Statistical methods – an introduction (SS 2022)

Problem set 6

Problem 1 [6 points] *Detector efficiency*

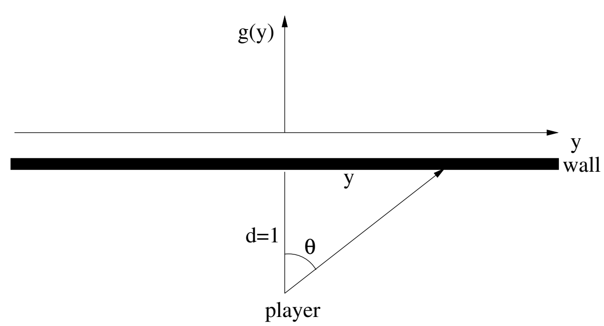
Solve the following problems (a and b) using a binomial and a Poisson-distribution, and compare the results!

a) A detector system is 98% efficient in registering the incidence of cosmic ray particles. What is the probability that it will register all of 100 incident particules? Calculate the probabilities both by considering the successes (in case, write a small script to evaluate the Poisson probability), and by considering the misses. What do you conclude?

b) How many particles must pass the detector to have a better than even (> 50%) chance that one or more particles are missed? Try to solve this problem "by hand".

Problem 2 [6 points] *Cauchy distribution*

Assume a tennis player in front of a long wall, at unit distance (whatever the units are). The player serves a large number of balls (always with sufficient force), with angles θ randomly chosen from a uniform distribution within [-$\pi/2 \ldots \pi/2$]. The balls hit the wall at positions y.



a) Determine the distribution (p.d.f.) $g(y)$ of the impact points on the wall, by an appropriate transformation of variables.

b) Confirm that the resulting *standard Cauchy* distribution is normalized within $[-\infty, \infty]$. (If you could not solve problem 1a), consult the literature for a definition of the Cauchy distribution).

c) Show that the Cauchy distribution has undefined variance, by calculating this quantity from the 2nd moment and the expectation value.

d) The so-called Lorentz distribution (sometimes also called Breit-Wigner distribution) used to describe the profile function of the majority of spectral lines is defined as

$$\Phi(\nu)\mathrm{d}\nu = \frac{\Gamma}{4\pi^2} \frac{1}{(\nu - \nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2} \mathrm{d}\nu,$$

with frequency ν, central=transition frequency ν_0 and 'damping parameter' Γ. Calculate the corresponding distribution function for the angular frequency $\omega = 2\pi\nu$. Show that the Cauchy distribution can be generated from a Lorentzian one (or vice versa) if $y = (\omega - \omega_0)/(\Gamma/2)$.

Note (this is just a comment, nothing to do here): It is easy to show that the damping parameter Γ describes the FWHM of the distribution w.r.t. ω.

e) Show that the characteristic function of a Cauchy distribution (according to problem a)) is $\Phi_y(t) = \exp(-|t|)$.

Use this result to prove that the arithmetic mean of Cauchy-distributed variates,

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$$

is also Cauchy-distributed.

Note: At first glance, this result seems to contradict the Central Limit Theorem; however, the CLT can be only applied for distributions with well-defined expectation value and variance!!!

f) Convince yourself that whenever you calculate the arithmetic mean for a sample of Cauchy-distributed random-variables, this does *not* converge (in a statistical sense) to a specific value (assumed to be zero) if you increase the sample-size. To this end, write a small IDL- or `Python-` script, and compare with a similar simulation for normally distributed variables ($\mu = 0, \sigma = 1$).

Have fun, and much success!