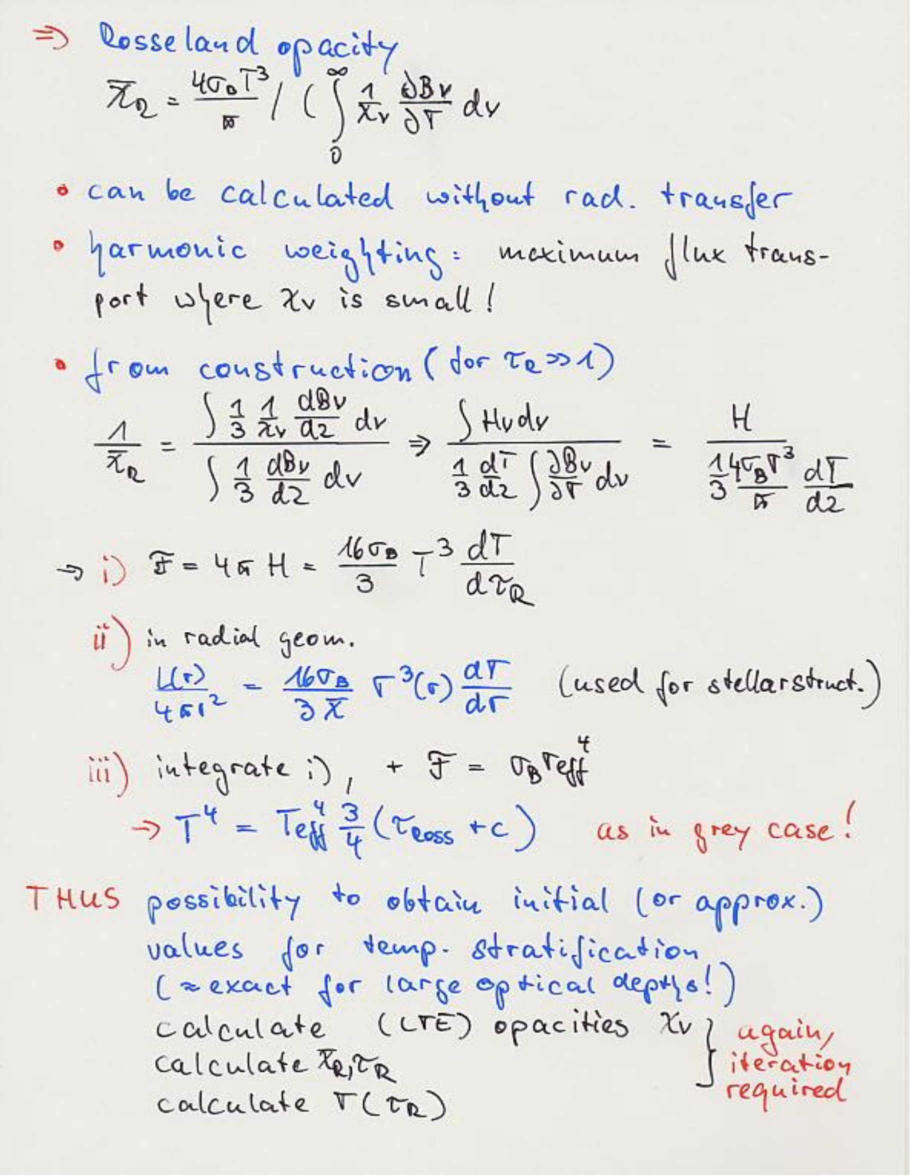
- thus, use additionally diffusion approx.

$$K_\nu = \frac{1}{3}B_\nu$$

$$\Rightarrow \bar\chi_R^{-1} = \int_0^\infty \frac{1}{\chi_\nu}\frac{1}{3}\frac{\partial B_\nu}{\partial T}\frac{dT}{dz}\,d\nu \Big/ \int_0^\infty \frac{1}{3}\frac{\partial B_\nu}{\partial T}\frac{dT}{dz}\,d\nu$$

$$= \int_0^\infty \frac{1}{\chi_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu \Big/ \left(\frac{4\sigma_B}{\pi}T^3\right)$$

$$\left[\int B_\nu\,d\nu = \frac{\sigma_B}{\pi}T^4 \Rightarrow \frac{\partial}{\partial T} = 4\frac{\sigma_B}{\pi}T^3\right]$$



⇒ Rosseland opacity

$$\bar\chi_R = \frac{4\sigma_B T^3}{\pi} \Big/ \left(\int_0^\infty \frac{1}{\chi_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu\right.$$

- can be calculated without rad. transfer
- harmonic weighting: maximum flux transport where χ_ν is small!
- from construction (for $\tau_R \gg 1$)

$$\frac{1}{\bar\chi_R} = \frac{\int \frac{1}{3}\frac{1}{\chi_\nu}\frac{dB_\nu}{dz}\,d\nu}{\int \frac{1}{3}\frac{dB_\nu}{dz}\,d\nu} \Rightarrow \frac{\int H_\nu\,d\nu}{\frac{1}{3}\frac{dT}{dz}\int\frac{\partial B_\nu}{\partial T}\,d\nu} = \frac{H}{\frac{1}{3}\frac{4\sigma_B T^3}{\pi}\frac{dT}{dz}}$$

→ i) $\mathcal{F} = 4\pi H = \frac{16\sigma_B}{3}T^3\frac{dT}{d\tau_R}$

ii) in radial geom.

$$\frac{L(r)}{4\pi r^2} = \frac{16\sigma_B}{3\bar\chi}T^3(r)\frac{dT}{dr} \quad \text{(used for stellar struct.)}$$

iii) integrate i), + $\mathcal{F} = \sigma_B T_{\text{eff}}^4$

$$\rightarrow T^4 = T_{\text{eff}}^4\,\frac{3}{4}(\tau_{Ross} + c) \quad \text{as in grey case!}$$

THUS possibility to obtain initial (or approx.) values for temp. stratification (≈ exact for large optical depths!)

calculate (LTE) opacities χ_ν
calculate $\bar\chi_R, \tau_R$
calculate $T(\tau_R)$

} again, iteration required

... back to Milne Eddington Model (page 75)

had $B_\nu(\tau_\nu) = a_\nu + b_\nu \tau_\nu$ linear approx

and $J_\nu(0) = \frac{b_\nu}{\sqrt{3}}$ for $\varepsilon_\nu = 0$ pure scattering

$= a_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{2}$ for $\varepsilon_\nu = 1$ purely thermal

$$\varepsilon_\nu = \frac{\kappa_\nu^+}{\kappa_\nu^+ + \sigma_e n_e}$$

- since temperature stratification known by now, can perform some estimates concerning continuum fluxes

had $T^4 \approx T_{eff}^4 \frac{3}{4}(\tau_R + 2/3)$

$T(0)^4 = T_{eff}^4 \frac{3}{4} \cdot 2/3$

$\Big\} \; T^4 = T^4(0)(1 + \frac{3}{2}\tau_R)$

$$B_\nu(\tau_R) \approx B_\nu(T_0) + \left(\frac{\partial B_\nu}{\partial \tau_R}\right)_0 \tau_R = B_0 + B_1 \tau_R$$

$$\Rightarrow B_1 = \frac{\partial B_\nu}{\partial T}\Big|_{T_0} \cdot \frac{\partial T}{\partial \tau_R}\Big|_{T_0} = B_\nu \frac{h\nu/kT \cdot \frac{1}{T} e^{h\nu/kT}}{(e^{h\nu/kT} - 1}\Big|_{T_0} \frac{\partial T}{\partial \tau_R}\Big|_{T_0}$$

$$= B_\nu \frac{u_0}{1 - e^{-u_0}} \frac{1}{T_0} \frac{\partial T}{\partial \tau_R}\Big|_0 \quad \text{with} \quad u_0 = \frac{h\nu}{kT_0}$$

$$4T^3 \frac{\partial T}{\partial \tau_R} = T^4(0) \frac{3}{2}, \quad \frac{\partial T}{\partial \tau_R}\Big|_{T_0} = \frac{3}{8} T_0$$

Thus $B_1 = B_0 \frac{u_0}{1 - e^{-u_0}} \frac{3}{8}$ $\rightarrow$ (Rayleigh-Jeans) $B_1 = \frac{3}{8} B_0$

$\rightarrow$ (Wien) $B_1 = \frac{3}{8} u_0 B_0$

example $T_{eff} = 40{,}000$ K, $\lambda = 500, 912$ Å

[Hydrogen $\genfrac{}{}{0pt}{}{\text{Balmer}}{\text{Lyman}}$ continuum, $\varepsilon_\nu \ll 1$)

$T_0 = 33{,}600$ K

$u_0 = \genfrac{}{}{0pt}{}{8.56}{4.70} \rightarrow B_1 \approx \genfrac{}{}{0pt}{}{3.21}{1.76} B_0$

$\rightarrow$ if $(\kappa_\nu^+ + \sigma_\nu) \approx \bar{\chi}_R$ $\quad J_\nu(0, \varepsilon_\nu = 1) \approx \genfrac{}{}{0pt}{}{1.42}{1.0} B_0$

$H_\nu(0) = \frac{1}{\sqrt{3}} J_\nu(0)$ $\quad J_\nu(0, \varepsilon_\nu \rightarrow 0) \approx \genfrac{}{}{0pt}{}{1.85}{1.01} B_0$

can look down deeper into atm.

- additional effect 1

T-stratification with respect to $\tau_R(\bar{\chi}_R)$, but radiation transfer with respect to freq. τ_ν

$J_\nu = B_\nu + \ldots = a_\nu + b_\nu \tau_\nu + \ldots$

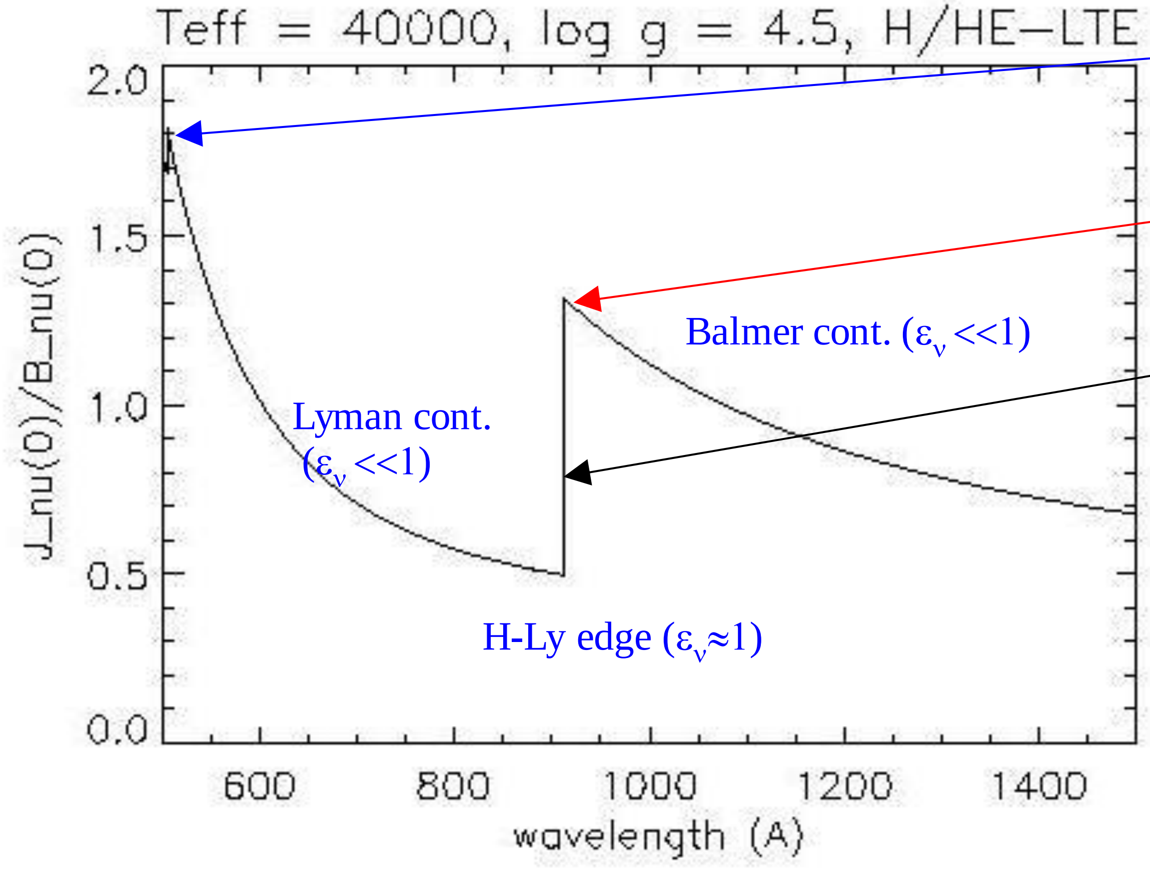
$$B_\nu = B_{\nu 0} + B_1 \tau_R = B_{\nu 0} + B_1 \tau_\nu \frac{\tau_R}{\tau_\nu} \approx B_{\nu_0} + \underbrace{B_1 \frac{\bar{\chi}_R}{\chi_\nu}}_{b_\nu} \cdot \tau_\nu$$

effective gradient increased, if κ_ν small compared to $\bar{\chi}_R$

- additional effect 2

far away from ionization edges (where ε_ν is small, anyway), also χ_ν small ($\kappa_\nu^+ \sim \left(\frac{\nu_0}{\nu}\right)^3$, cf Chapter 5) $\rightarrow$ additional enhancement

H/He continuum of a hot star around 1000 Å



Predictions

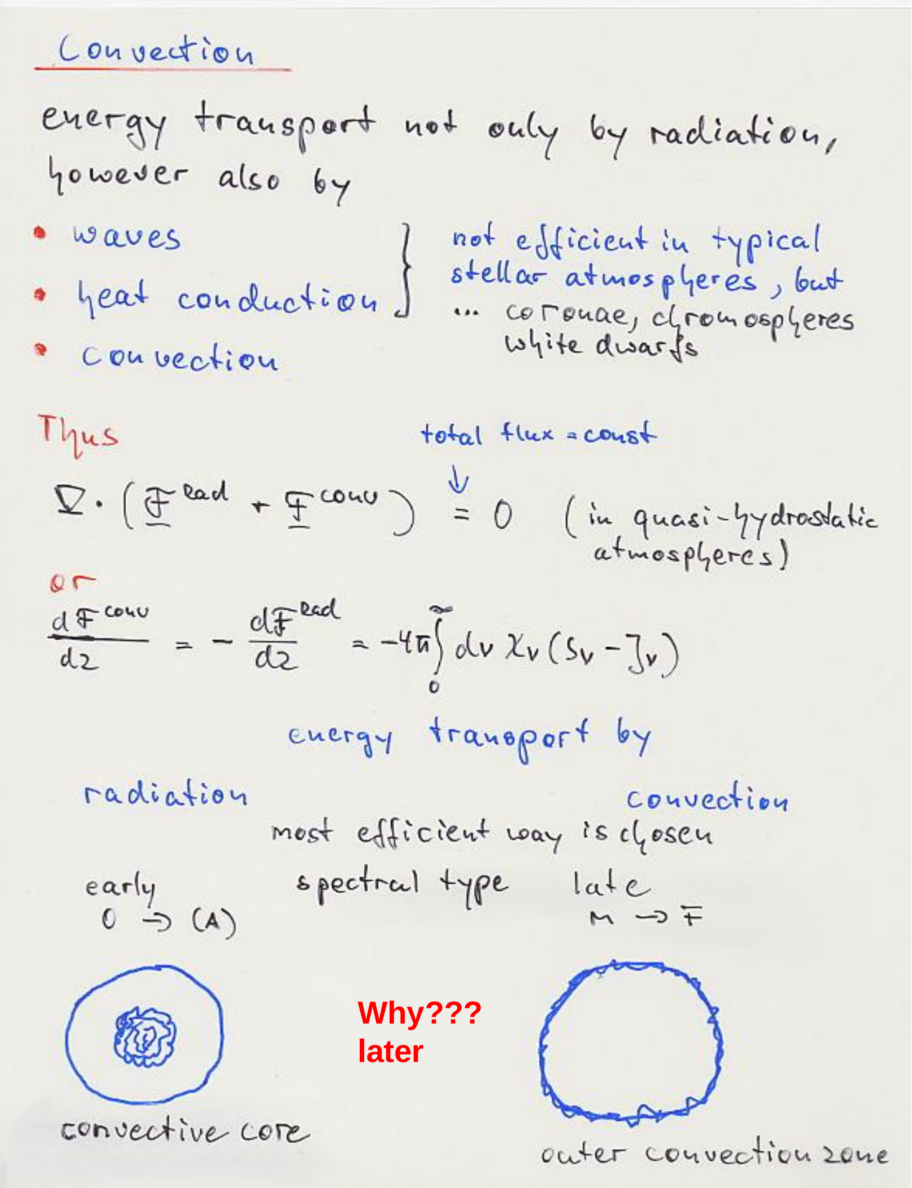
Lyman cont: $J_\nu / B_\nu \geq 1.85$, OK
(at 500 Å) $(\chi_\nu \approx \chi_R)$

Balmer cont: $J_\nu / B_\nu \geq 1.01$, OK
(at 912 Å) $(\chi_\nu < \chi_R)$

Lyman edge: $J_\nu / B_\nu \leq 1.0$, OK
(911 Å) $(\chi_\nu > \chi_R)$

note: large opacity leads to very small effective T-gradient, minimum value $J_\nu / B_\nu = 0.5$, (cf. page 81)

Convection (simplified)



Convection

energy transport not only by radiation, however also by

- waves
- heat conduction
- convection

} not efficient in typical stellar atmospheres, but ... coronae, chromospheres white dwarfs

Thus

total flux = const

$$\nabla\cdot(\underline{F}^{\mathrm{rad}} + \underline{F}^{\mathrm{conv}}) \overset{\downarrow}{=} 0 \quad \text{(in quasi-hydrostatic atmospheres)}$$

or

$$\frac{dF^{\mathrm{conv}}}{dz} = -\frac{dF^{\mathrm{rad}}}{dz} = -4\pi\int_0^\infty d\nu\,\chi_\nu(S_\nu - J_\nu)$$

energy transport by

radiation — convection

most efficient way is chosen

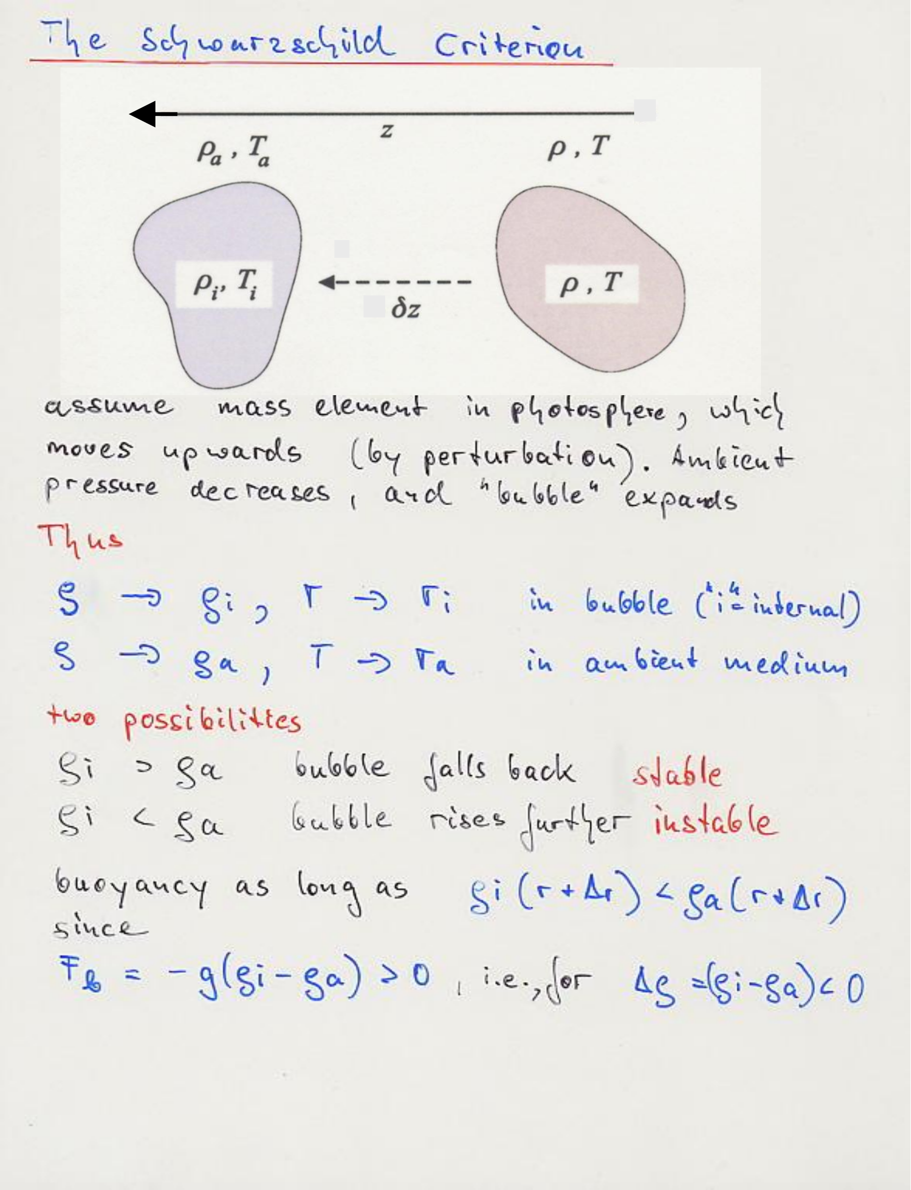
early — spectral type — late

O → (A) — M → F

Why??? later

convective core

outer convection zone



The Schwarzschild Criterion

assume mass element in photosphere, which moves upwards (by perturbation). Ambient pressure decreases, and "bubble" expands

Thus

$\rho \rightarrow \rho_i$, $T \rightarrow T_i$ in bubble ("i" = internal)

$\rho \rightarrow \rho_a$, $T \rightarrow T_a$ in ambient medium

two possibilities

$\rho_i > \rho_a$ bubble falls back stable

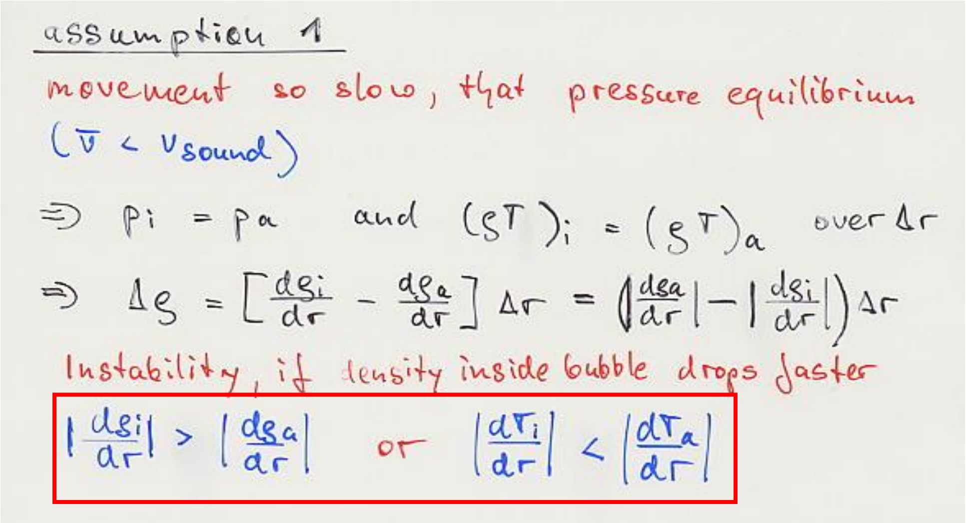
$\rho_i < \rho_a$ bubble rises further instable

buoyancy as long as $\rho_i(r+\Delta r) < \rho_a(r+\Delta r)$

since

$F_b = -g(\rho_i - \rho_a) > 0$, i.e., for $\Delta\rho = (\rho_i - \rho_a) < 0$

The Schwarzschild criterion



assumption 1

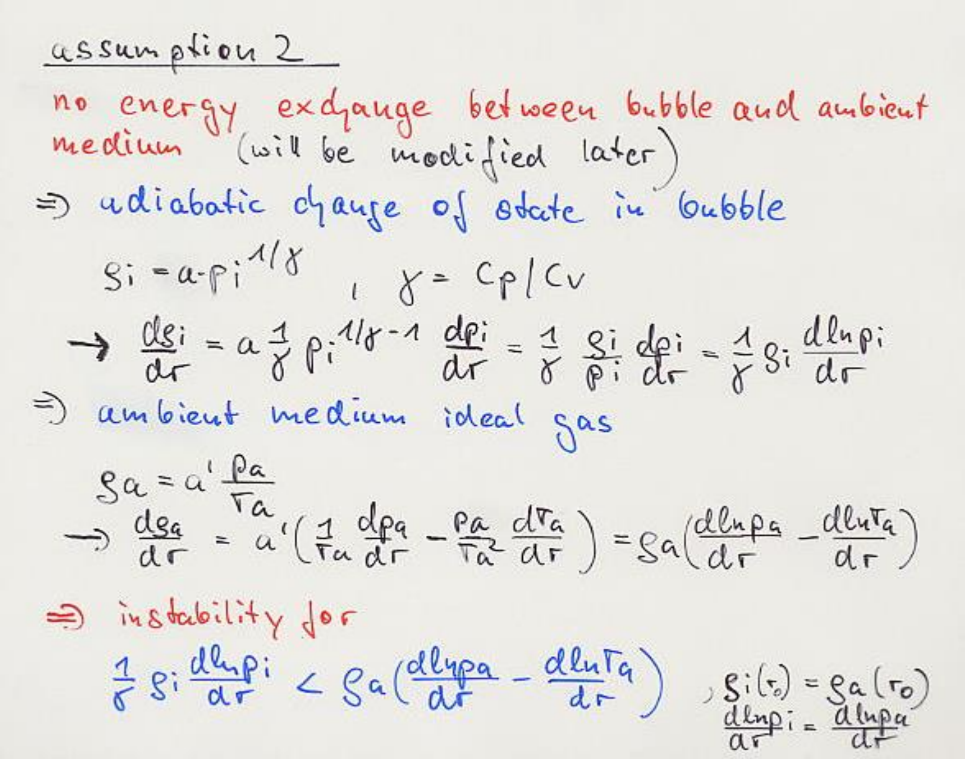
movement so slow, that pressure equilibrium ($\bar{v} < v_{sound}$)

$\Rightarrow p_i = p_a$ and $(\rho T)_i = (\rho T)_a$ over Δr

$\Rightarrow \Delta\rho = \left[\frac{d\rho_i}{dr} - \frac{d\rho_a}{dr}\right]\Delta r = \left(\left|\frac{d\rho_a}{dr}\right| - \left|\frac{d\rho_i}{dr}\right|\right)\Delta r$

Instability, if density inside bubble drops faster

$$\boxed{\left|\frac{d\rho_i}{dr}\right| > \left|\frac{d\rho_a}{dr}\right| \quad \text{or} \quad \left|\frac{dT_i}{dr}\right| < \left|\frac{dT_a}{dr}\right|}$$



assumption 2

no energy exchange between bubble and ambient medium (will be modified later)

$\Rightarrow$ adiabatic change of state in bubble

$\rho_i = a \cdot p_i^{1/\gamma}$, $\gamma = c_p / c_v$

$\rightarrow \frac{d\rho_i}{dr} = a \frac{1}{\gamma} p_i^{1/\gamma - 1} \frac{dp_i}{dr} = \frac{1}{\gamma} \frac{\rho_i}{p_i} \frac{dp_i}{dr} = \frac{1}{\gamma} \rho_i \frac{d\ln p_i}{dr}$

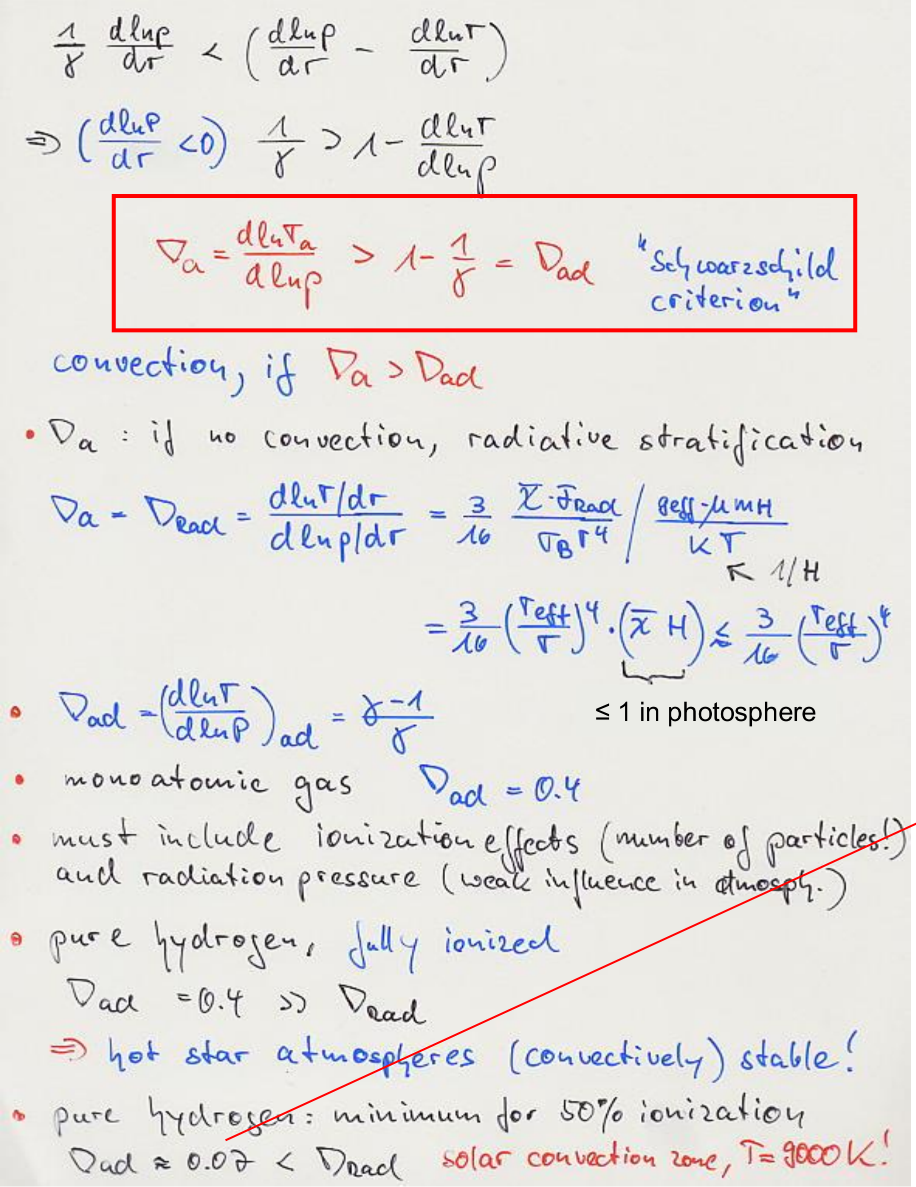
$\Rightarrow$ ambient medium ideal gas

$\rho_a = a' \frac{p_a}{T_a}$

$\rightarrow \frac{d\rho_a}{dr} = a'\left(\frac{1}{T_a}\frac{dp_a}{dr} - \frac{p_a}{T_a^2}\frac{dT_a}{dr}\right) = \rho_a\left(\frac{d\ln p_a}{dr} - \frac{d\ln T_a}{dr}\right)$

$\Rightarrow$ instability for

$\frac{1}{\gamma} \rho_i \frac{d\ln p_i}{dr} < \rho_a\left(\frac{d\ln p_a}{dr} - \frac{d\ln T_a}{dr}\right)$, $\rho_i(r_0) = \rho_a(r_0)$, $\frac{d\ln p_i}{dr} = \frac{d\ln p_a}{dr}$



$$\frac{1}{\gamma}\frac{d\ln p}{dr} < \left(\frac{d\ln p}{dr} - \frac{d\ln T}{dr}\right)$$

$$\Rightarrow \left(\frac{d\ln p}{dr} < 0\right) \quad \frac{1}{\gamma} > 1 - \frac{d\ln T}{d\ln p}$$

$$\boxed{\nabla_a = \frac{d\ln T_a}{d\ln p} > 1 - \frac{1}{\gamma} = \nabla_{ad} \quad \text{"Schwarzschild criterion"}}$$

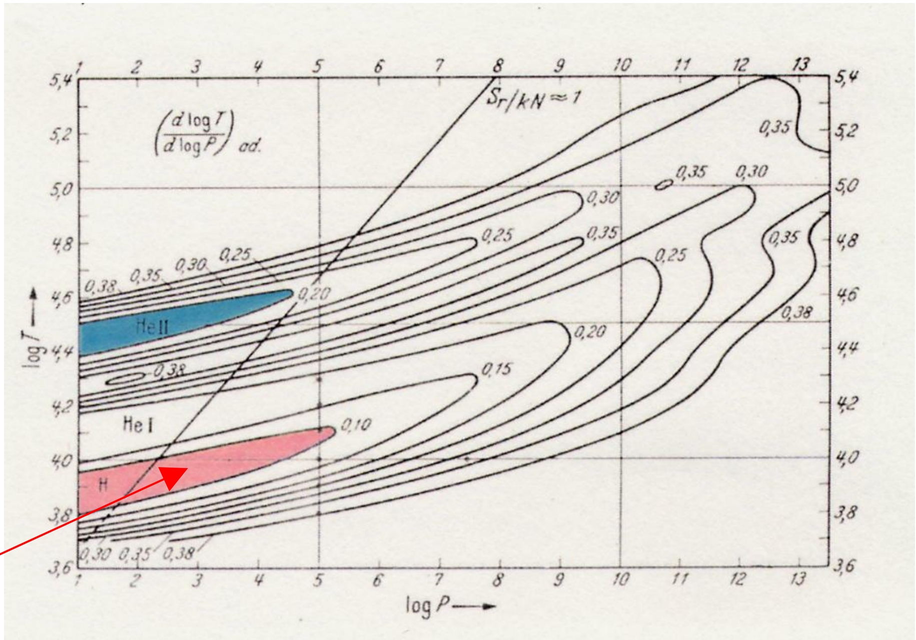
convection, if $\nabla_a > \nabla_{ad}$

- ∇_a : if no convection, radiative stratification

$$\nabla_a = \nabla_{rad} = \frac{d\ln T/dr}{d\ln p/dr} = \frac{3}{16}\frac{\bar{\chi} \cdot F_{rad}}{\sigma_B T^4} \Big/ \underset{\nwarrow\, 1/H}{\frac{g_{eff}\,\mu m_H}{kT}}$$

$$= \frac{3}{16}\left(\frac{T_{eff}}{T}\right)^4 \cdot \underbrace{\left(\bar{\chi}\, H\right)}_{\leq 1 \text{ in photosphere}} \lesssim \frac{3}{16}\left(\frac{T_{eff}}{T}\right)^4$$

- $\nabla_{ad} = \left(\frac{d\ln T}{d\ln p}\right)_{ad} = \frac{\gamma - 1}{\gamma}$
- monoatomic gas $\nabla_{ad} = 0.4$
- must include ionization effects (number of particles!) and radiation pressure (weak influence in atmosph.)
- pure hydrogen, fully ionized
  $\nabla_{ad} = 0.4 \gg \nabla_{rad}$
  ⇒ hot star atmospheres (convectively) stable!
- pure hydrogen: minimum for 50% ionization
  $\nabla_{ad} \approx 0.07 < \nabla_{rad}$ solar convection zone, T= 9000 K!



∇_{ad} as function of T and p

Mixing length theory

- most simplistic approach, however frequently used (reality is much too complex)
- suggested by Prandtl (1925)
- idea: - if atmosphere convective unstable at r_0, assume mass element rises until $r_0 + \ell$ (mixing length)
  - at $r_0 + \ell$, excess energy $\Delta E = c_p \rho \overline{\Delta T}$ is released into ambient medium, and temperature is increased. Always valid $\nabla_{ad} \le \nabla_i < \nabla_a < \nabla_{Rad}$
  - bubble cools, sinks down, absorbs energy, rises, etc...

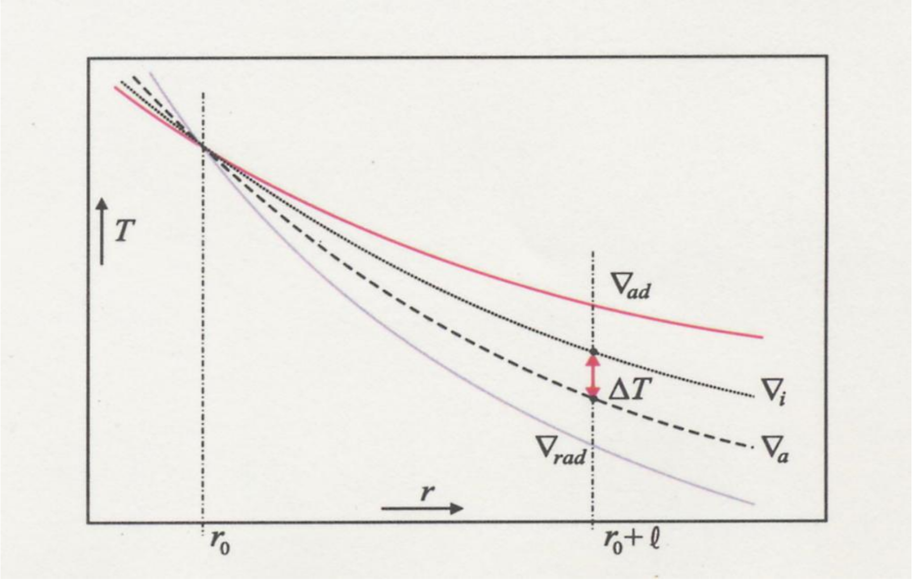
(continued on next page)

⇒ Energy is transported, temperature gradient becomes smaller

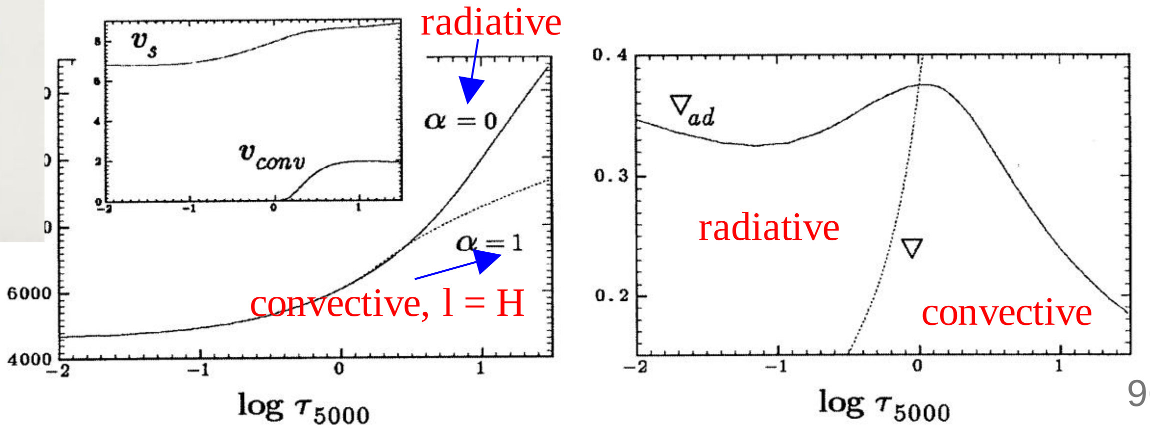
- Flux, temperature etc. calculated from simple arguments, $\ell = \alpha \cdot H$, $\alpha = 1, ...2$
- have to account for radiative losses during lifetime of element until energy is released
  ⇒ efficiency $\gamma = \frac{\text{excess energy lost}}{\text{radiative losses}}$
- γ large $\longrightarrow \nabla_a \approx \nabla_{ad}$; γ small $\longrightarrow \nabla_a \approx \nabla_{Rad}$

Note:
- mixing length theory only 0th order approach
- modern approach: calculate consistent hydrodynamic solution (e.g., solar convective layer+photosphere, Asplund and co-workers)

radiative vs. adiabatic T-stratification



Model for solar photosphere



Mixing length theory – some details

$\Delta E = \rho C_p \delta T$ is excess energy density delivered to ambient medium when bubble merges with surroundings.

C_p is specific heat per mass.

$\Rightarrow F_{conv} = \Delta E \bar{v} = C_p \delta T \rho \bar{v}$ is convective flux (transported energy) with $\bar{v}$ average velocity of rising bubble over distance Δr ($\rho \bar{v}$ mass flux).

δT is temperature difference between bubble and ambient medium.

$$\delta T = \left[\left(-\left.\frac{dT}{dr}\right|_a\right) - \left(-\left.\frac{dT}{dr}\right|_i\right)\right]\Delta r \ > 0$$ when convective instable,

since then $\left[(-\Delta T)_a - (-\Delta T)_i\right] > 0$

From the definiton of ∇,

$-\frac{dT}{dr} = \frac{T}{H}\nabla$, with pressure scale height H (see problem set 8),

assuming hydrostatic equilibrium and neglecting radiation pressure;
(inclusion of p_{rad} possible, of course)

Defining l as the **mixing length** after which element dissolves, and averaging over all elements (distributed randomly over their paths), we may write $\Delta r = \frac{l}{2}$.

$$\Rightarrow F_{conv} = C_p \rho \bar{v} (\nabla_a - \nabla_i) \frac{T}{H}\frac{l}{2} = \boxed{\frac{1}{2} C_p \rho \bar{v} T (\nabla_a - \nabla_i)\alpha,}$$ with

mixing length parameter $\alpha = \frac{l}{H}$ (from fits to observations, $\alpha = O(1)$)

The average velocity is calculated by assuming that the work done by the buoyant force is (partly) converted to kinetic energy, where the average of this work might be calculated via

$$\bar{w} = \int_0^{l/2} F_b(\Delta r) d(\Delta r),$$

and the upper limit results from averaging over elements passing the point under consideration. The buoyant force is given by (see page 93)

$$F_b = -g\delta\rho = -g(\rho_i - \rho_a) > 0$$

Using the equation of state, and accounting for pressure equilibrium ($p_i = p_a$),

we find $\frac{\delta\rho}{\rho} = -Q\frac{\delta T}{T}$ with $Q = \left(1 - \left.\frac{\partial \ln \mu}{\partial \ln T}\right|_p\right)$, to account for ionization effects.

$$\Rightarrow F_b = -g\delta\rho = gQ\frac{\rho}{T}\delta T = gQ\frac{\rho}{T}\left[\left(-\left.\frac{dT}{dr}\right|_a\right) - \left(-\left.\frac{dT}{dr}\right|_i\right)\right]\Delta r =$$

$gQ\frac{\rho}{H}(\nabla_a - \nabla_i)\Delta r := A\Delta r$. Thus, F_b is linear in Δr, and

$$\bar{w} = \int_0^{l/2} A\Delta r d(\Delta r) = A\frac{l^2}{8} = gQ\rho\frac{H}{8}(\nabla_a - \nabla_i)\left(\frac{l}{H}\right)^2$$

Mixing length theory – some details

Let's assume now that 50% of the work is lost to friction (pushing aside the turbulent elements), and 50% is converted into kinetic energy of the bubbles, i.e.,

$$\frac{1}{2}\bar{w} = \frac{1}{2}\rho\bar{v}^2 \quad \Rightarrow \quad \boxed{\bar{v} = \left(\frac{\bar{w}}{\rho}\right)^{1/2} = \left(\frac{gQH}{8}\right)^{1/2} (\nabla_a - \nabla_i)^{1/2}\,\alpha,}$$

and the convective flux is finally given by

$$\boxed{F_{conv} = \left(\frac{gQH}{32}\right)^{1/2} \left(\rho C_p T\right)(\nabla_a - \nabla_i)^{3/2}\,\alpha^2.}$$

NOTE : different averaging factors possible and actually found in different versions!

Remember that still $\nabla_{ad} \le \nabla_i < \nabla_a < \nabla_{rad}$.

The gradients ∇_i and ∇_a are calculated from the efficiency γ and the condition that the *total* flux remains conserved (outside the nuclear energy creating core), i.e.,

$$r^2(F_{conv} + F_{rad}) = r^2 F_{tot} = R_*^2 F_{rad}(R_*) = R_*^2 \sigma_B T_{eff}^4 = \frac{L}{4\pi}$$

or from the condition that

$(F_{conv} + F_{rad}) = \dfrac{L_r}{4\pi r^2}$ with L_r the luminosity at r.

Usually, a tricky iteration cycle is necessary. An example for a simple case will be discussed in problem set 8.

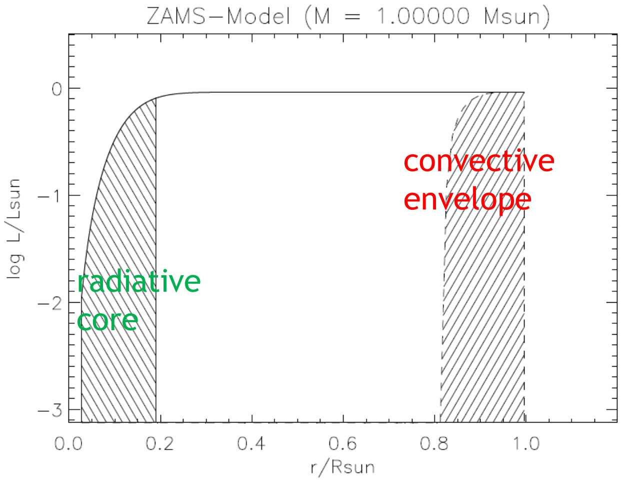
Convective vs. radiative energy transport

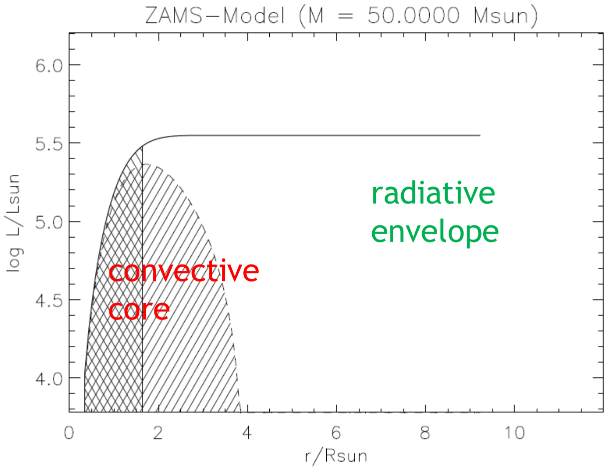
- major difference in internal structure at MS – convective vs. radiative energy transport:
  - if T-stratification shallow (compared to adiabatic gradient) → radiative energy transport;
  - else convective energy transport
- cool (low-mass stars) during MS:
  - interior: p-p chain, shallow dT/dr → radiative core
  - outer layers: H/He recombines → large opacities → steep dT/dr, low adiabatic gradient → convective envelope
- hot (massive) stars during MS:
  - interior: CNO cycle, steep dT/dr → convective core
  - outer layers: H/He ionized → low opacities → shallow dT/dr, large adiabatic gradient → radiative envelope

Note: (i) transition from p-p chain to CNO cycle around 1.3 to 1.4 M_{sun} at ZAMS

(ii) most massive stars have a sub-surface convection zone due to iron opacity peak

(iii) evolved objects (red giants and supergiants) and brown dwarfs **are fully convective**





Chap. 7 Microscopic theory

Absorption- and emission coefficients

- can calculate now a lot, if absorption- and emission-coefficients given, e.g.

$$\chi_\nu \sim \sigma_{lu} \bullet \phi(\nu) \bullet n_l(r)$$

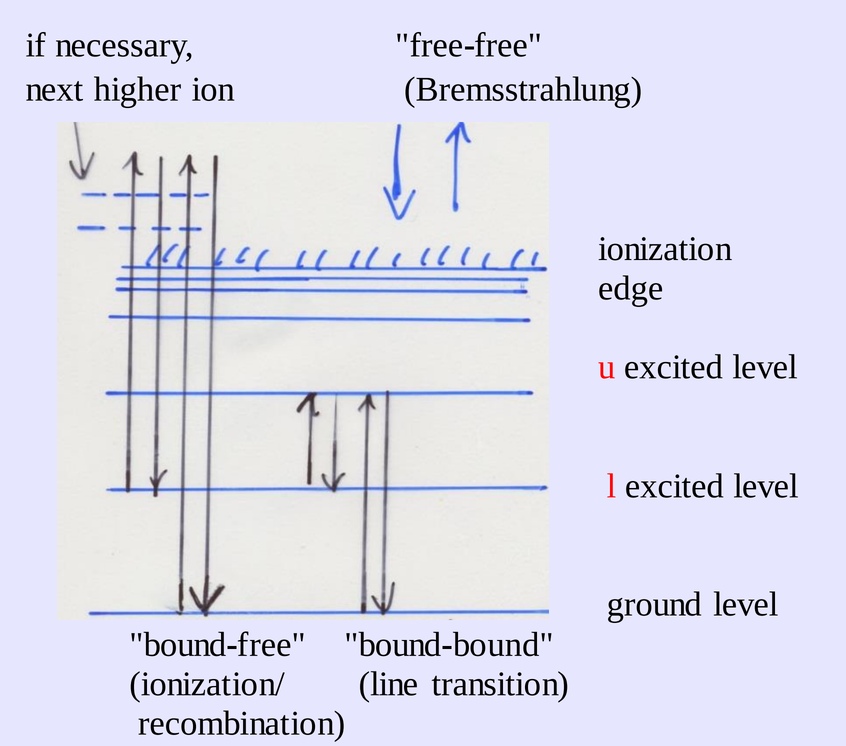
cross-section (quantum mech.)

profile function (normalized)

occupation numbers

LTE Saha-Boltzmann

NLTE detailed calculation



absorption

$h\nu + E_l \rightarrow E_u$

emission

- spontaneous ← isotropic

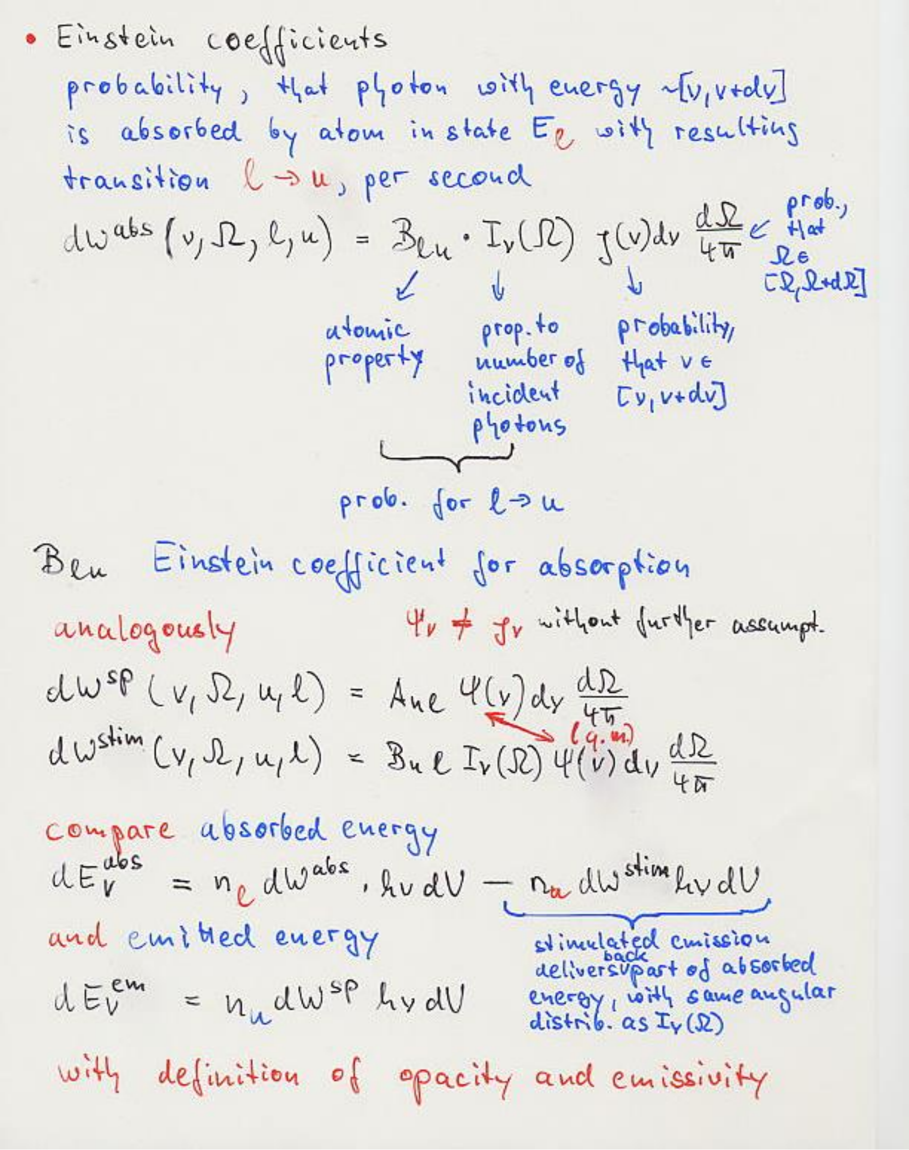
  $E_u \rightarrow E_l + h\nu$

- stimulated

  $h\nu + E_u \rightarrow E_l + 2\,h\nu$

same angular distribution

Line transitions



- Einstein coefficients

probability, that photon with energy $\sim[\nu,\nu+d\nu]$ is absorbed by atom in state E_l with resulting transition $l \to u$, per second

$$dW^{abs}(\nu,\Omega,l,u) = \underbrace{\underset{\text{atomic property}}{B_{lu}} \cdot \underset{\text{prop. to number of incident photons}}{I_\nu(\Omega)}}_{\text{prob. for } l \to u} \; \underset{\text{probability, that } \nu \in [\nu,\nu+d\nu]}{\varphi(\nu)d\nu} \; \frac{d\Omega}{4\pi} \leftarrow \text{prob., that } \Omega \in [\Omega,\Omega+d\Omega]$$

B_{lu} Einstein coefficient for absorption

analogously $\qquad \psi_\nu \neq \varphi_\nu$ without further assumpt.

$$dW^{sp}(\nu,\Omega,u,l) = A_{ul}\,\psi(\nu)d\nu\,\frac{d\Omega}{4\pi}$$

$$dW^{stim}(\nu,\Omega,u,l) = B_{ul}\,I_\nu(\Omega)\,\psi(\nu)d\nu\,\frac{d\Omega}{4\pi}$$

(q.m.)

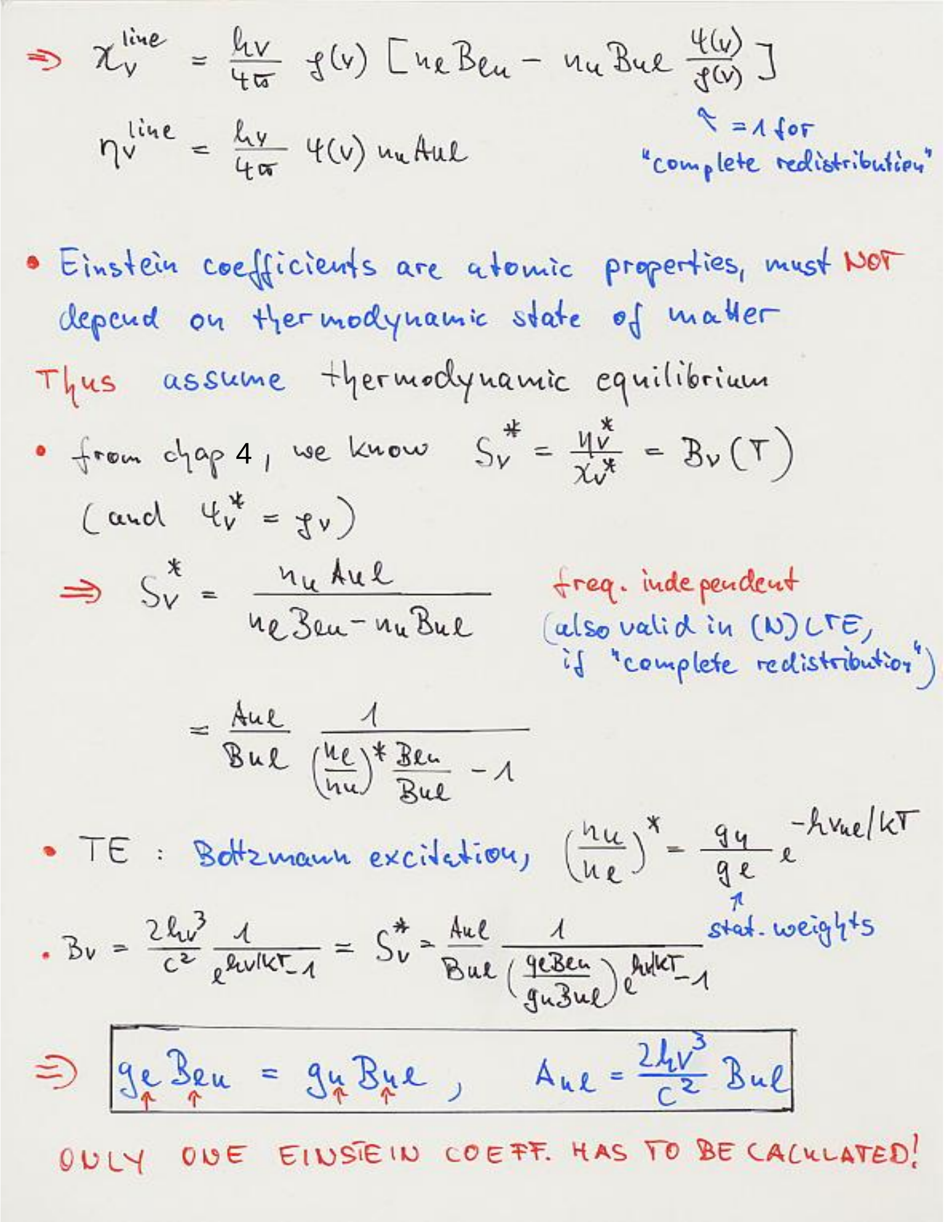
compare absorbed energy

$$dE_\nu^{abs} = n_l\,dW^{abs}\cdot h\nu\,dV - \underbrace{n_u\,dW^{stim}h\nu\,dV}_{\text{stimulated emission delivers back part of absorbed energy, with same angular distrib. as } I_\nu(\Omega)}$$

and emitted energy

$$dE_\nu^{em} = n_u\,dW^{sp}\,h\nu\,dV$$

with definition of opacity and emissivity



$$\Rightarrow \chi_\nu^{line} = \frac{h\nu}{4\pi}\varphi(\nu)\left[n_l B_{lu} - n_u B_{ul}\frac{\psi(\nu)}{\varphi(\nu)}\right]$$

$$\eta_\nu^{line} = \frac{h\nu}{4\pi}\psi(\nu)\,n_u A_{ul}$$

$\uparrow = 1$ for "complete redistribution"

- Einstein coefficients are atomic properties, must NOT depend on thermodynamic state of matter

Thus assume thermodynamic equilibrium

- from chap 4, we know $S_\nu^* = \frac{\eta_\nu^*}{\chi_\nu^*} = B_\nu(T)$

(and $\psi_\nu^* = \varphi_\nu$)

$$\Rightarrow S_\nu^* = \frac{n_u A_{ul}}{n_l B_{lu} - n_u B_{ul}}$$

freq. independent (also valid in (N)LTE, if "complete redistribution")

$$= \frac{A_{ul}}{B_{ul}} \frac{1}{\left(\frac{n_l}{n_u}\right)^* \frac{B_{lu}}{B_{ul}} - 1}$$

- TE : Boltzmann excitation, $\left(\frac{n_u}{n_l}\right)^* = \frac{g_u}{g_l} e^{-h\nu_{ul}/kT}$

$\uparrow$ stat. weights

- $B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} = S_\nu^* = \frac{A_{ul}}{B_{ul}} \frac{1}{\left(\frac{g_l B_{lu}}{g_u B_{ul}}\right)e^{h\nu/kT}-1}$

$$\Rightarrow \boxed{\underset{\uparrow}{g_l}\,\underset{\uparrow}{B_{lu}} = \underset{\uparrow}{g_u}\,\underset{\uparrow}{B_{ul}}, \qquad A_{ul} = \frac{2h\nu^3}{c^2}B_{ul}}$$

ONLY ONE EINSTEIN COEFF. HAS TO BE CACULATED!

- has to be calculated from quantum mechanics (from "dipoloperator")
- result

$$\frac{h\nu}{4\pi} B_{\ell u} = \frac{\pi e^2}{m_e c} f_{\ell u}$$

f "oscillator strength", dimensionless

↑ classical result, from electrodynamics

"strong" transitions have $f \approx 0.1 \dots 10$

and "selection rules", e.g. $\Delta \ell = \pm 1$

"forbidden transitions": magnetic dipole, electr. quadrupol: f very low, 10^{-5} and lower

- THUS

$$\chi_\nu = \frac{\pi e^2}{m_e c} f_{\ell u} \left(n_\ell - \frac{g_\ell}{g_u} \cdot n_u \right) \cdot \varphi_\nu$$

$$= \frac{\pi e^2}{m_e c} (gf)_{\ell u} \cdot \left(\frac{n_\ell}{g_\ell} - \frac{n_u}{g_u} \right) \cdot \varphi_\nu$$

↑ "gf-value" $= g_\ell \cdot f_{\ell u}$

with $\int_{-\infty}^{\infty} \varphi(\nu) d\nu = 1$

$$\frac{\pi e^2}{m_e c} \approx 0.02654 \frac{cm^2}{s}$$

Profile function?

Line broadening

1. Radiation damping ("natural" line broadening)

- QED effect
- heuristic   finite life time with respect to spontaneous emission

$\tau = \frac{1}{A_{ul}}$ (e.g., 10^{-8}s for H 2 ⇒ 1)

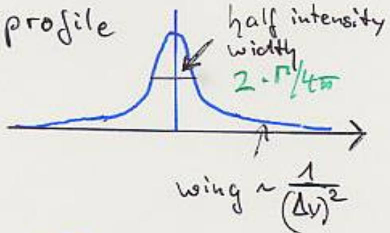
and uncertainty principle

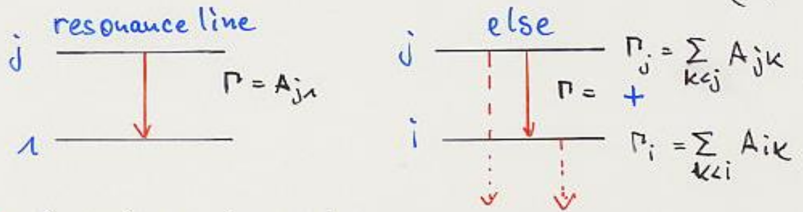
$\Delta E \cdot \tau \gtrsim \hbar$

⇒ broadening (classical theory: damping by radiation)

→ dispersion (Lorentzian) profile

$$\phi(\nu) = \frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2}$$



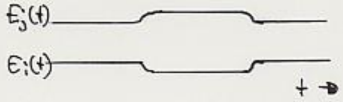


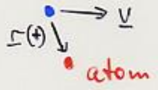
- of primary importance for strong lines (res. lines) in low density environment (no other broadening mechanisms), e.g. L_α in interstellar medium

2. Collisional broadening

- radiating atoms perturbed by passing particles
- brief perturbation, close perturbers

"impact theory"





$\Delta E(t) \sim \frac{1}{r^n(t)}$

n=2   linear Stark effect

for levels with degenerate angular momentum, e.g., H I, He II

$\Delta E \sim F = \frac{q}{r^2}$

↑ field strength

very important, if many electrons:

photospheres of hot stars, $n_e \gtrsim 10^{12}$ cm^{-3}

n = 3   resonance broadening

atom A is perturbed by atom A' of same species in "cool" stars, e.g. Balmer lines in sun

n = 4   quadratic Stark effect

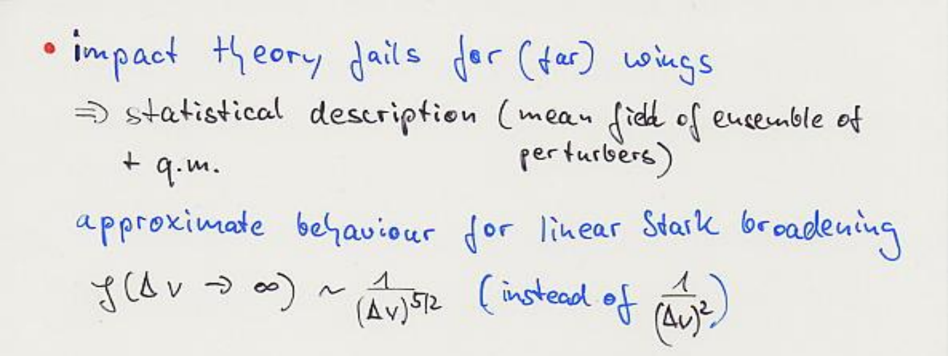
metal ions in photospheres of hot stars

$\Delta E \sim F^2$

n = 6   van der Vaals broadening

atom A perturbed by atom B in cool stars, e.g. Na perturbed by H in sun

resulting profiles are dispersion profiles!

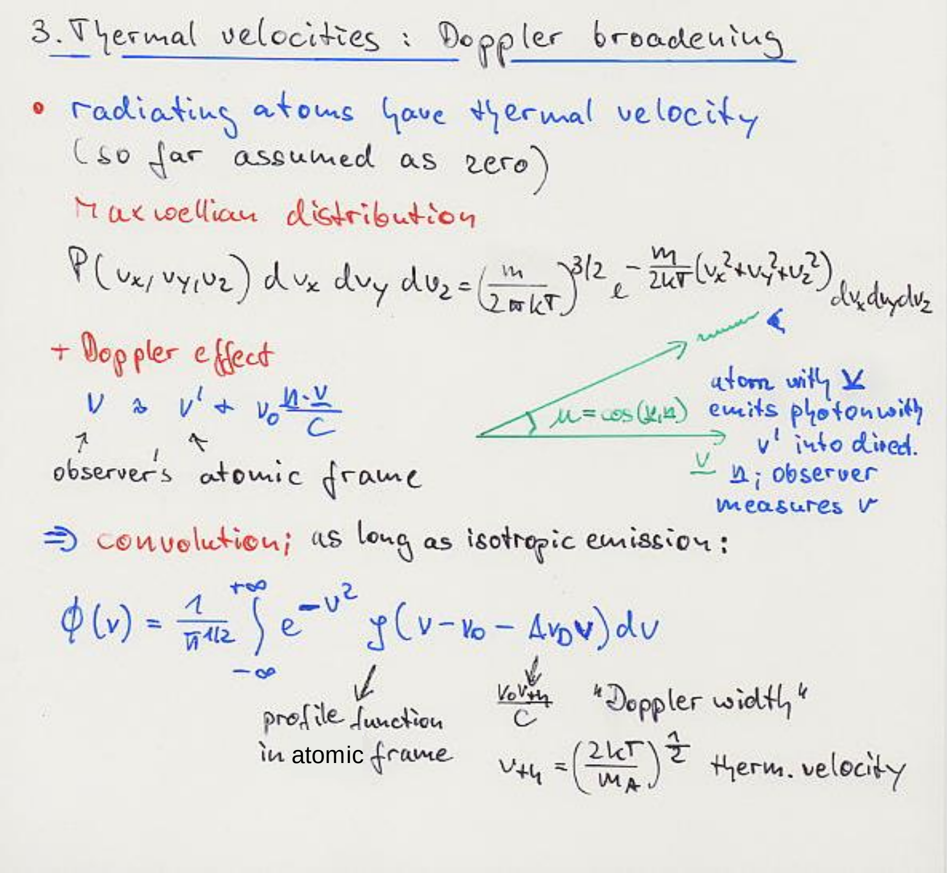


- impact theory fails for (far) wings

$\Rightarrow$ statistical description (mean field of ensemble of perturbers)
+ q.m.

approximate behaviour for linear Stark broadening

$\varphi(\Delta\nu \rightarrow \infty) \sim \frac{1}{(\Delta\nu)^{5/2}}$ (instead of $\frac{1}{(\Delta\nu)^2}$)



3. Thermal velocities : Doppler broadening

- radiating atoms have thermal velocity (so far assumed as zero)

Maxwellian distribution

$$P(v_x, v_y, v_z)\, dv_x\, dv_y\, dv_z = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}(v_x^2+v_y^2+v_z^2)} dv_x dv_y dv_z$$

+ Doppler effect

$$\nu \approx \nu' + \nu_0 \frac{\underline{n}\cdot\underline{v}}{c}$$

↑ observer's ↑ atomic frame

$\mu = \cos(\underline{v}, \underline{n})$

$\underline{v}$

atom with $\underline{v}$ emits photon with ν' into direct. $\underline{n}$; observer measures ν

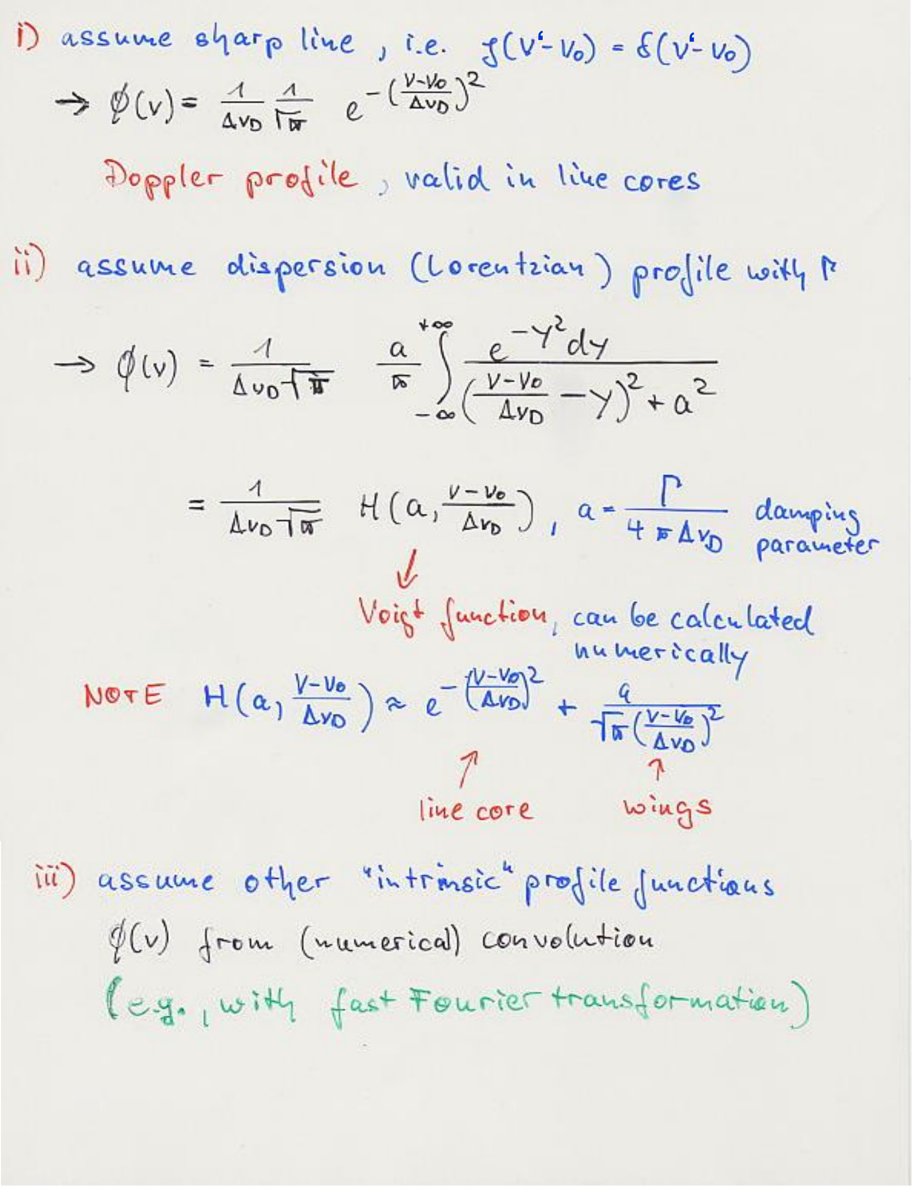
$\Rightarrow$ convolution; as long as isotropic emission:

$$\phi(\nu) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{+\infty} e^{-v^2} \varphi(\nu - \nu_0 - \Delta\nu_D v)\, dv$$

profile function in atomic frame

$\frac{\nu_0 v_{th}}{c}$ "Doppler width"

$v_{th} = \left(\frac{2kT}{m_A}\right)^{\frac{1}{2}}$ therm. velocity



i) assume sharp line, i.e. $\xi(\nu'-\nu_0) = \delta(\nu'-\nu_0)$

$$\rightarrow \phi(\nu) = \frac{1}{\Delta\nu_D}\frac{1}{\sqrt{\pi}}\; e^{-\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2}$$

Doppler profile, valid in line cores

ii) assume dispersion (Lorentzian) profile with Γ

$$\rightarrow \phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}}\;\frac{a}{\pi}\int_{-\infty}^{+\infty}\frac{e^{-y^2}dy}{\left(\frac{\nu-\nu_0}{\Delta\nu_D}-y\right)^2+a^2}$$

$$= \frac{1}{\Delta\nu_D\sqrt{\pi}}\; H\left(a,\frac{\nu-\nu_0}{\Delta\nu_D}\right),\quad a = \frac{\Gamma}{4\pi\Delta\nu_D}$$ damping parameter

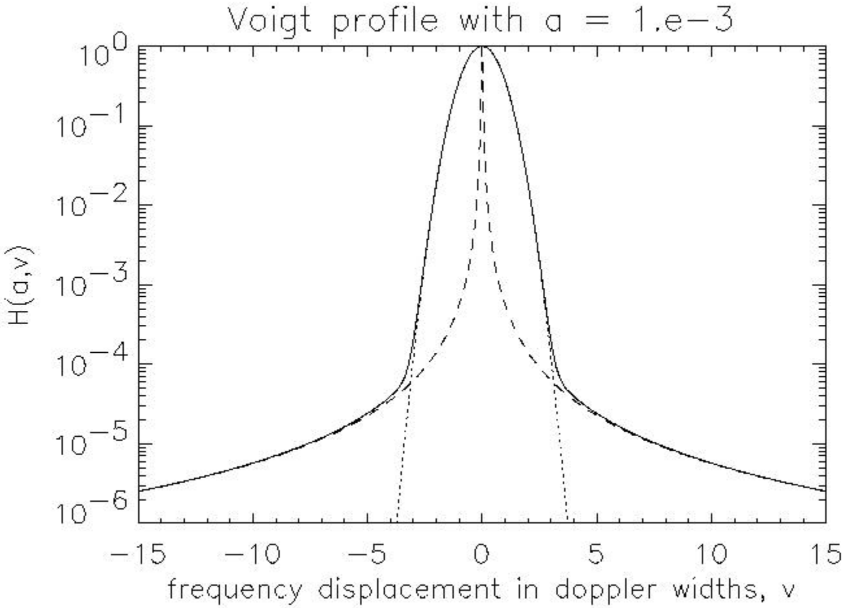
↙ Voigt function, can be calculated numerically

NOTE $H\left(a,\frac{\nu-\nu_0}{\Delta\nu_D}\right) \approx \underbrace{e^{-\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2}}_{\text{line core}} + \underbrace{\frac{a}{\sqrt{\pi}\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2}}_{\text{wings}}$

iii) assume other "intrinsic" profile functions

$\phi(\nu)$ from (numerical) convolution

(e.g., with fast Fourier transformation)



fully drawn: Voigt profile H(a,v)
dotted : exp($-v^2$), Doppler profile (core)
dashed: a / ($\sqrt{\pi}\,v^2$), dispersion profile (wings)

Curve of growth method

Theoretical curve of growth

- standard diagnostic tool to determine metal abundances in cool stars in a simple way
- assumptions
  pure absorption line
  Milne Eddington model, LTE, $\varepsilon_\nu = 1$ (no scattering)

$$\chi_\nu = \chi_c + \underbrace{\bar{\chi}_L \phi_\nu}_{\chi_\nu^{Line}} = \chi_c(1+\beta_\nu), \quad \underset{\substack{\uparrow \\ \text{depthindependent}}}{\beta_\nu} = \frac{\bar{\chi}_L}{\chi_c}\phi_\nu$$

$B_\nu(\tau) = a + b\,\tau_c$ defined on continuum scale

$$= a + b\frac{\chi_c}{\chi_\nu}\tau_\nu = a + b\underbrace{\frac{1}{1+\beta_\nu}}_{\hat{=}\, b_\nu \text{ in Milne-Edd. model}}\tau_\nu$$

(result of advanced reading)

- From Milne Edd. model we have

$$H_\nu^{Line}(0),\ \varepsilon_\nu = 1 = \frac{1}{\sqrt{3}} J_\nu(0) = \frac{1}{\sqrt{3}}\left(a + \frac{\frac{1}{1+\beta_\nu}\frac{b}{\sqrt{3}} - a}{2}\right)$$

$$H_\nu^{cont}(0),\ \varepsilon_\nu = 1 = (\beta_\nu = 0) = \frac{1}{\sqrt{3}}\left(a + \frac{b/\sqrt{3} - a}{2}\right)$$

$\Rightarrow$ residual intensity ("line profile"

$$R_\nu = \frac{H_\nu^{Line}}{H_\nu^{cont}} = \frac{b\frac{1}{1+\beta_\nu} + \sqrt{3}a}{b + \sqrt{3}\,a}$$

$$\beta_\nu = \frac{\pi e^2}{m_e c} f_{lu}\frac{n_e}{\chi_c}\left(1 - e^{-h\nu/kT}\right)\phi(\nu) = \beta_0\phi(\nu)$$

$\swarrow$ Voigt profile!

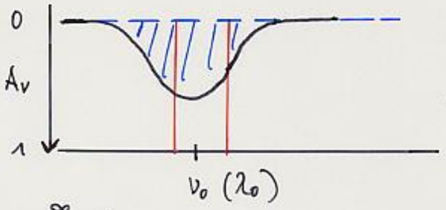
line depth $A_\nu = 1 - R_\nu$

$$= \frac{\beta_0\phi_\nu}{1+\beta_0\phi_\nu}\underbrace{\left(\frac{b}{b+\sqrt{3}\,a}\right)}_{}$$

A_0 central depth of line with $\beta_0 \to \infty$

$$A_\nu = A_0\beta_0\frac{\phi_\nu}{1+\beta_0\phi_\nu}$$

equivalent width $W_\nu = \int_0^\infty A_\nu d\nu$ area below continuum (see also p. 8.3)



W_λ width of line in Å, if line would have depth "1"

$$\Rightarrow W_\nu = A_0\beta_0\int_0^\infty \frac{\phi_\nu}{1+\beta_0\phi_\nu}d\nu$$

$$W_\lambda = \int_0^\infty A(\lambda)d\lambda \approx \left(\int_0^\infty A_\nu d\nu\right)\frac{\lambda_0^2}{c} \qquad W_\lambda = \frac{\lambda_0^2}{c}\cdot W_\nu$$

with Voigt profile H (Doppler core + Lorentz wings)

$$W_\nu = A_0\beta_0\frac{1}{\sqrt{\pi}\Delta\nu_D}\int_0^\infty \frac{H\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)d\nu}{1 + \frac{\beta_0}{\sqrt{\pi}\Delta\nu_D}H\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)}$$

$v = \frac{\nu-\nu_0}{\Delta\nu_D}$

$d\nu = dv\,\Delta\nu_D$

$= \ldots$

$$W_\nu = \frac{A_0\beta_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{H(v)\,dv}{1+\frac{\beta_0}{\sqrt{\pi}\Delta\nu_D} H(v)}$$

3 regimes

a) linear regime: Doppler core not saturated,

$$H(a,v) = e^{-v^2}$$

$$\Rightarrow W_\nu \approx \frac{A_0\beta_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-v^2}\,dv}{1+\frac{\beta_0}{\sqrt{\pi}\Delta\nu_D} e^{-v^2}}$$

$$\rightarrow (\beta_0/\Delta\nu_D < 1) \quad \frac{A_0\beta_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-v^2}\left(1-\frac{\beta_0}{\Delta\nu_D\sqrt{\pi}} e^{-v^2}+\dots\right)dv$$

$\approx A_0\beta_0 \sim \beta_0$, independent on $\Delta\nu_D$

b) saturation part: line reaches maximum depth ($=A_0$), however wings still unimportant

as above, i.e. $\phi_\nu \sim e^{-v^2}$, however $\beta_0/\Delta\nu_D > 1$

$\Rightarrow$ (integration tricky)

$$W_\nu = 2 A_0 \Delta\nu_D \sqrt{\ln\beta^*} \left(1 - (\pi^2/24(\ln\beta^*)^2 - \dots\right)$$

with $\beta^* = \beta_0/\sqrt{\pi}\Delta\nu_D$

flat growth with $\sqrt{\ln\beta^*}$, $W_\nu \sim \Delta\nu_D$

c) damping (square-root) part

line wings dominate equivalent width

$$\Rightarrow W_\nu \approx \frac{A_0\beta_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{a/(\sqrt{\pi}v^2)\,dv}{1+\frac{\beta_0}{\sqrt{\pi}\Delta\nu_D}\frac{a}{\sqrt{\pi}v^2}}$$

a damping parameter

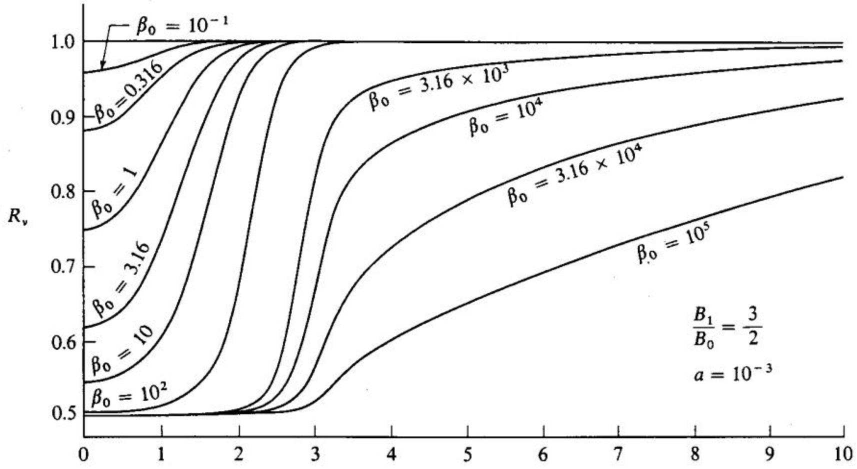
$$= \frac{A_0\beta_0}{\pi} a \int_{-\infty}^{+\infty} \frac{dv}{v^2+\frac{\beta_0 a}{\pi\Delta\nu_D}}$$

$$= A_0 (a\pi\Delta\nu_D\beta_0)^{\frac{1}{2}}$$

(attention: typo in Mihalas)

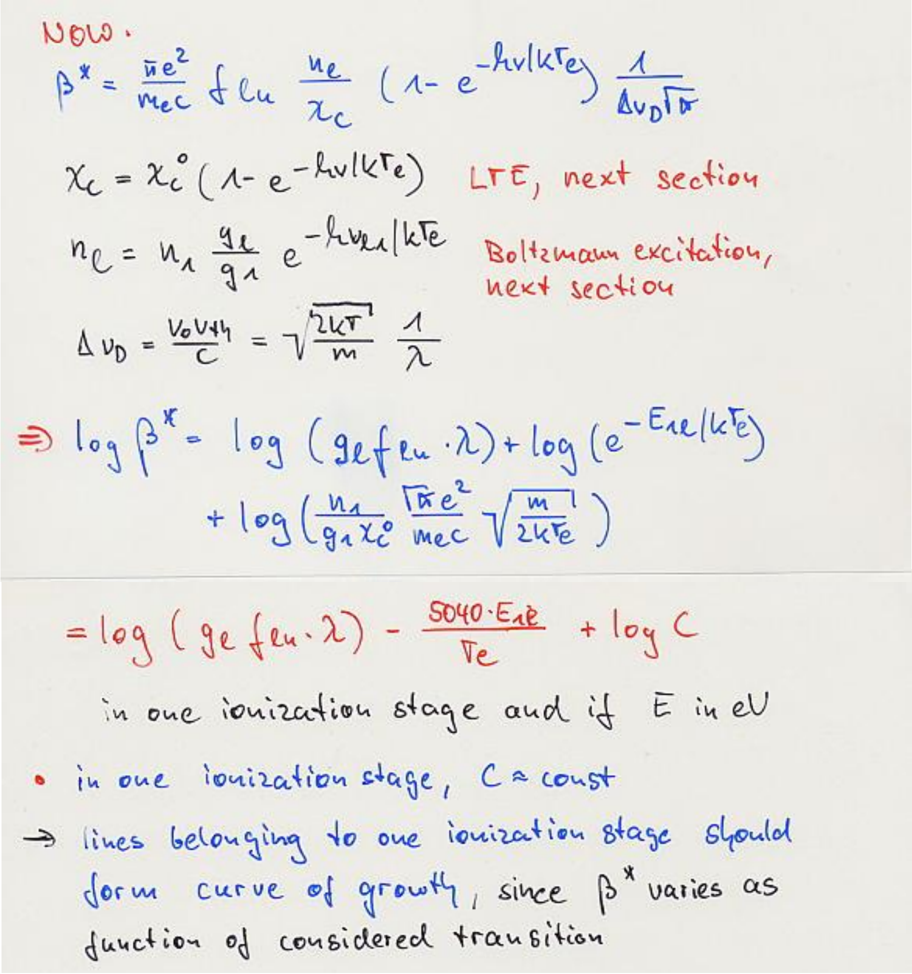
growth with $\beta_0^{\frac{1}{2}}$

in total, we have $W_\nu = f(\beta_0)$ or $f\left(\frac{\beta_0}{\Delta\nu_D\sqrt{\pi}}\right) = f(\beta^*)$



Voigt profile with $A_O = 0.5$, $\beta_o =: \beta^*$

Development of a spectrum line with increasing number of atoms along the line of sight. The line is assumed to be formed in pure absorption. For $\beta_0 \lesssim 1$, the line strength is directly proportional to the number of absorbers. For $30 \lesssim \beta_0 \lesssim 10^3$ the line is saturated, but the wings have not yet begun to develop. For $\beta_0 \gtrsim 10^4$ the line wings are strong and contribute most of the equivalent width.



Now:

$$\beta^* = \frac{\pi e^2}{m_e c} f_{lu} \frac{n_l}{\chi_c} \left(1 - e^{-h\nu/kT_e}\right) \frac{1}{\Delta\nu_D \sqrt{\pi}}$$

$$\chi_c = \chi_c^0 \left(1 - e^{-h\nu/kT_e}\right)$$ LTE, next section

$$n_l = n_1 \frac{g_l}{g_1} e^{-h\nu_{1l}/kT_e}$$ Boltzmann excitation, next section

$$\Delta\nu_D = \frac{\nu_0 v_{th}}{c} = \sqrt{\frac{2kT}{m}} \frac{1}{\lambda}$$

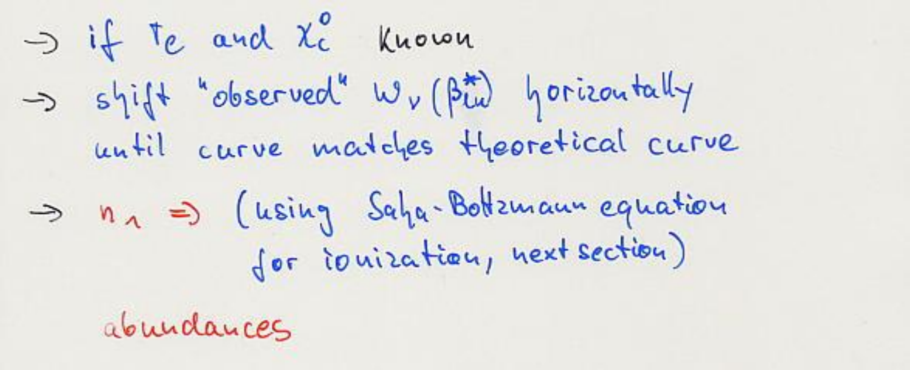
$$\Rightarrow \log \beta^* = \log(g_l f_{lu} \cdot \lambda) + \log\left(e^{-E_{1l}/kT_e}\right) + \log\left(\frac{n_1}{g_1 \chi_c^0} \frac{\sqrt{\pi} e^2}{m_e c} \sqrt{\frac{m}{2kT_e}}\right)$$

$$= \log(g_l f_{lu} \cdot \lambda) - \frac{5040 \cdot E_{1l}}{T_e} + \log C$$

in one ionization stage and if E in eV

- in one ionization stage, $C \approx$ const

→ lines belonging to one ionization stage should form curve of growth, since β^* varies as function of considered transition



→ if T_e and χ_c^0 known

→ shift "observed" $W_\nu(\beta^*_{lu})$ horizontally until curve matches theoretical curve

→ n_1 ⇒ (using Saha-Boltzmann equation for ionization, next section)

abundances

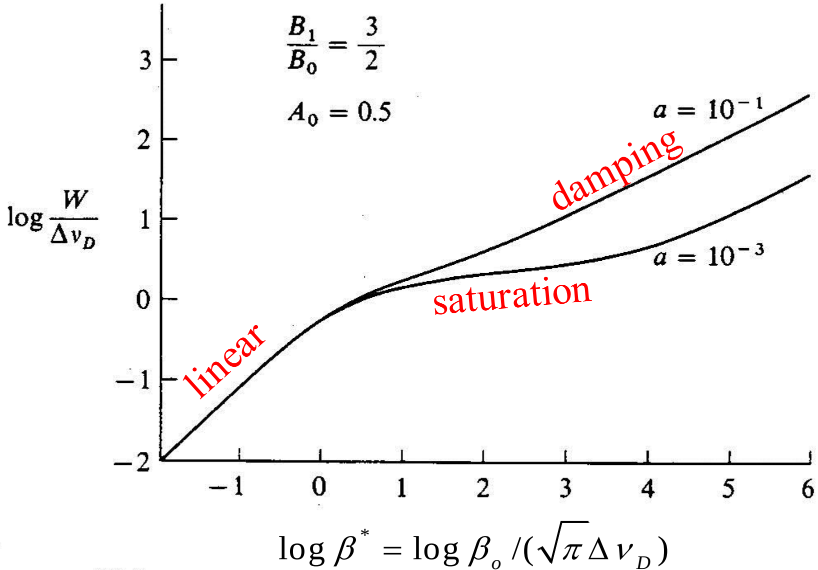
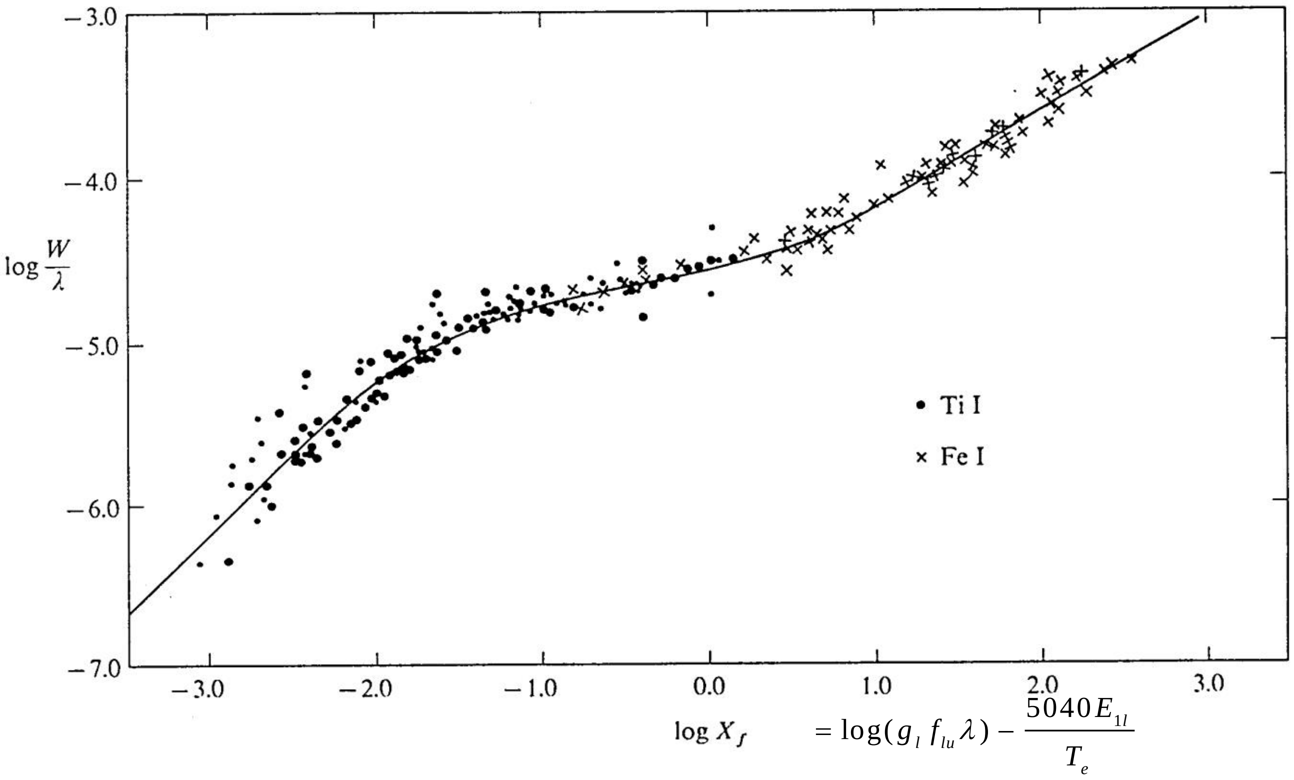


FIGURE 10-2
Curves of growth for pure absorption lines. Note that the larger the value of a, the sooner the square-root part of the curve rises away from the flat part.

measure W(λ) for different lines (with different strengths) of one ionization stage

plot as function of $\log(g_l f_{lu} \lambda) - \frac{5040 E_{1l}}{T_e} + \log C$, with "C" fit-quantity

shift horizontally until *theoretical curve of growth* W(β^*) is matched => log C => $\frac{n_1}{\chi_c^0}$ => n_1

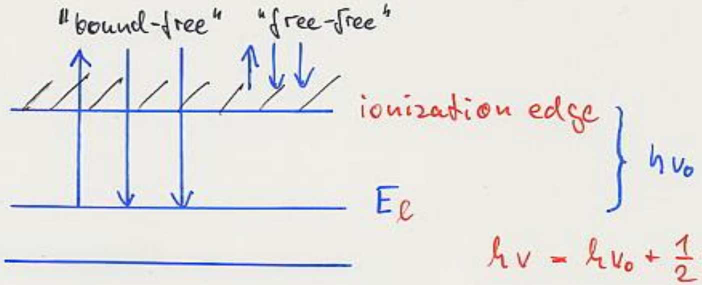


Empirical curve of growth for solar Fe I and Ti I lines. Abscissa is based on laboratory *f*-values. From **(686)**.

Ti I lines shifted horizontally to define a unique relation

Continous processes

Continous absorption / emission and scattering



$h\nu = h\nu_0 + \frac{1}{2} m_e v^2$

- bound free processes

"one" transition : $\chi_\nu^{bf} = n_\ell \sigma_{\ell k}(\nu)$, $\nu > \nu_0$

↑ absorption cross section ↑ threshold

in total : many processes at one frequency

$$\chi_\nu^{bf} = \sum_{\text{elements}} \sum_{\text{ions}} \sum_{\ell} n_\ell \sigma_{\ell k}(\nu)$$

hydrogenic ions $\sigma_{\ell k}(\nu) = \sigma_0(\ell) \left(\frac{\nu_0}{\nu}\right)^3 \cdot g_{bf}(\nu)$

↑ "gaunt-factor", ≈ 1

EINSTEIN-MILNE relations

$$\chi_\nu^{bf} = \sum_{\substack{\text{elements,}\\ \text{ions}}} \sum_{\ell} \sigma_{\ell k}(\nu) \left(n_\ell - n_\ell^* e^{-h\nu/kT} \right)$$

↑ stim. emission

$$\eta_\nu^{bf} = \sum \sum_{\ell} \sigma_{\ell k}(\nu) \frac{2h\nu^3}{c^2} n_\ell^* e^{-h\nu/kT}$$

↑ spontaneous emission

$n_\ell^* =$ LTE value

NOTE : $n_\ell = n_\ell^* \Rightarrow S_\nu^{bf} = \frac{\eta_\nu^{bf}}{\chi_\nu^{bf}} = B_\nu(T)$!

free-free processes

(emission process: "bremsstrahlung", decelerated charges radiate!)

$$\chi_\nu^{ff} = n_e n_k^{ion} \sigma_{kk}(\nu) \left(1 - e^{-h\nu/kT}\right)$$

↑ stim. emission

$\sigma_{kk} \sim \frac{\lambda^3}{\sqrt{T}}$, important in IR and radio!

$$\eta_\nu^{ff} = n_e n_k^{ion} \sigma_{kk}(\nu) \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

NOTE $S_\nu^{ff} = B_\nu(T)$ always!

Scattering

1. electron scattering

- important for hot stars
- difference to f-f processes

f-f: photon interacts with e^- in ion's central field
$\Rightarrow$ absorption $\Rightarrow$ photon destruction, i.e. "true" process

scattering: without influence of central field, i.e., no "third" partner in collisional process
$\Rightarrow$ no absorption possible, since energy and momentum conservation cannot be fulfilled simultaneously
$\Rightarrow$ scattering

- very high energies (many MeVs)
  Klein Nishina (Q.E.D.)
- high energies
  Compton / inverse Compton scattering
  $\downarrow$ $\downarrow$
  e^- has low / has high kinetical energy
- low energies (< 12.4 keV ≙ 1 Å)
  Thomson scattering

classical e^- radius

$$\sigma^{TH} = n_e \sigma_T \, ; \quad \sigma_T = \sigma_{class} = \frac{8\pi}{3} \overset{\downarrow}{r_0^2} = \frac{8\pi}{3} \frac{e^4}{m_e^2 c^4}$$

$$= 6.65 \cdot 10^{-25} \text{cm}^2$$

2. Rayleigh-scattering

actually: line absorption / emission of atoms / molecules far from resonance frequency

$\Rightarrow$ from q.m., Lorentzprofile with $|\nu - \nu_0| \gg \nu_0$

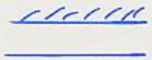
$$\sigma(\nu) = f_{lu} \sigma_T \cdot \left(\frac{\nu}{\nu_0}\right)^4 \sim \lambda^{-4}$$ for $\nu << \nu_0$

- if line transition strong, λ^{-4} decrease of far wings can be of major importance

example: Ly-α in cool stars, Rayleigh wings are visible in optical!

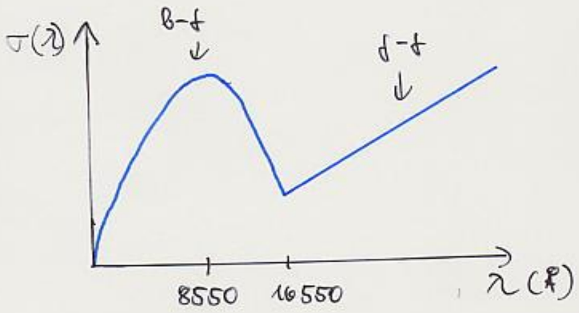
The H$^-$ ion

- for cool stars (e.g., the sun), one bound state of H$^-$ ($1p + 2e^-$)



} 0.75 eV ≙ 16550 Å

- dominant bf-opacity (also ff component)
- only by inclusion of H$^-$ (Pannekoek + Wildt, 1939) the solar continuum could be explained



Total opacities and emissivities

$$\chi_\nu^{tot} = \chi^{Line} \phi(\nu) + \Sigma \chi_\nu^{bf} + \Sigma \chi_\nu^{ff} + n_e \sigma_T$$

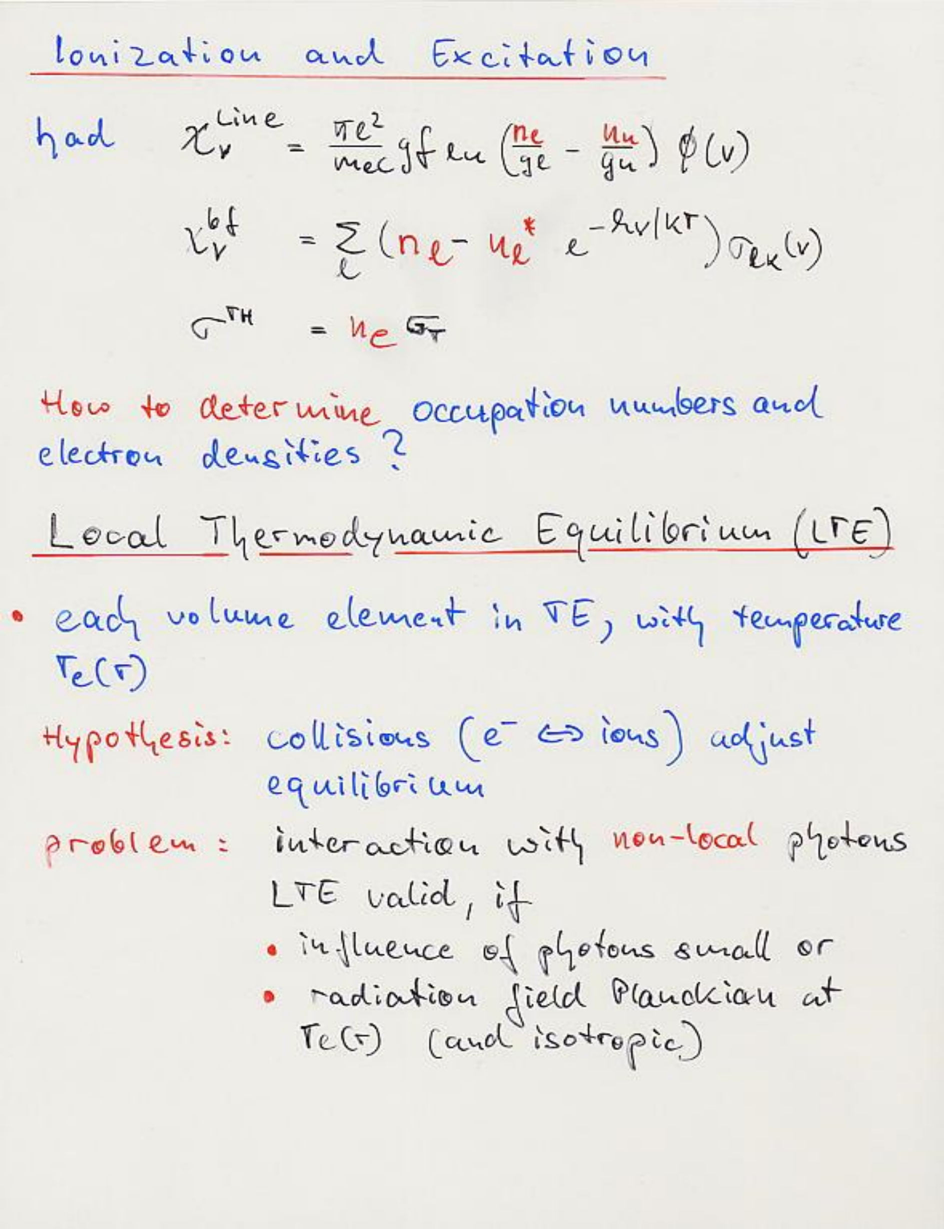
$$\eta_\nu^{tot} = \chi^{Line} \phi(\nu) S_L + \Sigma \eta_\nu^{bf} + \Sigma \eta_\nu^{ff} + n_e \sigma_T J_\nu$$

NOTE: for LTE ($n_i - n_i^*$) and $J_\nu = B_\nu$

we have always

$$\frac{\eta_\nu^{tot}}{\chi_\nu^{tot}} = B_\nu(T)$$, good test!

Ionization and Excitation



Ionization and Excitation

had $\chi_\nu^{\text{Line}} = \frac{\pi e^2}{m_e c} g f_{lu} \left(\frac{n_l}{g_l} - \frac{n_u}{g_u}\right) \phi(\nu)$

$\chi_\nu^{bf} = \sum_l \left(n_l - n_l^* e^{-h\nu/kT}\right) \sigma_{lk}(\nu)$

$\sigma^{TH} = n_e \sigma_T$

How to determine occupation numbers and electron densities?

Local Thermodynamic Equilibrium (LTE)

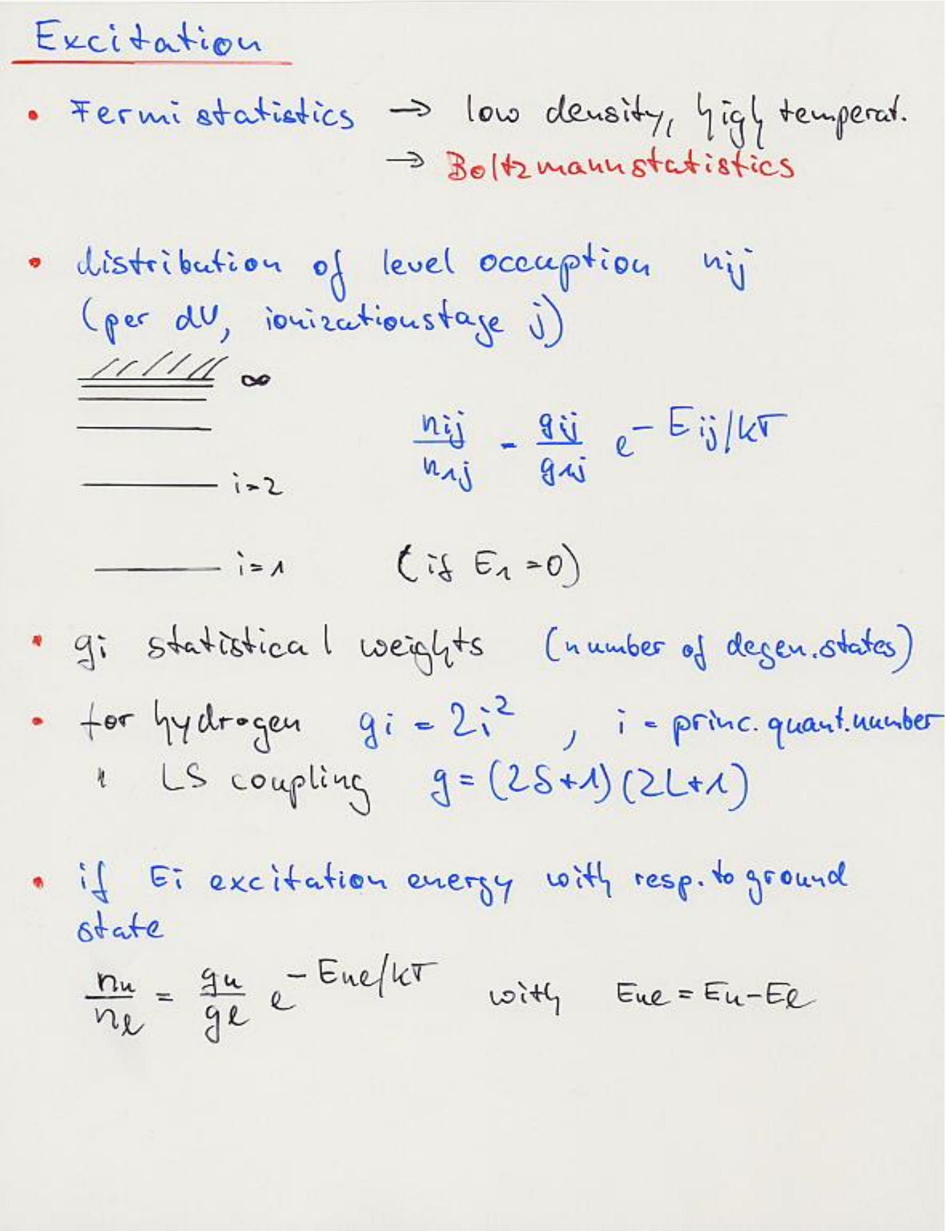
- each volume element in TE, with temperature $T_e(\tau)$

Hypothesis: collisions ($e^- \Leftrightarrow$ ions) adjust equilibrium

problem: interaction with non-local photons

LTE valid, if

- influence of photons small or
- radiation field Planckian at $T_e(\tau)$ (and isotropic)



Excitation

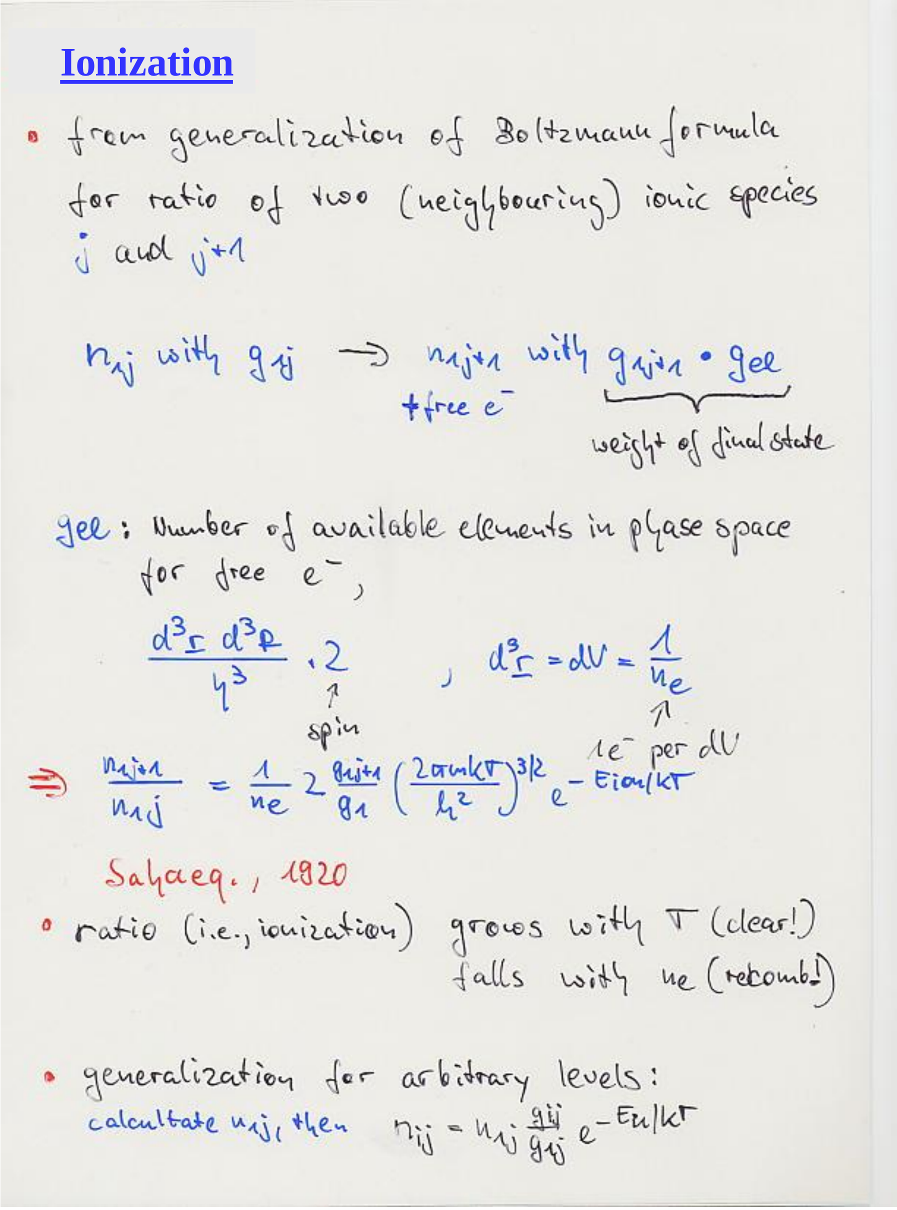
- Fermi statistics → low density, high temperat.
  → Boltzmann statistics
- distribution of level occuption n_{ij} (per dV, ionizationstage j)

$$\frac{n_{ij}}{n_{1j}} = \frac{g_{ij}}{g_{1j}} e^{-E_{ij}/kT}$$

(if $E_1 = 0$)

- g_i statistical weights (number of degen. states)
- for hydrogen $g_i = 2i^2$, i = princ. quant. number
  " LS coupling $g = (2S+1)(2L+1)$
- if E_i excitation energy with resp. to ground state

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-E_{ul}/kT} \quad \text{with} \quad E_{ul} = E_u - E_l$$



Ionization

- from generalization of Boltzmann formula for ratio of two (neighbouring) ionic species j and $j+1$

$$n_{1j} \text{ with } g_{1j} \Rightarrow \underset{+\text{free } e^-}{n_{1j+1}} \text{ with } \underbrace{g_{1j+1} \cdot g_{el}}_{\text{weight of final state}}$$

g_{el}: Number of available elements in phase space for free e^-,

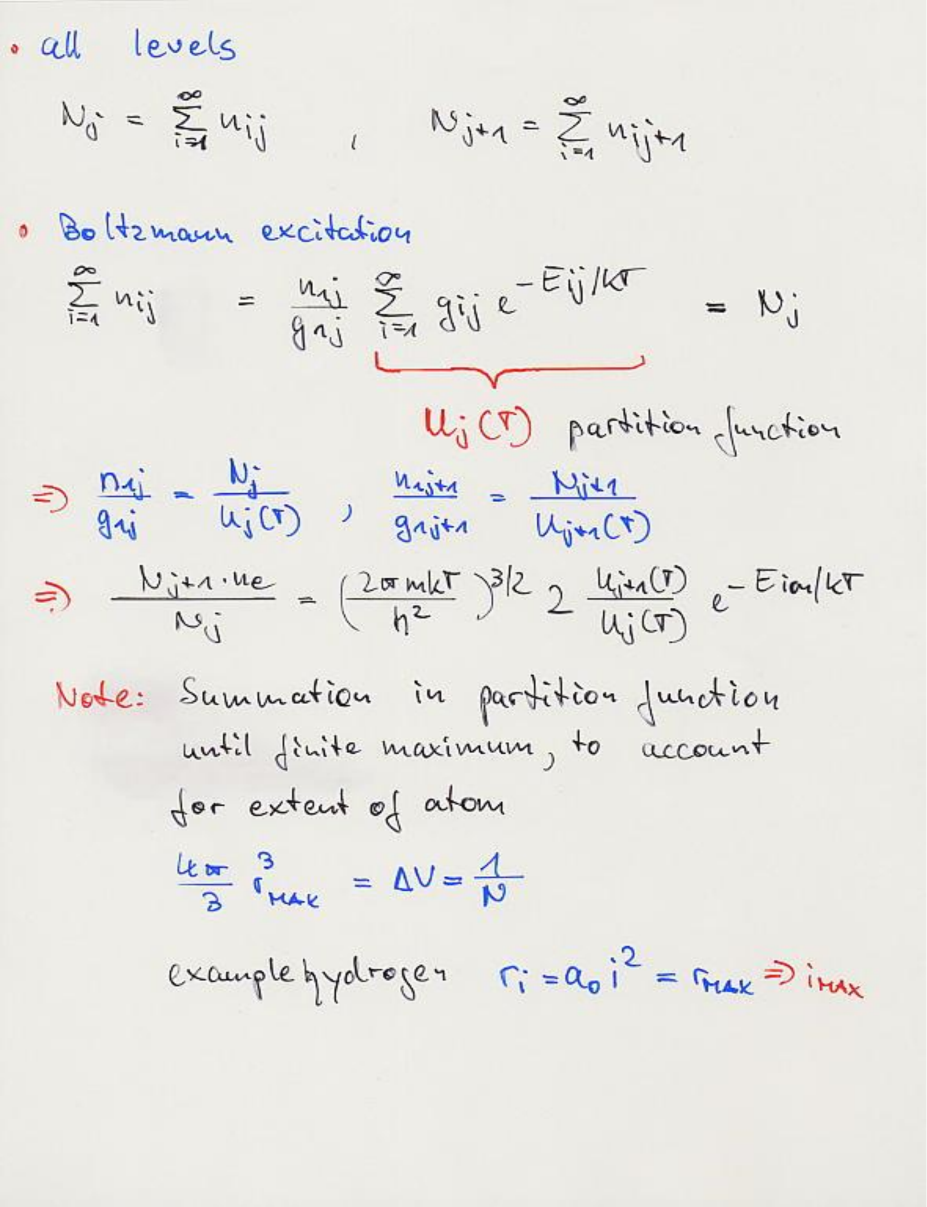
$$\frac{d^3\underline{r}\, d^3\underline{p}}{h^3} \cdot \underset{\overset{\uparrow}{\text{spin}}}{2} \quad , \quad d^3\underline{r} = dV = \underset{\overset{\uparrow}{1e^- \text{ per } dV}}{\frac{1}{n_e}}$$

$$\Rightarrow \frac{n_{1j+1}}{n_{1j}} = \frac{1}{n_e} 2 \frac{g_{1j+1}}{g_1} \left(\frac{2\pi mkT}{h^2}\right)^{3/2} e^{-E_{ion}/kT}$$

Saha eq., 1920

- ratio (i.e., ionization) grows with T (clear!)
  falls with n_e (recomb.!)

- generalization for arbitrary levels:
  calcultate n_{1j}, then $n_{ij} = n_{1j} \frac{g_{ij}}{g_{1j}} e^{-E_{i1}/kT}$



- all levels

$$N_j = \sum_{i=1}^{\infty} n_{ij} \quad , \quad N_{j+1} = \sum_{i=1}^{\infty} n_{ij+1}$$

- Boltzmann excitation

$$\sum_{i=1}^{\infty} n_{ij} = \frac{n_{1j}}{g_{1j}} \underbrace{\sum_{i=1}^{\infty} g_{ij}\, e^{-E_{ij}/kT}}_{U_j(T) \text{ partition function}} = N_j$$

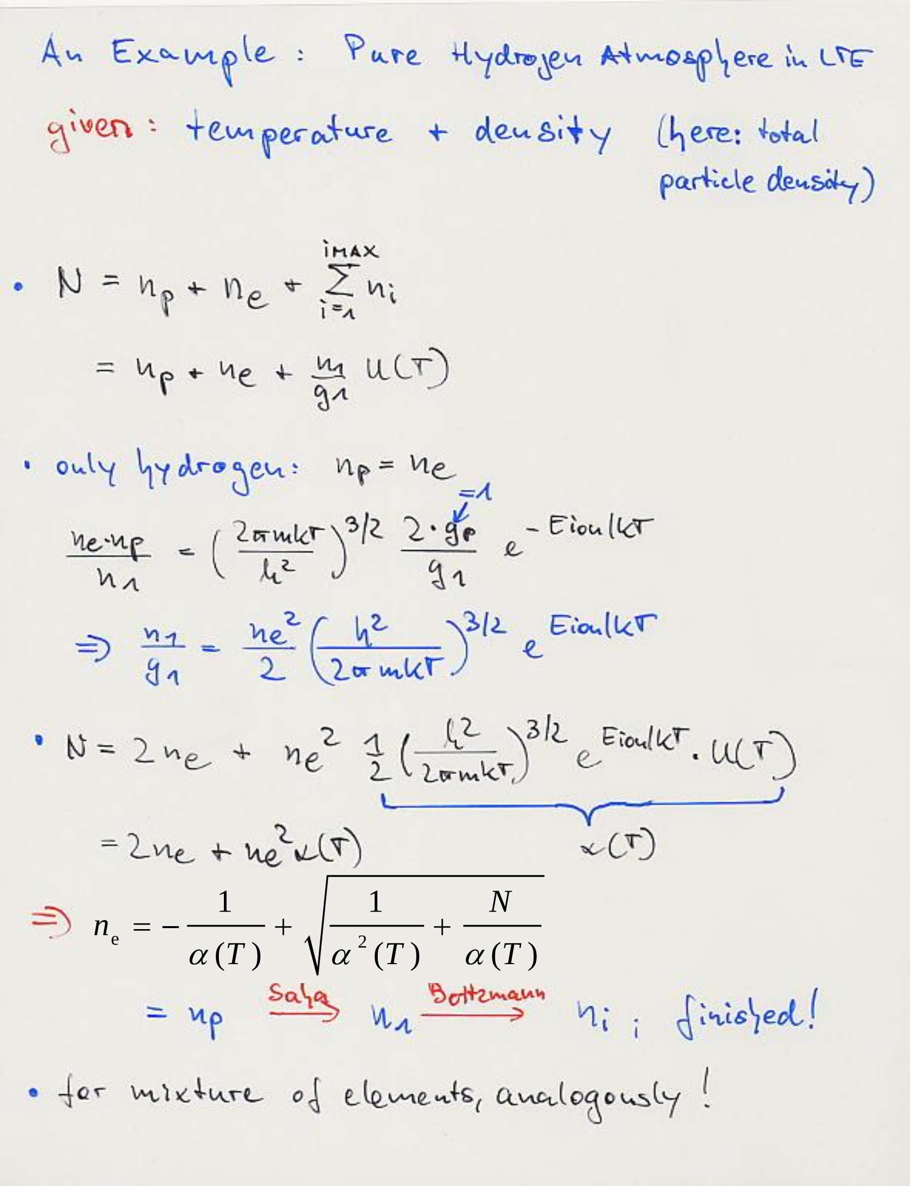
$$\Rightarrow \frac{n_{1j}}{g_{1j}} = \frac{N_j}{U_j(T)} \quad , \quad \frac{n_{1j+1}}{g_{1j+1}} = \frac{N_{j+1}}{U_{j+1}(T)}$$

$$\Rightarrow \frac{N_{j+1} \cdot n_e}{N_j} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} 2 \frac{U_{j+1}(T)}{U_j(T)} e^{-E_{ion}/kT}$$

Note: Summation in partition function until finite maximum, to account for extent of atom

$$\frac{4\pi}{3} r_{MAX}^3 = \Delta V = \frac{1}{N}$$

example hydrogen $r_i = a_0 i^2 = r_{MAX} \Rightarrow i_{MAX}$



An Example: Pure Hydrogen Atmosphere in LTE

given: temperature + density (here: total particle density)

- $N = n_p + n_e + \sum_{i=1}^{i_{MAX}} n_i$

$$= n_p + n_e + \frac{n_1}{g_1} U(T)$$

- only hydrogen: $n_p = n_e$

$$\frac{n_e \cdot n_p}{n_1} = \left(\frac{2\pi m k T}{h^2}\right)^{3/2} \frac{2 \cdot \overset{=1}{g_p}}{g_1} e^{-E_{ion}/kT}$$

$$\Rightarrow \frac{n_1}{g_1} = \frac{n_e^2}{2} \left(\frac{h^2}{2\pi m k T}\right)^{3/2} e^{E_{ion}/kT}$$

- $N = 2 n_e + n_e^2 \underbrace{\frac{1}{2}\left(\frac{h^2}{2\pi m k T}\right)^{3/2} e^{E_{ion}/kT} \cdot U(T)}_{\alpha(T)}$

$$= 2 n_e + n_e^2 \alpha(T)$$

$$\Rightarrow n_e = -\frac{1}{\alpha(T)} + \sqrt{\frac{1}{\alpha^2(T)} + \frac{N}{\alpha(T)}}$$

$$= n_p \xrightarrow{\text{Saha}} n_1 \xrightarrow{\text{Boltzmann}} n_i \text{; finished!}$$

- for mixture of elements, analogously!

LTE bf and ff opacities for hydrogen

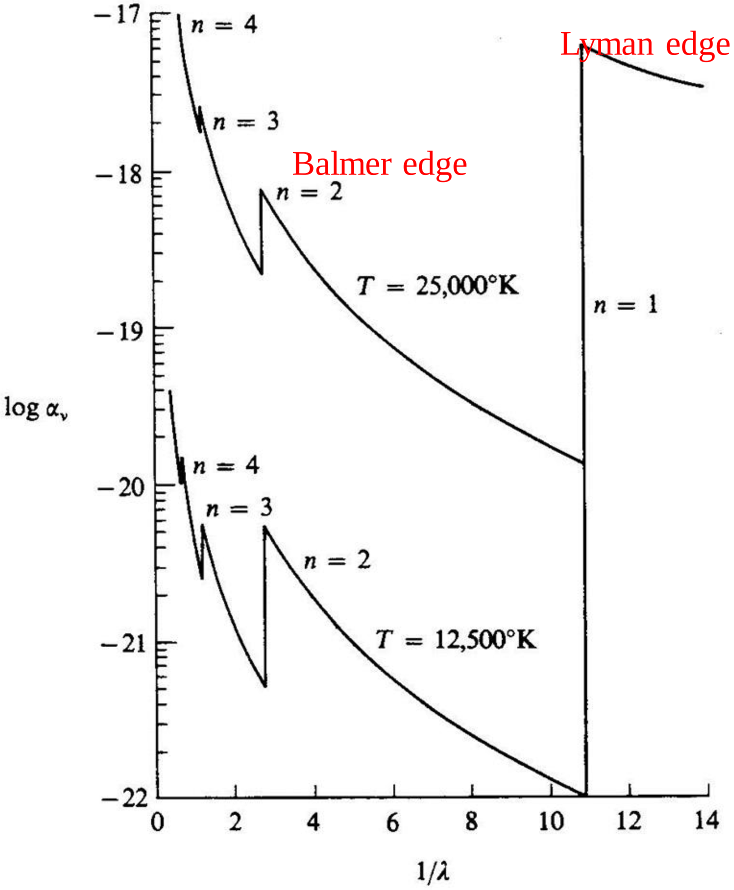


FIGURE 4-1

Opacity from neutral hydrogen at T = 12,500°K and T = 25,000°K, in LTE; photoionization edges are labeled with the quantum number of state from which they arise/neutral atom *Ordinate*: sum of bound–free and free–free opacity in cm^2/atom; *abscissa*: 1/λ where λ is in microns.

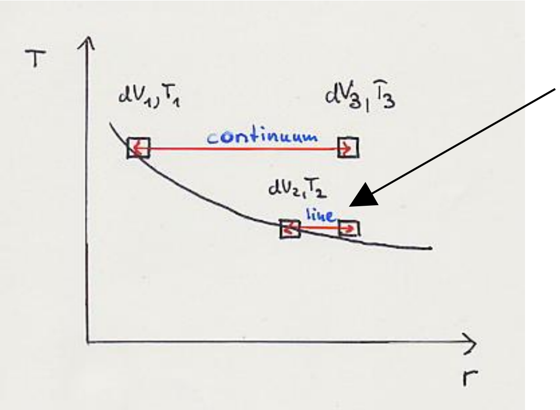
LTE and NLTE

(L)TE: for each process, there exists an inverse process with identical transition rate

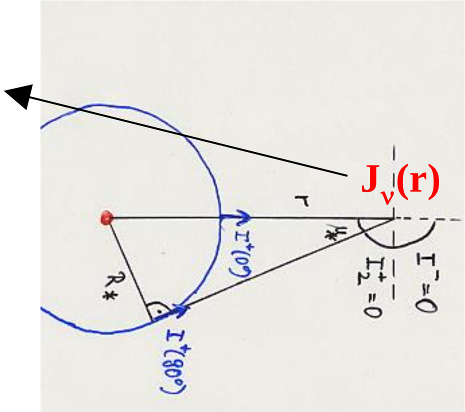
LTE = detailed balance for all processes!

processes = radiative + collisional

- **collisional** processes (and those which are essentially collisional in character, e.g., radiative recombination, ff-emission) in detailed balance, if velocity distribution of colliding particles is **Maxwellian** (valid in stellar atm., see below)
- **radiative** processes: photoionization, photoexcitation (= bb absorption) in detailed balance only if radiation field **Planckian** and **isotropic** (approx. valid only in innermost atmosphere)



$J_\nu(r) = W(r)\, B_\nu$



radiative processes couple regions with different temperatures, as a function of frequency: $\Delta\tau_\nu \le 1$

anisotropy

Question: **is f(v) dv Maxwellian?**

- elastic collisions -> establish equilibrium
- inelastic collisions/recombinations disturb equilibrium

inelastic collisions: involve electrons only in certain velocity ranges, tend to shift them to lower velocities

recombinations : remove electrons from the pool, prevent further elastic collisions

- can be shown: in *typical* stellar plasmas, $t_{el} / t_{rec} \approx 10^{-5} \dots 10^{-7} \approx t_{el} / t_{inel}$

  => Maxwellian distribution

- under certain conditions (solar chromosphere, corona), certain deviations in high-energy tail of distribution possible

Question: **is T(electron) = T(atom/ion)?**

- equality can be proven for stellar atmospheres with 5,000 K < Te < 100,000 K

When is LTE valid???

roughly: electron collisions >> photoabsorption rates

$\propto n_e T^{1/2}$ $\qquad \propto I_\nu(T) \propto T^x, \; x \geq 1$

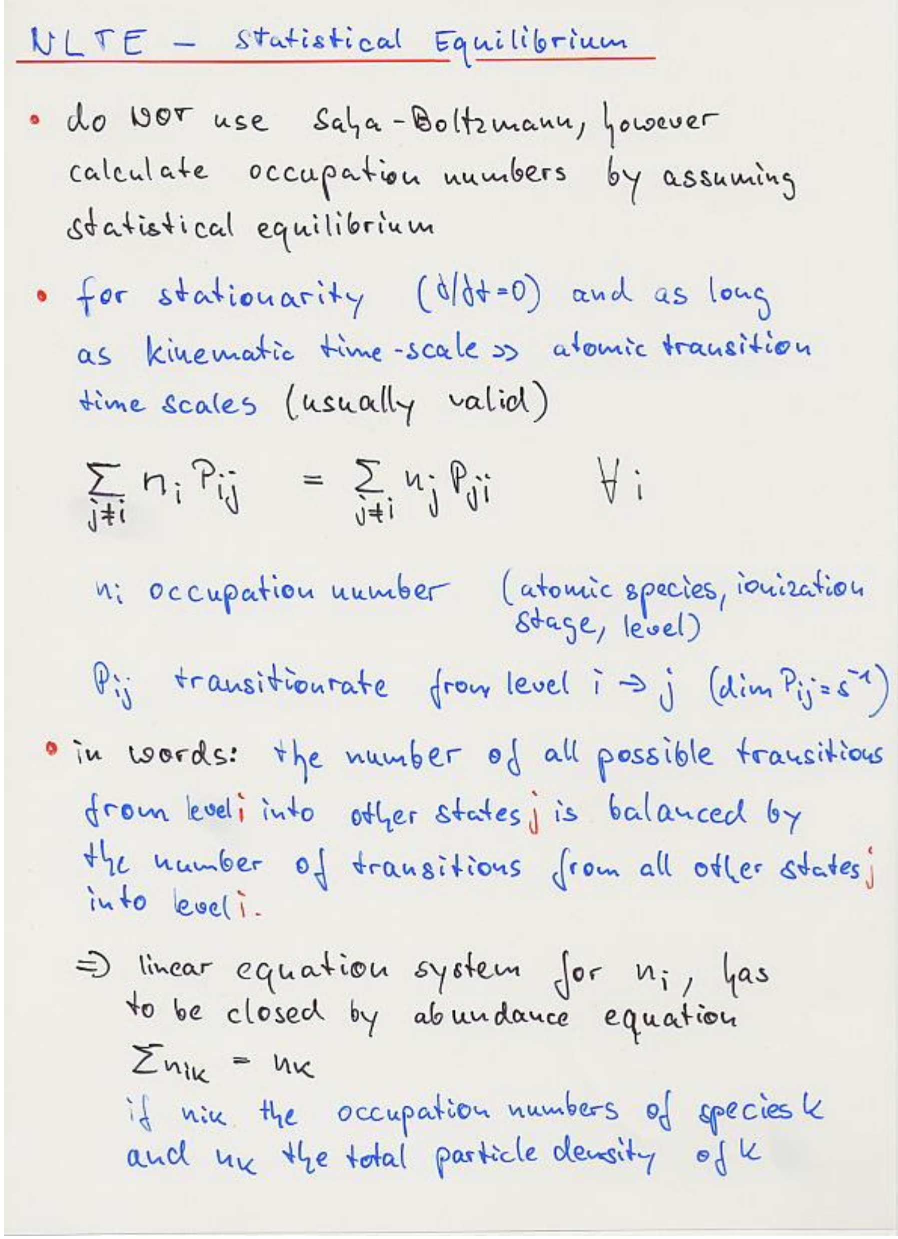
LTE:	T low,	n_e high	dwarfs (giants), late B and cooler
NLTE:	T high,	n_e low	all supergiants + rest

however:
NLTE-effects also in cooler stars, e.g.. iron in sun

TE – LTE – NLTE : a summary

	TE	LTE	NLTE
velocity distribution of particles **Maxwellian ($T_e=T_i$)**	✓	✓	✓
excitation **Boltzmann**	✓	✓	no
ionization **Saha**	✓	✓	no
source function	$B_\nu(T)$	$B_\nu(T)$, except scattering component	only $S_\nu^{ff} = B_\nu(T)$
radiation field	$J_\nu = B_\nu(T)$	$J_\nu \neq B_\nu(T)$, equality only for $\tau_\nu \geq \left(\frac{1}{\varepsilon_\nu}\right)^{1/2}$	$J_\nu \neq B_\nu(T)$ dito

Statistical equilibrium



NLTE – Statistical Equilibrium

- do NOT use Saha-Boltzmann, however calculate occupation numbers by assuming statistical equilibrium
- for stationarity ($\partial/\partial t = 0$) and as long as kinematic time-scale >> atomic transition time scales (usually valid)

$$\sum_{j \neq i} n_i P_{ij} = \sum_{j \neq i} n_j P_{ji} \qquad \forall i$$

n_i: occupation number (atomic species, ionization stage, level)

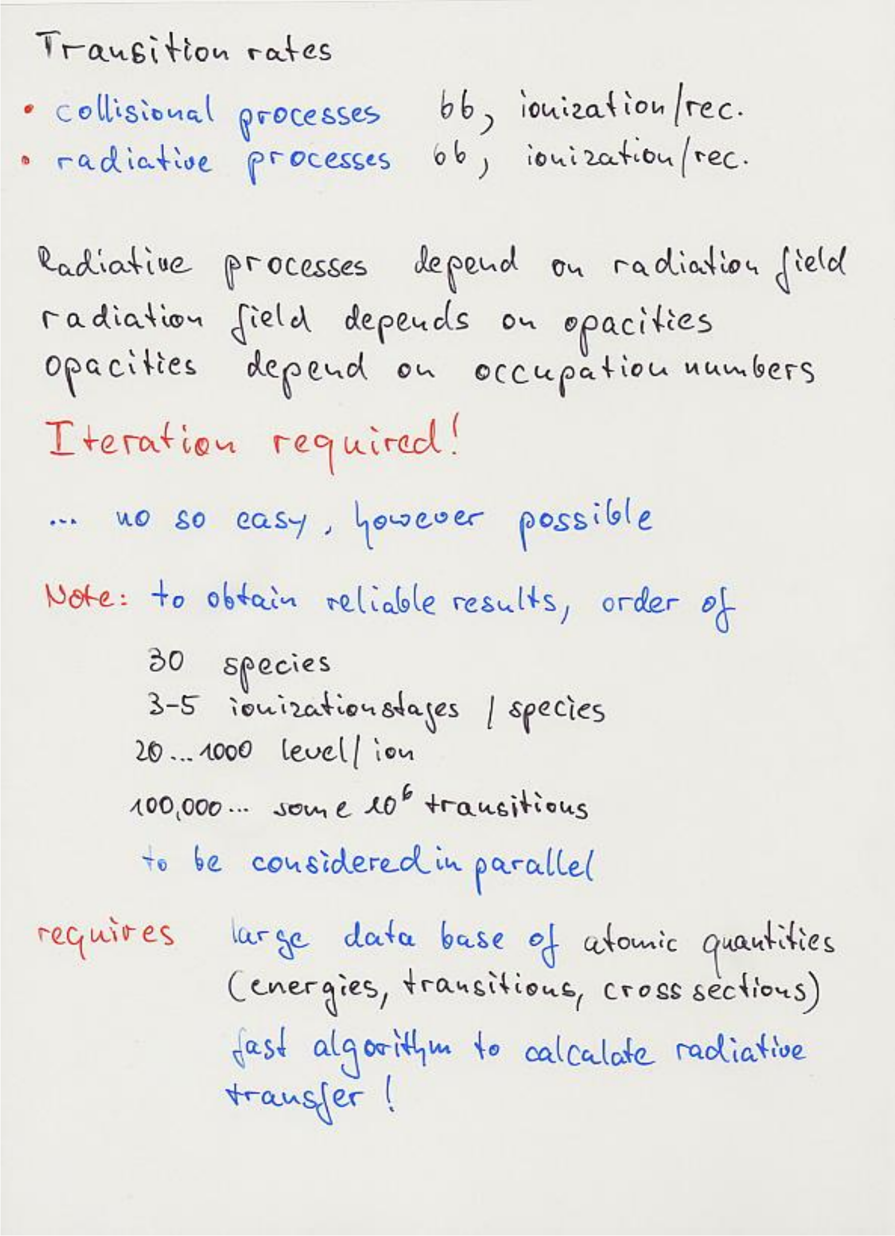
P_{ij} transitionrate from level $i \rightarrow j$ ($\dim P_{ij} = s^{-1}$)

- in words: the number of all possible transitions from level i into other states j is balanced by the number of transitions from all other states j into level i.

$\Rightarrow$ linear equation system for n_i, has to be closed by abundance equation

$\sum n_{ik} = n_k$

if n_{ik} the occupation numbers of species k and n_k the total particle density of k



Transition rates

- collisional processes bb, ionization/rec.
- radiative processes bb, ionization/rec.

Radiative processes depend on radiation field
radiation field depends on opacities
opacities depend on occupation numbers

Iteration required!

... no so easy, however possible

Note: to obtain reliable results, order of

30 species
3-5 ionizationstages / species
20 ... 1000 level / ion
100,000 ... some 10^6 transitions

to be considered in parallel

requires large data base of atomic quantities (energies, transitions, cross sections)

fast algorithm to calcalate radiative transfer!

Solution of the rate equations – a simple example

HAD: for each atomic level, the sum of all populations must be equal to the sum of all depopulations (for stationary situations)

example: 3-niveau atom with continuum

assume: all rate coefficients are known (i.e., also the radiation field)

=> rate equations (equations of statistical equilibrium)

$$-n_1\left[R_{1k}+C_{1k}+R_{12}+C_{12}+R_{13}+C_{13}\right]+n_2(R_{21}+C_{21})+n_3(R_{31}+C_{31})+n_k(R_{k1}+C_{k1})=0$$

$$n_1(R_{12}+C_{12})-n_2\left[R_{2k}+C_{2k}+R_{21}+C_{21}+R_{23}+C_{23}\right]+n_3(R_{32}+C_{32})+n_k(R_{k2}+C_{k2})=0$$

$$n_1(R_{13}+C_{13})+n_2(R_{23}+C_{23})-n_3\left[R_{3k}+C_{3k}+R_{31}+C_{31}+R_{32}+C_{32}\right]+n_k(R_{k3}+C_{k3})=0$$

$$n_1(R_{1k}+C_{1k})+n_2(R_{2k}+C_{1k})+n_3(R_{3k}+C_{1k})-n_k\left[R_{k1}+C_{k1}+R_{k2}+C_{k2}+R_{k3}+C_{k3}\right]=0$$

with

R_{ij}, radiative bound-bound transitions (lines!)

R_{ik} radiative bound-free transitions (ionizations)

R_{ki} radiative free-bound transitions (recombinations)

C_{ij} collisional bound-bound transitions

C_{ik} collisional bound-free transitions

C_{ki} collisonal free-bound transitions

in matrix representation =>

$$P = \underbrace{\begin{pmatrix} -(R_{1k}+C_{1k}+R_{12}+C_{12}+R_{13}+C_{13}) & (R_{21}+C_{21}) & (R_{31}+C_{31}) & (R_{k1}+C_{k1}) \\ (R_{12}+C_{12}) & -(R_{2k}+C_{2k}+R_{21}+C_{21}+R_{23}+C_{23}) & (R_{32}+C_{32}) & (R_{k2}+C_{k2}) \\ (R_{13}+C_{13}) & (R_{23}+C_{23}) & -(R_{3k}+C_{3k}+R_{31}+C_{31}+R_{32}+C_{32}) & (R_{k3}+C_{k3}) \\ (R_{1k}+C_{1k}) & (R_{2k}+C_{2k}) & (R_{3k}+C_{3k}) & -(R_{k1}+C_{k1}+R_{k2}+C_{k2}+R_{k3}+C_{k3}) \end{pmatrix}}_{\text{rate matrix, diagonal elements sum of all depopulations}}$$

$$P * \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 (= n_k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Rate matrix is singular, since, e.g., last row linear combination of other rows (negative sum of all previous rows)

THUS: LEAVE OUT arbitrary line (mostly the last one, corresponding to ionization equilibrium) and REPLACE by inhomogeneous, linearly independent equation for all n_i, to obtain unique solution

particle number conservation for considered atom:

$$\sum_{i=1}^{N} n_i = \alpha_k N_{\mathrm{H}}, \text{ with } \alpha_{\mathrm{k}} \text{ the abundance of element k}$$

NOTE 1: numerically stable equation solver required, since typically hundreds of levels present, and (rate-) coefficients of highly different orders of magnitude

NOTE 2: occupation numbers n_i depend on radiation field (via radiative rates), and radiation field depends (non-linearly) on n_i (via opacities and emissivities)
=> Clever iteration scheme required!!!!

Example for extreme NLTE condition
Nebulium (= [OIII] 5007, 4959) in Planetary Nebulae

mechanism suggested by *I. Bowen* (1927):

- low-lying meta-stable levels of OIII(2.5 eV) **collisionally excited** by free electrons (resulting from photoionization of hydrogen via "hot", *diluted* radiation field from central star)
- Meta-stable levels become **strongly populated**
- **radiative decay** results in **very strong** [OIII] emission lines
- impossible to observe suggested process in laboratory, since *collisional deexitation* (no photon emitted)) *much stronger than radiative decay under terrestrial conditions.*
- Thus, after detection new element proposed , "nebulium"

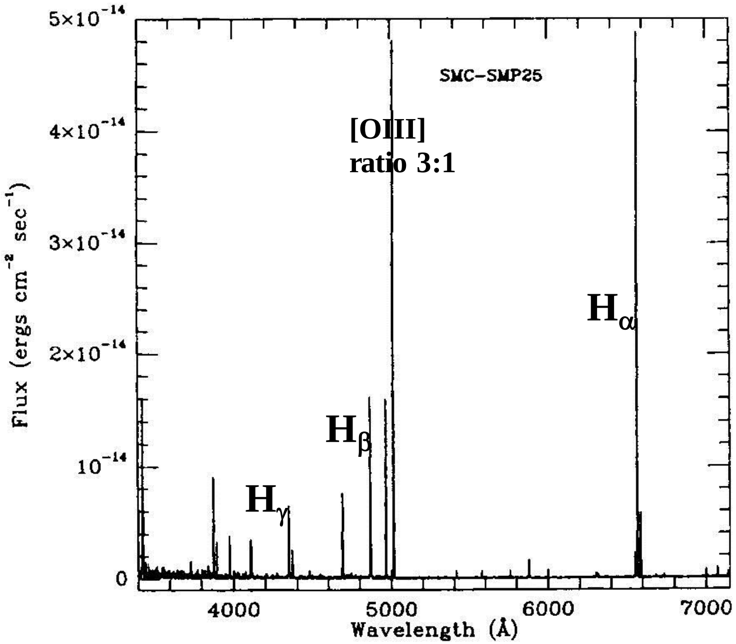


FIG. 1*a*

Condition for radiative decay

NOTE: $A_{\text{ml}} \le 10^{-2}$ (typical values are 10^{7})

$n_m A_{ml} \gg n_m n_e q_{ml}(T_e)$, with metastable level m

$\rightarrow n_e \ll n_e(\text{crit})$,

$$n_e(\text{crit}) = \frac{A_{ml}}{q_{ml}(T_e)}, \quad q_{\text{ml}} = 8.63 \cdot 10^{-6} \frac{\Omega(l,m)}{g_m \sqrt{T_e}}$$

$\Omega(l,m)$ collisional strength, order unity

for typical temperatures $T_e \approx 10{,}000\text{K}$ and [OIII] 5007,

we have $n_e(\text{crit}) \approx 4.9 \cdot 10^{5} \text{cm}^{-3}$,

much larger than typical nebula densities