Radiative processes, stellar atmospheres and winds (WS 2021/2022)

LAST Problem set (9)

Problem 1 [3 points] *Natural line width*

One can show (e.g., in the classical approach by using the equation of motion for an oscillating electron inclusive damping), that the radiated *power* from an excited atom decays with $\exp(-\Gamma t)$.

a) Interpret this as a probability distribution function, $p(t) \propto \exp(-\Gamma t)$, and show that the mean life time of an excited atom, $\tau = < t > = \int tp(t)dt = 1/\Gamma$.

b) Assume a transition between two excited states with energies E_i and E_f, and mean life times τ_i and τ_f. Calculate the corresponding line-width (with respect to frequency and wavelength) from the uncertainty principle.

c) Compare the result from 1b) with the full-width at half maximum from a corresponding Lorentz profile.

d) Calculate the natural line-width (see 1b/c) for the Balmer-α transition of hydrogen in units of Å, assuming $\tau_{n=2} = \tau_{n=3} = 10^{-8}$s.

Problem 2 [3.5 points] *Doppler broadening*

For the following problem and nomenclature, see script page 119.

In order to account for the thermal velocities of the radiating atoms, we have to convolve the 'atomic' profile function with the corresponding velocity distribution, $P(v_x, v_y, v_z)$ (Dopplershifts!). Thus, if the emission is isotropic, we need to evaluate

$$\Phi(\nu) = \int\int\int P(v_x, v_y, v_z)\phi(\nu' - \nu_0)dv_x, dv_y, dv_z.$$

$\Phi(\nu)$ is the resulting profile function at observer's (rest) frequency ν, and ϕ is the intrinsic ('atomic') profile in dependence of $(\nu' - \nu_0)$, with $\nu' = \nu'(\nu, \vec{n} \cdot \vec{v})$ the frequency in the atomic frame and ν_0 the transition frequency.

a) Derive the equation for $\Phi(\nu)$ as quoted on page 119.

   Hint: Assume (without loss of generality) that the x-axis of the $\vec{v}$ coordinate system is aligned with $\vec{n}$ (direction from atom towards observer), and that in this geometry only the v_x components contribute to the Dopplershifts (no transversal Dopplershift, because $v \ll c$).

b) Assume that the intrinsic profile, $\phi(\nu' - \nu_0)$, is a delta function, and derive the Doppler-profile quoted on page 120.

c) Assume now that the intrinsic profile, $\phi(\nu' - \nu_0)$, is a Lorentzian, and derive the Voigt-profile quoted on page 120.

d) Compare the natural line-width from problem 1d) (actually, half of this width) with the corresponding thermal Dopplerwidth, $\Delta\lambda_{\rm D}$ (also in Å), assuming a thermal speed of 10 km/s. In view of this result, interpret the parameter a appearing in the Voigt-profile.

Problem 3 [3 points] *Electrons in the solar photosphere*

According to the Holweger-Müller model of the solar photosphere, at $\tau_{5000} = 0.04$ (the optical depth at 5,000 Å) there is a temperature of 5,000 K, and a gas pressure of $2.63 \cdot 10^4$ dyn/cm^2.

Adopt LTE conditions, and calculate the corresponding electron-density n_e, by assuming that the photosphere consists of hydrogen and helium only ($\rm N_{He}/N_H = 0.1$), and that Helium is completely neutral. Adopt a partition function for neutral hydrogen, $U(\textrm{H}\,\textsc{i}) = 2$ (why?). Note: the statistical weight of a proton, $g_p = 1$.

If you have made no error, you should have obtained $n_e \approx 7.52 \cdot 10^{11}$ cm^{-3}. Calculate the corresponding electron pressure, P_e, and compare it with the value of 2.54 dyn/cm^2 from the Holweger-Müller model. Try to explain the discrepancy.

Problem 4 [2.5 points] *Approximate rate equations*

NOTE: The following nomenclature refers to the script, page 133/134.

a) Detailed balance in the resonance lines of a stellar wind

To estimate the occupation numbers (particularly, the ground-state) of an ion in the supersonic part of an expanding hot-star atmosphere (wind), one might apply the following approximations

(i) Because of the low densities, all collisional rates can be neglected.

(ii) The resonance lines (radiative transitions connected to the ground state) are optically thick throughout the wind, and the corresponding radiative bound-bound rates (upwards and downwards) cancel each other, i.e., $n_1 R_{1j} = n_j R_{j1}$. In other words, these rates do not appear in the rate equations.

Write down the corresponding approximate rate equations for an ion with four bound levels, in the form

$$\text{matrix} \cdot (n_1, n_2, n_3, n_4)^T = \vec{b},$$

assuming that the ground-state population of the next higher ion, n_k, is known, and that $\vec{b}$ is a vector containing all rates proportional to n_k.

Which processes control the ground-state population of the considered ion, and what is the result for n_1?

b) Nebular approximation

The situation in a Planetary Nebula or an HII region illuminated by a hot star is similar to the conditions from a), except that because of the much lower densities the radiative bound-bound rates for the resonance lines do no longer cancel each other, and that *generally* (i.e., for all lines) only the spontaneous emission terms 'survive'. With respect to page 133/134, in this situation we then have

$$n_i R_{ij} \rightarrow 0, \text{and } n_j R_{ji} \rightarrow n_j A_{ji}, \tag{1}$$

with A_{ji} the Einstein coefficient for spontaneous decay. Moreover, the ionization rates, $n_i R_{ik}$, can be neglected for all *excited* levels, because of the very small dilution factor (sizes of order 0.1 to 1 pc for PNe, and 10 to 100 pc for HII regions).

Formulate the corresponding approximate rate equations similar to problem 4a), and compare the structure of both systems.

Have fun, and much success!