

Radiative processes, stellar atmospheres and winds (WS 2021/2022)

Problem set 4

Problem 1 [9 points] *Radiation pressure*

- a) Calculate all components of the radiation pressure tensor (with respect to Cartesian or spherical coordinates) for the general case, i.e., when the specific intensity depends both on θ and ϕ . For P_{11} (where the indices '1,2,3' refer to x, y, z or Θ, Φ, r), the corresponding result is

$$P_{11} = \frac{1}{c} \int_0^{2\pi} \cos^2 \phi d\phi \int_{-1}^{+1} (1 - \mu^2) I_\nu(\vec{r}, \mu, \phi, t) d\mu.$$

- b) Prove the result provided on page 67 of the script (bottom of left part) for the radiation pressure tensor in plane-parallel/spherical symmetry.
- c) Show that the radiation pressure tensor is not only isotropic for an isotropic radiation field, but also if the angular dependence of the radiation field is given by $I(\mu) = I_0 + I_1\mu$. This result is important because such a description of the radiation field is appropriate in the so-called diffusion limit, reached at large depths in the atmosphere.
- d) Show that for an isotropic radiation field in thermal equilibrium, the total (i.e., frequency integrated) radiation pressure scalar is given by

$$p_R^*(r, t) = \frac{1}{3} a T^4(r)$$

where a is the so-called radiation constant. How does a relate to the Stefan-Boltzmann constant, σ_B ?

- e) The most extreme case of an anisotropic radiation field is associated with a planar wave, where we can write (see exercise 3.1) $I_\nu(\mu, \phi) = I_0 \delta(\mu - \mu_0) \delta(\phi - \phi_0)$.
1. Calculate the individual components of the radiation pressure tensor, for arbitrary μ_0 ($-1 < \mu_0 < 1$) and ϕ_0 ($0 \leq \phi_0 < 2\pi$).
 2. What is the result for the radiation pressure tensor P (expressed w.r.t. Cartesian coordinates) if the (planar) wave moves along one of the coordinate axes?
 3. Now assume that the amplitude of the planar wave has a z -dependence. Calculate the components of $\nabla \cdot P$ (again w.r.t. Cartesian coordinates) when the planar wave propagates into an arbitrary direction.
- f) An extreme anisotropy of the radiation field is reached far away from a star, e.g., in a nebula, when the radiation field originates from a stellar surface that occupies only a very small solid angle. To show this explicitly, suppose an observer located

at distance r from the center of a star with radius R_* . Assume that the (spherically symmetric) star has a uniformly bright surface, i.e., I shall be independent of μ , and that there is negligible absorption between stellar surface and observer. Show that the corresponding (half) opening angle of the star as seen from the observer, μ_* , is (exactly) given by

$$\mu_* = \left(1 - \left(\frac{R_*}{r}\right)^2\right)^{1/2},$$

and that all moments of the radiation field, $1/2 \int_{-1}^1 I \mu^n d\mu$, with $n = 0, 1, 2, \dots$ become almost identical for $r \gg R_*$. Prove that in this situation the radiation pressure tensor has only one non-vanishing component w.r.t. spherical coordinates.

Problem 2 [3 points] *Solar sail*

- a) A 10-tons spacecraft is launched from Earth, and is to be accelerated radially away from the sun by using a circular solar sail, where the weight of the sail is included in the above mass. The initial acceleration of the spacecraft shall be $1 g$ (when it is roughly at 1 AU away from the sun). Assuming a flat sail, determine the required radius of the sail if it is black, so that it absorbs all of the sun's light (for a reflective sail, the force becomes a factor of two larger).

Hint(1): In view of the results from exercise 1f), the force, F , on the sail can be computed, as in hydrostatics, from $F = pA$, with pressure p and area A . Show that the radiative acceleration is given by $g_{\text{rad}} = (f_{\odot}/c) A/m$, when f_{\odot} is the solar irradiance.

Hint(2): The spacecraft, like the earth, is orbiting the sun. Should you include the sun's gravity in your calculations?

- b) Derive the radial dependence of the radiative acceleration (initially $1 g$) as the spacecraft propagates away from the earth and sun.

Have fun, and much success!