Gravity, Entropy & Holography

Oliver Friedrich

Fakultät für Physik, Ludwig-Maximilians Universität München,
Geschwister-Scholl-Platz 1, 80539 München, Germany

E-mail: oliver.friedrich@physik.lmu.de

Abstract. Test

Contents

1 Horizons, horizon entropy & the holographic principle - an incomplete overview

1.1 Units

In this lecture we will for the most part use natural units, where the reduced Planck constant $\hbar$, the speed of light c and Boltzmann's constant k are all set to 1. We will keep the appropriate factors of Newton's constant G in most expressions. The reason for this is that G has mass dimension

$$[G] = \frac{1}{\text{mass}^2} \ , \tag{1.1}$$

so multiplication by G can change the mass dimension of an expression.

1.2 Null surfaces and black holes

Consider a 4-dimensional manifold $\mathcal{M}$ equipped with coordinates $x = (x^0, x^1, x^2, x^3)$ and a pseudo-Riemannian line element

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu \ , \tag{1.2}$$

where $g_{\mu\nu}$ are the components of the (pseudo-Riemannian) metric tensor on $\mathcal{M}$. Any real-values function $s : \mathcal{M} \to \mathbb{R}$ induces a family of hypersurfaces on $\mathcal{M}$ via

$$\Sigma_{s=r} := \{x \in \mathcal{M} \ \text{ s.t. } \ s(x) = r\} \ . \tag{1.3}$$

Except for potentially existing local extrema of the function s, these surfaces will be 3-dimensional (since the condition $s(x) = r$ "removes" one of the dimensions).

All "horizons" that we will talk about in this lecture are *null surfaces*. These are surfaces whose normal vectors are null vectors (i.e. light-like). At any location $x \in \Sigma_{s=r}$ the components of the normal vector to $\Sigma_{s=r}$ are given by $\partial_\mu S(x)$, and in order for the normal vector to be null these components need to satisfy

$$g^{\mu\nu}\,\partial_\mu S\,\partial_\nu S = 0 \ . \tag{1.4}$$

We could choose a coordinate system where, e.g. , $S(x) \equiv x^1$. In this case, Equation 1.4 becomes a condition for the 11-component of the inverse metric, i.e.

$$g^{11} = 0 \ . \tag{1.5}$$

This will come in handy in a minute, when we try to identify null surfaces of black holes.

In general, a reason why null surfaces are interesting because they *bifurcate* a given spacetime into two regions $\mathcal{A}$ and $\mathcal{B}$ such that light-like geodesics cannot ("light rays") cannot travel from $\mathcal{A}$ to $\mathcal{B}$ (though they may still be able to travel from $\mathcal{B}$ to $\mathcal{A}$). This in itself is not a unique feature of null surfaces, because any space-like boundary that separates $\mathcal{M}$ into a "past" region and a "future" region would also act as a on-way-surface for light. But for a null-surface the region $\mathcal{B}$ may be one with an infinite past and future, i.e. an observer may have lived in $\mathcal{B}$ since forever and continue to live there forever but still never receive signals from the region $\mathcal{A}$. In Section 3 we will be more nuanced about our definition of "horizon" and distinguish between *event horizons*, *particle horizons* and *Killing horizons*. But for now let us stick with the notion that a horizon is a null surface and identify such a surface in a black hole spacetime.

We consider a stationary, non-rotating black hole of mass M. The latter is e.g. described by the *Schwarzschild metric*, whose line element is

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{r_s}{r}}\,\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\ , \tag{1.6}$$

where we defined the *Schwarzschild radius* as

$$r_s := 2GM\ . \tag{1.7}$$

Note that in the above coordinates the metric is diagonal, such that $g^{rr} = 1/g_{rr}$. Remembering Equation 1.5, this allows us to identify as null the surface where $r = r_s$ which is of course the event horizon of the black hole.

A coordinate system that makes the null-nature of the hypersurface $r \equiv r_s$ more transparent are *Kruskal-Szekeres coordinates*. They are given in terms of the Schwazschild coordinates t and r as

$$T \ = \left|1 - \tfrac{r}{r_s}\right|^{\frac{1}{2}} \exp\left(\tfrac{r}{2r_s}\right)\cdot\begin{cases}\sinh\left(\frac{t}{2r_s}\right) & \text{for } r > r_s\\ \cosh\left(\frac{t}{2r_s}\right) & \text{for } r \le r_s\end{cases} \tag{1.8}$$

$$R = \left|1 - \tfrac{r}{r_s}\right|^{\frac{1}{2}} \exp\left(\tfrac{r}{2r_s}\right)\cdot\begin{cases}\cosh\left(\frac{t}{2r_s}\right) & \text{for } r > r_s\\ \sinh\left(\frac{t}{2r_s}\right) & \text{for } r \le r_s\end{cases}\ , \tag{1.9}$$

and with these coordinates the line element becomes

$$\mathrm{d}s^2 = \frac{4r_s^3}{r(R,T)}\exp\left(-\frac{r(R,T)}{r_s}\right)\left(-\mathrm{d}T^2 + \mathrm{d}R^2\right) + r(R,T)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\ . \tag{1.10}$$

(We do not replace r by its explicit expression in terms of R and T because it is tedious and because it doesn't offer any additional insight at the moment.) A convenient feature of Kruskal-Szekeres coordinates is the fact that T is a timelike coordinate and R is a spacelike coordinate throughout spacetime. In contrast the Schwarzschild coordinates r and t switch their roles and temporal and spatial coordinates when crossing the horizon. Additionally, any light-like geodesic that is radial, i.e. for which $\mathrm{d}\theta = 0 = \mathrm{d}\phi$, will have $\mathrm{d}T = \pm\mathrm{d}R$. Hence, such geodesics will appear as lines of 45° in a diagram of R and T. Together with the following exercise this enables us to see that $r \equiv r_s$ is indeed a null surface.

Exercise 1

Show that the black hole horizon $r \equiv r_s$ is located at the diagonal $T = R$ of a diagram of the Kruskal-Szekeres coordinates T and R. Hint: you cannot directly take the limit $r \to r_2$ in Equations 1.8 and 1.9. Find a useful way to combine the two equations first.

In Figure 1 we compare the locations of the horizon $r = r_s$ and the location of an observer outside the horizon in Schwarzschild and Kruskal-Szekeres coordinates. In the latter, the horizon does not in fact appear as a very special surface. The fact that g_{rr} diverges in the Schwarzschild picture is thus only a so called coordinate singularity, and one can for example show that spacetime curvature is completely well behaved at the horizon surface. The only true singularity (i.e. point where spacetime curvature itself diverges) appears at $r = 0$. In Kruskal-Szekeres coordinates this is a future boundary for observers entering the black hole.

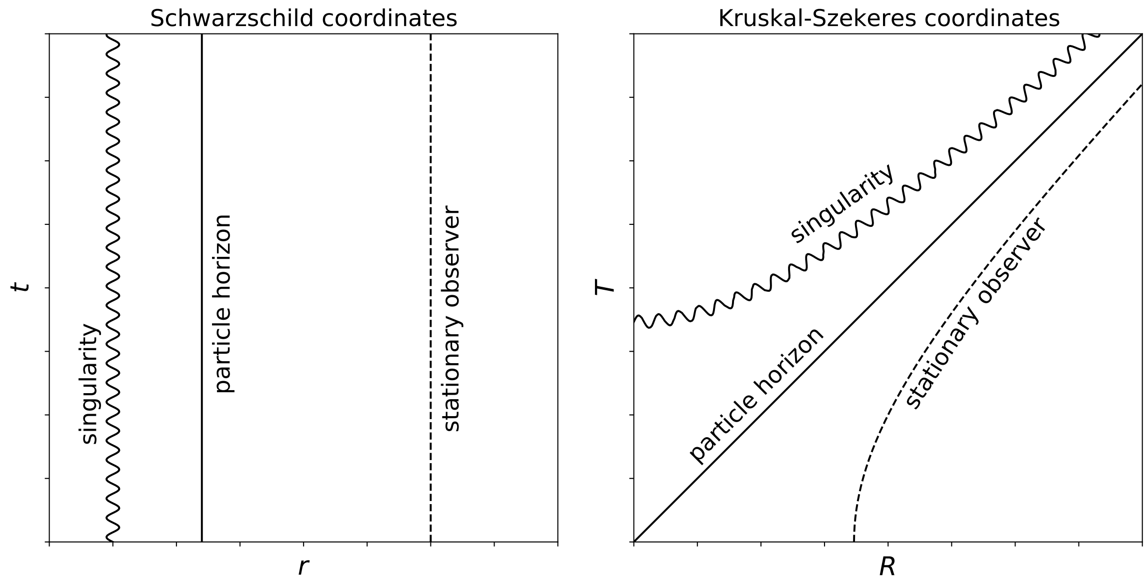

Figure 1: Sketching the location of singularity, particle horizon and a stationary observer outside the horizon in Schwarzschild coordinates and Kruskal-Szekeres coordinates of a non-rotating black hole. The Kruskal-Szekeres coordinate T is time-like everywhere in spacetime, and R is space-like everywhere in spacetime. Another benefit of Kruskal-Szekeres coordinates is the fact that light-like (and radial) geodesics would appear as lines of a 45° angle wrt. the coordinate axes.

Exercise 2

Derive an equation $T = R(T)$ for the location of the black hole singularity (i.e. for the "wavy" line in Figure 1) in Kruskal-Szekeres coordinates.

1.3 Key events in the development of horizon thermodynamics

It is almost common knowledge that a black hole with horizon area A_{bh} supposedly carries an entropy

$$S_{\mathrm{bh}} = \frac{A_{\mathrm{bh}}}{4\,\ell_{\mathrm{Planck}}^2}\ , \tag{1.11}$$

where ℓ_{Planck} is the Planck-length. And sentences like the following are popular when trying to motivate this entropy assignment: "If some amount of matter, together with the entropy that it carries, crosses the horizon of a black hole, then this decreases the entropy of the observable Universe and thus breaks the 2nd law of thermodynamics. The horizon entropy then restores the 2nd law."

We sketch this situation in Figure 2, and in Kruskal-Szekeres coordinates such a proclaimed breakdown of the 2nd law becomes questionable. At any $T = \mathrm{const.}$ surface of spacetime, the entropy of the matter falling into the black hole is still perfectly present in the Universe (except potentially for the moment, when the matter falls onto the singularity, but let us not bother with this moment of which there is no common sense understanding yet). Claiming that matter crossing the horizon breaks the 2nd law is a somewhat observer-centric

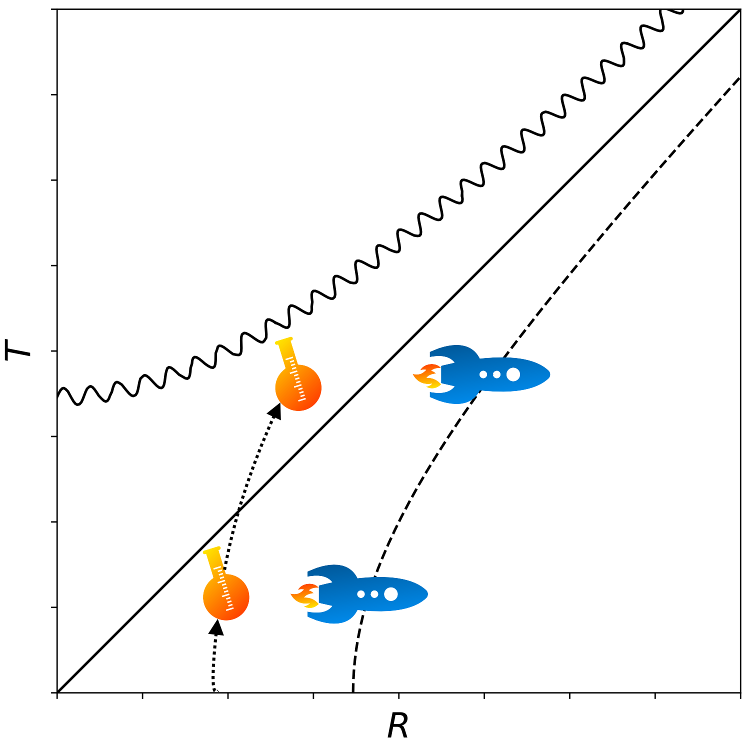

Figure 2: Information can travel at most at the speed of light, which corresponds to lines of 45° in Kruskal-Szekeres coordinates. When a container of matter - and the corresponding entropy - moves across a black hole horizon, then an outside observer loses the ability to receive any information about that container. Does that constitute a violation of the 2nd law of thermodynamics? This question is somewhat missing the point of the holographic principle.

point-of-view (which, to be honest, your lecturer holds). But it is also somewhat missing the point of the assignment of entropy to a black hole horizon! The fact that horizons seem to carry entropy is interesting not primarily because it restores the 2nd law, but e.g. for the following reasons:

A) The fact Equation 1.11 restores the 2nd law for outside observers means that for some reason the horizon area A_{bh} is an upper bound for how much entropy has been thrown into the horizon! This is actually how Bekenstein first motivated his entropy formula: he tried to through entropy into a black hole in a way that minimizes the growth of A_{bh} (cf. Section 2).

B) The entropy assignment in Equation 1.11 together with the horizon temperature that was later derived by Hawking actually makes it so that the combined system of black holes and ordinary matter don't just satisfy a 2nd law but also a *1st law*! And it was later shown by Jacobson, that demanding the validity of this 1st law for all local horizons is in fact *equivalent to the full Einstein equations!* We will investigate this in more detail in Section 4, but let us already get a glimpse of such a 1st law by looking at the Schwarzschild black hole. The inner energy of the latter will be given by its mass M, while its entropy is given by

$$S = \frac{A_{\rm bh}}{4G} = \frac{4\pi r_s^2}{4G} = 4\pi G M^2 \ . \tag{1.12}$$

We will further more see in Section 3, that the temperature of the Hawking radiation emitted by the black hole horizon is given by

$$T = 1/8\pi G M \ . \tag{1.13}$$

If we now drop an infinitesimal mass $\mathrm{d}M$ into the black hole, then temperature, entropy and inner energy satisfy the equation

$$T\mathrm{d}S = \mathrm{d}M \ . \tag{1.14}$$

This equation describes the reaction of spacetime geometry to a flow of matter across the horizon, but it also seems to represent a 1st law of horizon thermodynamics.

C) The fact that horizon area (and not e.g. the volume enclosed by the horizon) bounds the entropy inside the horizon seems to be in conflict with standard particle physics! The entropy that can be carried by a quantum field inside a given volume V is typically proportional to V and not to the boundary area of that volume.

D) Hawking radiation is an effect of quantum field theory on a fixed curved spacetime background. In particular, its derivation does not require that this background spacetime satisfies the Einstein equations. How then does quantum field theory know to produce horizon radiation with a temperature that exactly makes the Einstein equations look like a 1st law of thermodynamics? This apparent coincidence has sparked the hope that horizon thermodynamics is a pathway towards understanding quantum gravity itself. This hope is e.g. summarized by the following quote from professor Padmanabhan [15]:

> *"... if spacetime can be hot, it must have microstructure."*

In Table 1 you can find a (very incomplete!) timeline of important scientific work that lead to the above realisations A) - D).

Exercise 3

In Appendix A it is shown that a scalar quantum field in side a box of side length $\ell_{\rm box}$ can be interpreted as a collection of quantum harmonic oscillators.

a) Show that the number of these oscillators scales as the volume of the box, i.e.

$$N_{\rm osc} \sim \ell_{\rm box}^3 \ .$$

The dimension of the Hilbert space of a quantum harmonic oscillator is infinite, so the Hilbert space of a quantum field in a box must also be infinite. Assume that quantum gravitational effects regularize this Hilbert space dimension such that each oscillator only lives in a Hilbert space of dimension d (independent of Fourier mode $\boldsymbol{k}$).

b) What is the dimension of the full Hilbert space of the quantum field as a function of d?

A generalized notion of a quantum state is given by a density matrix*. This is a Hermitian operator $\hat{\rho}$ that is positive (semi-)definite and has* $\mathrm{Tr}\,\hat{\rho} = 1$*. If $\hat{\rho}$ is defined on a Hilbert space $\mathcal{H}$ of dimension d and of $\{\lambda_1\ ,\ \ldots\ ,\ \lambda_d\}$ are the eigenvalues of $\hat{\rho}$, then the von-Neumann entropy of $\hat{\rho}$ is defined as*

$$S(\hat{\rho}) := -\sum_j \lambda_j \ln(\lambda_j)\ . \tag{1.15}$$

c) *Show that the maximum entropy that any $\hat{\rho}$ on $\mathcal{H}$ can have is* $\ln(d)$*.* (Hint: the condition $\mathrm{Tr}(\hat{\rho}) = 1$ means that $\sum_j \lambda_j = 1$, which you can add to your optimization problem via the method of Lagrange multipliers.)

d) *What is the maximum entropy of a scalar quantum field in a box (assuming that the dimension of the individual oscillators is again regularized to be d)? Discuss your result in the light of Bekenstein's proposal that the maximum entropy that can be stored in a given volume is proportional to the boundary area of that volume.*

Script incomplete - will be updated regularly (last time: 28 April 25)

Table 1 (incomplete!) timeline of key developments in horizon thermodynamics

1971	Demonstration that no classical process can reduce the sum of black hole horizon areas: • Hawking 1971, "Gravitational Radiation from Colliding Black Holes"
1972-1974	Bekenstein proposes that black holes have an entropy, and that this entropy is proportional to horizon area: • Bekenstein 1972, "Black holes and the second law" • Bekenstein 1973, "Black holes and entropy" • Bekenstein 1974, "Generalized second law of thermodynamics in black-hole physics"
1974	Derivation that black hole horizons have a temperature and emit thermal radiation: • Hawking 1974, "Black hole explosions?"
1977	Discovery that entropy and temperature can also be assigned to the cosmological horizon: • Gibbons & Hawking 1977, "Cosmological event horizons, thermodynamics, and particle creation"
1981	It is proposed that the maximum entropy that can be accumulated in a spherical volume is bounded by the surface area of that volume: • Bekenstein 1981: "Universal upper bound on the entropy-to-energy ratio for bounded systems"
1995	Discovery that the entire Einstein equations follow from demanding the validity of a 1st law for the combined system of horizons & outside matter: • Jacobson 1995: "Thermodynamics of Spacetime: The Einstein Equation of State" • many works of Padmanabhan; see e.g. https://arxiv.org/pdf/0706.1654 for an overview
1999	Formulation of a generally covariant version of Bekenstein's original entropy bound (the *holographic principle*): • Bousso 1999: "A Covariant Entropy Conjecture"

1.4 Exercises & feedback form for lecture 1

Lecture 1 is accompanied by exercises 1 - 3. Also, you can find the feedback form for this lecture here:
https://cloud.physik.lmu.de/index.php/apps/forms/s/eLefQDX4LrZjweCsKtmRdD8S

Part I:
The classical holographic principle

2 Bekenstein entropy and the Bekenstein bound

3 Hawking radiation

4 The Einstein equations as a 1st law

Acknowledgments

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A The scalar field as a collection of oscillators

We closely follow the notation of [4], see also [13, 14]. The action of a massive scalar field in Minkowski space is given by

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x \,\left[\dot{\phi}^2 - (\boldsymbol{\nabla}\phi)^2 - m^2\phi^2\right]$$
$$= \frac{1}{2}\int \frac{\mathrm{d}t\mathrm{d}^3k}{(2\pi)^3}\,\left[|\dot{\phi}_{\boldsymbol{k}}|^2 - (|\boldsymbol{k}|^2 + m^2)|\phi_{\boldsymbol{k}}|^2\right] \ , \tag{A.1}$$

where in the second line we moved to Fourier space, with $\boldsymbol{k}$ labeling the Fourier modes. Expressing the Fourier transform of the field in terms of real and imaginary parts, $\phi_{\boldsymbol{k}} = A_{\boldsymbol{k}} + iB_{\boldsymbol{k}}$, we must have $A_{\boldsymbol{k}} = A_{-\boldsymbol{k}}$ and $B_{\boldsymbol{k}} = -B_{-\boldsymbol{k}}$ because ϕ is real. This allows us to define a new field

$$q_{\boldsymbol{k}} = \sqrt{2}\begin{cases} A_{\boldsymbol{k}} \text{ for } k_1 \leq 0 \\ \\ B_{\boldsymbol{k}} \text{ for } k_1 > 0 \end{cases} \tag{A.2}$$

such that the action can be re-written as [14]

$$S = \int \frac{\mathrm{d}t\mathrm{d}^3k}{(2\pi)^3}\,\left[\frac{1}{2}\dot{q}_{\boldsymbol{k}}^2 - \frac{|\boldsymbol{k}|^2+m^2}{2}q_{\boldsymbol{k}}^2\right] \ . \tag{A.3}$$

To make it explicit that this can be interpreted as a collection of harmonic oscillators, let us restrict the field ϕ to a finite box of side length ℓ_{box}, imposing periodic boundary conditions. This modifies the action to

$$S_{\text{box}} = \int \mathrm{d}t\,\frac{1}{\ell_{\text{box}}^3}\sum_{\boldsymbol{k}}\left[\frac{1}{2}\dot{q}_{\boldsymbol{k}}^2 - \frac{|\boldsymbol{k}|^2+m^2}{2}q_{\boldsymbol{k}}^2\right] \equiv \int \mathrm{d}t\; L_{\text{box}}\left(\{q_{\boldsymbol{k}}\},\{\dot{q}_{\boldsymbol{k}}\},t\right) \ , \tag{A.4}$$

where we have replaced $\mathrm{d}^3k \to \Delta k^3 = (2\pi/\ell_{\text{box}})^3$ and the sum is over all $\boldsymbol{k} = (k_1, k_2, k_3)$ with $k_i \in \{2\pi n/\ell_{\text{box}} \mid n \in \mathbb{Z}\}$. The second equality serves as a definition of the Lagrangian L_{box} of the discretized field. It is literally the Lagrangian of a set of harmonic oscillators with masses $1/\ell_{\text{box}}^3$ and frequencies $\sqrt{|\boldsymbol{k}|^2+m^2}$. The corresponding Hamiltonian is given by

$$H_{\text{box}}\left(\{q_{\boldsymbol{k}}\},\{p_{\boldsymbol{k}}\},t\right) = \sum_{\boldsymbol{k}}\left[\frac{\ell_{\text{box}}^3}{2}\,p_{\boldsymbol{k}}^2 + \frac{1}{\ell_{\text{box}}^3}\frac{(|\boldsymbol{k}|^2+m^2)}{2}q_{\boldsymbol{k}}^2\right] \ , \tag{A.5}$$

where we introduced the conjugate momenta $p_{\boldsymbol{k}} = \partial L_{\text{box}}/\partial \dot{q}_{\boldsymbol{k}}$. To obtain the quantum theory of this field one would usually promote $q_{\boldsymbol{k}}$ and $p_{\boldsymbol{k}}$ to conjugate Hermitian operators satisfying the Heisenberg commutation relations

$$[\hat{q}_{\boldsymbol{k}}, \hat{p}_{\boldsymbol{k}'}] = i\delta_{\boldsymbol{k},\boldsymbol{k}'} \tag{A.6}$$

such that the Hamiltonian operator of the field becomes

$$\hat{H}(t) = \sum_{\boldsymbol{k}}\left[\frac{\ell_{\text{box}}^3}{2}\,\hat{p}_{\boldsymbol{k}}^2 + \frac{1}{\ell_{\text{box}}^3}\frac{(|\boldsymbol{k}|^2+m^2)}{2}\hat{q}_{\boldsymbol{k}}^2\right] \ . \tag{A.7}$$

which has the minimum eigenvalue

$$\lambda_{\min}\left[\hat{H}(t)\right] = \sum_{|\boldsymbol{k}|<k_{\max}}\frac{\sqrt{|\boldsymbol{k}|^2+m^2}}{2} \ . \tag{A.8}$$

Here we have introduced an ultra-violet cut-off $k_{\rm max}$ in order to regularise this otherwise divergent expression. To obtain the vacuum energy density of the field we need to divide this eigenvalue by the volume of the box, i.e.

$$\begin{aligned}\epsilon_{\rm vac} &= \frac{1}{\ell_{\rm box}^3} \sum_{|\boldsymbol{k}|<k_{\rm max}} \frac{\sqrt{|\boldsymbol{k}|^2+m^2}}{2} \\ &\approx \int_{|\boldsymbol{k}|<k_{\rm max}} \frac{{\rm d}^3k}{(2\pi)^3} \frac{\sqrt{|\boldsymbol{k}|^2+m^2}}{2} \\ &\approx \int_{|\boldsymbol{k}|<k_{\rm max}} 2\pi \frac{{\rm d}k}{(2\pi)^3} k^3 \quad \text{for} \quad k_{\rm max} \gg m \\ &= \frac{1}{4} \frac{k_{\rm max}^4}{(2\pi)^2} \ . \end{aligned} \tag{A.9}$$

The sharp cut-off we used in the above expressions has been criticized because it breaks Lorentz symmetry [1, 11]. There are however reasons to believe that the breaking of Lorentz symmetry is physical [2, 5, 12]. What's more relevant to us: the sharp cutoff does not sufficiently regularize quantum fields to make them consistent with holography.

B The Weyl field as a collection of qubits

This section is taken from [3] and serves as a reminder of the Weyl field, which is the simples Fermionic quantum field. We also make explicit in which sense the Weyl field can be decomposed into a collection of qubits.

B.1 Weyl field basics

The (left-handed) Weyl spinor is a two-component field ψ with the Lagrangian

$$\mathcal{L} = i\psi^\dagger \sigma^\mu \partial_\mu \psi \ , \tag{B.1}$$

where $\sigma^0 \equiv \mathbf{1}$ and σ^i are e.g. the Pauli matrices (we are following here the notation of [10, 16]). The above Lagrangian leads to the equations of motion

$$\sigma^\mu \partial_\mu \psi = 0 \ . \tag{B.2}$$

A general solution to these equations can be expressed as

$$\psi(\boldsymbol{x}, t) = \int \frac{{\rm d}^3p}{(2\pi)^3 E_p} \ \left\{ a_{\boldsymbol{p}}(t) u(\boldsymbol{p}) \ e^{i\boldsymbol{px}} + b^*_{\boldsymbol{p}}(t) u(\boldsymbol{p}) \ e^{-i\boldsymbol{px}} \right\} \ , \tag{B.3}$$

where the time evolution of the coefficients $a_{\boldsymbol{p}}$ and $b^*_{\boldsymbol{p}}$ is given by

$$a_{\boldsymbol{p}}(t) = a_{\boldsymbol{p},0} \ e^{-iE_pt} \ , \ b_{\boldsymbol{p}}(t)^* = b^*_{\boldsymbol{p},0} \ e^{iE_pt} \tag{B.4}$$

with $E_p \equiv |\boldsymbol{p}|$, and where $u(\boldsymbol{p})$ are eigenvectors of the matrix $\sigma^j p_j$ with eigenvalues $+E_p$ [3] ,

$$\sigma^j p_j \cdot u(\boldsymbol{p}) = +E_p \ u(\boldsymbol{p}) \ . \tag{B.5}$$

[3]Note that $u(-\boldsymbol{p})$ is then an eigenvector with eigenvalue $-E_p$. This is why only a single family of functions $u(\boldsymbol{p})$ appears in the expansion of Equation B.3.

Note that in the following we will keep the time dependence of $a_{\boldsymbol{p}}$ and $b^*_{\boldsymbol{p}}$ implicit in our notation. The reason is that this dependence will deviate from Equation B.4 once we consider overlapping degrees of freedom.

Normalising the $u(\boldsymbol{p})$ such that

$$u(\boldsymbol{p})^\dagger \cdot u(\boldsymbol{p}) = E_p \tag{B.6}$$

ensures that they are orthogonal wrt. the Lorentz invariant momentum space measure

$$\mathrm{d}\tilde{p} := \frac{\mathrm{d}^3 p}{(2\pi)^3\ E_p}\ . \tag{B.7}$$

Up to an irrelevant phase factor we can e.g. choose $u(\boldsymbol{p})$ as [16]

$$u(\boldsymbol{p}) = \sqrt{E_p}\begin{pmatrix} e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\end{pmatrix}\ , \tag{B.8}$$

where $\boldsymbol{p} = (p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)^T$.

In the quantum version of the above field theory we consider the operator valued field

$$\hat{\psi}(\boldsymbol{x}, t) = \int \mathrm{d}\tilde{p}\ \left\{\hat{a}_{\boldsymbol{p}}(t)u(\boldsymbol{p})\ e^{i\boldsymbol{px}} + \hat{b}_{\boldsymbol{p}}(t)^\dagger u(\boldsymbol{p})\ e^{-i\boldsymbol{px}}\right\}\ , \tag{B.9}$$

where the operator $\hat{b}^\dagger_{\boldsymbol{p}}$ can be thought of as creating an anti-spinor of momentum $\boldsymbol{p}$ while $\hat{a}_{\boldsymbol{p}}$ is destroying a spinor with momentum $\boldsymbol{p}$. At equal times these operators satisfy the anti-commutation relations

$$0 = \{\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\} = \{\hat{b}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{q}}\} = \{\hat{a}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{q}}\} = \{\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\} \tag{B.10}$$

$$\{\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\} = \{\hat{b}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\} = (2\pi)^3 E_p\ \delta_D(\boldsymbol{p}-\boldsymbol{q})\ . \tag{B.11}$$

This ensures that the field operators satisfy the equal-time anti-commutation relation

$$\{\hat{\psi}(\boldsymbol{x}), i\hat{\psi}(\boldsymbol{y})^\dagger\} \tag{B.12}$$

$$= i\int \mathrm{d}\tilde{p}\mathrm{d}\tilde{q}\ \left[\{\hat{a}_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\}\ u(\boldsymbol{p})u(\boldsymbol{q})^\dagger + \{\hat{b}^\dagger_{-\boldsymbol{p}}, b_{-\boldsymbol{q}}\}\ u(-\boldsymbol{p})u(-\boldsymbol{q})^\dagger\right] e^{i\boldsymbol{px}-i\boldsymbol{qy}}$$

$$= i\int \frac{\mathrm{d}^3p}{(2\pi)^3 E_p}\ \left[u(\boldsymbol{p})u(\boldsymbol{p})^\dagger + u(-\boldsymbol{p})u(-\boldsymbol{p})^\dagger\right] e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}$$

$$= i\ \mathbf{1}_{2\mathrm{D}}\ \delta_D(\boldsymbol{x}-\boldsymbol{y})\ , \tag{B.13}$$

as is needed because $i\hat{\psi}^\dagger$ is the conjugate momentum of the field $\hat{\psi}$.

B.2 Decomposition into qubits

To make it explicit that the above field can be considered as a collection of qubits, let us constrict $\psi(\boldsymbol{x})$ to a box of finite size L . This means that we have to perform the substitutions

$$\mathrm{d}^3 p \rightarrow \frac{(2\pi)^3}{L^3}\ ,\ \delta_D(\boldsymbol{p}-\boldsymbol{q}) \rightarrow \frac{L^3}{(2\pi)^3}\delta_{\boldsymbol{p},\boldsymbol{q}}\ , \tag{B.14}$$

and replace integrals with sums over the discrete grid $\boldsymbol{p} \in \{\ \frac{2\pi}{L}(n_1, n_2, n_3)\ |\ n_i \in \mathbb{Z}\ \}$. For convenience, we will also consider redefined mode operators

$$\hat{c}_{\boldsymbol{p}} = \frac{\hat{a}_{\boldsymbol{p}}}{(E_p V)^{\frac{1}{2}}}\ ,\ \hat{d}_{\boldsymbol{p}} = \frac{\hat{b}_{\boldsymbol{p}}}{(E_p V)^{\frac{1}{2}}}\ , \tag{B.15}$$

such that the new operators satisfy the anti-commutation relations

$$0 = \{\hat{c}_{\boldsymbol{p}}, \hat{c}_{\boldsymbol{q}}\} = \{\hat{d}_{\boldsymbol{p}}, \hat{d}_{\boldsymbol{q}}\} = \{\hat{c}_{\boldsymbol{p}}, \hat{d}_{\boldsymbol{q}}\} = \{\hat{c}_{\boldsymbol{p}}, \hat{d}^\dagger_{\boldsymbol{q}}\} \tag{B.16}$$

$$\{\hat{c}_{\boldsymbol{p}}, \hat{c}^\dagger_{\boldsymbol{q}}\} = \{\hat{d}_{\boldsymbol{p}}, \hat{d}^\dagger_{\boldsymbol{q}}\} = \delta_{\boldsymbol{p},\boldsymbol{q}} \ . \tag{B.17}$$

Our field can now be decomposed in terms of these operators as

$$\hat{\psi}(\boldsymbol{x}, t) = \sum_{\boldsymbol{p}} \frac{1}{(E_p V)^{\frac{1}{2}}} \left\{ \hat{c}_{\boldsymbol{p}}(t)\ u(\boldsymbol{p})\ e^{i\boldsymbol{px}} + \hat{d}_{\boldsymbol{p}}(t)^\dagger\ u(\boldsymbol{p})\ e^{-i\boldsymbol{px}} \right\} \ . \tag{B.18}$$

Usually, each of the grid points $\boldsymbol{p}$ in the above sum represents a 4-dimensional Hilbert space factor

$$\mathcal{H}_{\boldsymbol{p}} = \mathcal{H}^c_{\boldsymbol{p}} \otimes^{\mathrm{JW}} \mathcal{H}^d_{\boldsymbol{p}} \tag{B.19}$$

and the total Hilbert space is the tensor product over all these factors,

$$\mathcal{H} = \bigotimes_{\boldsymbol{p}}^{\mathrm{JW}} \mathcal{H}_{\boldsymbol{p}} \ , \tag{B.20}$$

where the superscript "JW" again indicates that operators in the individual Hilbert spaces need to be embedded into the product space via Jordan-Wigner-factors in order for them to anti-commute (as opposed to commute). The $\hat{c}_{\boldsymbol{p}}$, $\hat{d}_{\boldsymbol{p}}$ and their Hermitian conjugates act non-trivially only on the factors $\mathcal{H}^c_{\boldsymbol{p}}$ and $\mathcal{H}^d_{\boldsymbol{p}}$ respectively. On the factor $\mathcal{H}^c_{\boldsymbol{p}}$ (and similarly for $\mathcal{H}^d_{\boldsymbol{p}}$) we can define

$$\hat{\sigma}^c_{x,\boldsymbol{p}} = \hat{c}_{\boldsymbol{p}} + \hat{c}^\dagger_{\boldsymbol{p}} \tag{B.21}$$

$$\hat{\sigma}^c_{y,\boldsymbol{p}} = i\left(\hat{c}_{\boldsymbol{p}} - \hat{c}^\dagger_{\boldsymbol{p}}\right) \tag{B.22}$$

$$\hat{\sigma}^c_{z,\boldsymbol{p}} = -i\hat{\sigma}^c_{x,\boldsymbol{p}}\hat{\sigma}^c_{y,\boldsymbol{p}} = 2\hat{c}^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}} - 1 \ . \tag{B.23}$$

These operators constitute a Pauli algebra on the qubit Hilbert space $\mathcal{H}^c_{\boldsymbol{p}}$. Note however, that the labels x, y, z are simply notation, and not meant to indicate directions in physical space. The Hamiltonian of the field can be expressed in terms of these operators as

$$\begin{aligned} \hat{H} &= \textstyle\sum_{\boldsymbol{p}} E_p \left\{ \left(\hat{c}^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}} - \frac{1}{2}\right) + \left(\hat{d}^\dagger_{\boldsymbol{p}}\hat{d}_{\boldsymbol{p}} - \frac{1}{2}\right) \right\} \\ &= \textstyle\sum_{\boldsymbol{p}} \frac{E_p}{2} \left\{ \hat{\sigma}^c_{z,\boldsymbol{p}} + \hat{\sigma}^d_{z,\boldsymbol{p}} \right\} \ . \end{aligned} \tag{B.24}$$

So our field behaves like a set of non-interacting spins in a $\boldsymbol{p}$-dependent magnetic field. Of course, the occupation of different Fourier modes $\boldsymbol{p}$ of the Weyl field does not measure the state of any actual spins, but rather the existence or non-existence of particles with momentum $\boldsymbol{p}$.