Radiative processes, stellar atmospheres and winds (WS 2021/2022)

Problem set 6

Problem 1 [3 points] *Radiative transfer with linear source function*

On page 81 of the script, we quoted that the emergent intensity for a linear source function, $S = a + b\tau$, is given by

$$I_\nu^{\mathrm{em}}(\mu) = a + b\mu$$

a) Prove this relation, by providing the missing steps as indicated by '...' on page 81 of the script.

b) Calculate the corresponding mean intensity at $\tau = 0$. Interpret your result, by considering the ratio between J and S (distinguish the limits of a weak and strong gradient of S).

Problem 2 [5 points] Λ *operator with linear source function*

In problem 1b, we calculated the mean intensity at the outer boundary of a slab or atmosphere, $\tau = 0$, given a source function that is linear in τ. Now we want to calculate the total run of the the mean intensity, for all τ. To this end, we make use of the Λ operator as derived in the script, and calculate $J(\tau)$ assuming a linear source function $S = a + b\tau$.

After accounting for useful calculation rules for exponential integrals with real arguments:

$$\begin{aligned}
E_1(t) &= \int_t^\infty \frac{\mathrm{e}^{-x}}{x}\mathrm{d}x = \int_1^\infty \frac{\mathrm{e}^{-tx}}{x}\mathrm{d}x \\
E_n(t) &= \int_1^\infty \frac{\mathrm{e}^{-tx}}{x^n}\mathrm{d}x, \quad n = 0, 1, 2, \ldots \\
E_n(0) &= \frac{1}{n-1}, \quad n > 1; \qquad E_n(t) \to \frac{\mathrm{e}^{-t}}{t} \text{ for } t \to \infty \\
\frac{dE_n(t)}{dt} &= -E_{n-1}(t), \quad n > 1 \quad (\to \text{integration of exp. integrals}) \\
\int_a^b tE_n(t)\mathrm{d}t &= E_{n+2}(a) - E_{n+2}(b) + aE_{n+1}(a) - bE_{n+1}(b) \quad \text{from partial integration}
\end{aligned}$$

we obtain the following result (remember: *linear* source function!)

$$J(\tau) = a + b\tau - \frac{a}{2}E_2(\tau) + \frac{b}{2}E_3(\tau). \tag{1}$$

a) Prove, by evaluating the Lambda-operator and by using the above calculation rules, that for a spatially constant source function $S = a$ (i.e., $b = 0$),

$$\Lambda_\tau(S = a) = J(\tau) = a\Big(1 - \frac{1}{2}E_2(\tau)\Big),$$

consistent with Eq. 1.

b) Evaluate the result for the linear source function, Eq. 1, at $\tau = 0$, and check it by comparing with your result from problem 1b. Compare also with the corresponding result from the Milne Eddington model, and explain the difference.

c) What do you expect for $J(\tau \to \infty)$? Check whether this limiting value is reached by Eq. 1.

d) Plot the ratio $J(\tau)/S(\tau)$, for $\tau = [0, 2]$ and parameters {a=1,b=0.5} and {a=1,b=4}, and explain the difference of the two curves. Under which condition does the Milne-Eddington model predict $J(\tau) = S(\tau)$ for *all* τ? Plot the corresponding result when using the "exact" solution, Eq. 1.

   To calculate and/or plot your results, you might use MATHEMATICA, MAPLE, IDL, PYTHON, or GNUPLOT.

Problem 3 [4 points] *Eddington and two-stream approximation*

The Milne Eddington model uses the Eddington approximation (Eddington, 1929, MNRAS, 'On the formation of absorption lines'), which adopts a ratio of $K_\nu/J_\nu = 1/3$ everywhere throughout a plane-parallel atmosphere (the ratio K_ν/J_ν is nowadays called Eddington-factor).

Eddington's approximation is consistent with other flavors of approximation (e.g., the diffusion approximation, cf. page 85 of the script), and we will consider some of them in the following. For all problems, assume a plane-parallel atmosphere for consistency.

a) *Angle-independent intensity.* Assume that the outward- and inward-directed specific intensities, I_ν^+ and I_ν^-, are angle independent, and shall depend only on height z, i.e.,

$$I_\nu(z, \mu) =: I_\nu^+(z) \text{ for } \mu \ge 0 \qquad \text{and} \qquad I_\nu(z, \mu) =: I_\nu^-(z) \text{ for } \mu < 0$$

   Calculate the corresponding 0th to 2nd moment of the radiation field, plus radiative flux and radiation pressure scalar, and show that the Eddington factor is indeed 1/3 for this approximation.

   What is the ratio H_ν/J_ν at the outer boundary, $z = 0$?

b) Show that the Eddington factor is also 1/3, if the specific intensity can be described by

$$I_\nu(\mu, z) =: I_0(z) + \sum_{i=1,\mathrm{odd}}^{\infty} \mu^i I_i(z)$$

   where the sum extends only over *odd* indices.

An example is the assumption of an intensity which is linear in μ, i.e., $I_\nu(\mu) =: I_0 + \mu I_1$ (providing, e.g., the most simple description of limb-darkening, see next problem sheet). Another example for such a law is the diffusion approximation.

c) In the *two-stream approximation*, the radiation shall propagate in only two representative, discrete directions μ_0, usually inwards and outwards in opposite ones (i.e., $\mu_0^- = -\mu_0^+$). Thus,

$$I_\nu(z,\mu) =: I_\nu^+(z)\delta(\mu - |\mu_0|) \text{ for } \mu \geq 0 \qquad \text{and} \qquad I_\nu(z,\mu) =: I_\nu^-(z)\delta(\mu + |\mu_0|) \text{ for } \mu < 0$$

Calculate the corresponding 0th to 2nd moment of the radiation field in dependence of $|\mu_0|$, and determine $|\mu_0|$ from the condition that the Eddington approximation shall be fulfilled.

What is now the ratio H_ν/J_ν at the outer boundary?