

Fortuitous Lines of Sight:
The Role of Gravitational Lensing
in Cosmological Predictions

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Abstract

The studies presented here begin with an investigation into strong lensing on cluster scales. The observed frequency of multiple imaging can be used as a statistic for tests of structure formation and cosmology. However, the interpretation of this statistic is subject to complex issues surrounding the modelling of the lens. How do the triaxial shapes of clusters affect the frequency of multiple imaging? What is the role of the filamentary structures that feed cluster haloes? Do they enhance the orientation dependence? I explore these issues by characterising the relevant statistic by the cross section for multiple imaging, σ_{MI} , for a sample of simulated cluster haloes.

The cosmological applications of gravitational lensing extend to predicting the weak lensing properties of large scale structure, in particular, the magnification probability distribution for background sources. Such studies generally require ray-tracing through cosmological N-body simulations to generate model predictions, but standard methods apply the multiple thin-lens approximation. I take the first steps in quantifying the accuracy of standard methods by developing an innovative code that performs ray-tracing without the use of this approximation. The efficiency and accuracy of this computationally challenging approach can be improved by careful choices of numerical parameters; therefore, the results are analysed for the behaviour of the ray-tracing code in the vicinity of Schwarzschild and Navarro-Frenk-White lenses. Preliminary comparisons to a multiple lens-plane algorithm are drawn in the context of cosmological mass distribution for a source redshift of $z_s = 0.5$.

Finally, I analyse the extent to which gravitational lensing could affect future cosmological tests. The evolution of the expansion rate of the Universe results in a drift in the redshift of distant sources over time. The redshift-drift experiment has been touted as a direct test of cosmological expansion, and carries with it the potential to distinguish between various cosmological models. The Lyman α forest has been recognized by Loeb as the best candidate for this experiment, but the signal would be weak and it will take next generation large telescopes coupled with ultra-stable high resolution spectrographs to reach the cm s^{-1} resolution required. I assess the effect of gravitational lensing by Lyman α absorbers, particularly the collapsed structures associated with Lyman limit systems and damped Lyman α systems; the transverse component of their motion varies the gravitational potential in the line of sight and subsequently shifts the positions of background absorption lines, an extension of the the transverse moving lens effect determined by Birkinshaw & Gull. Will it prevent detection of the cosmic signal?

Statement of Originality

This thesis describes work carried out in the Sydney Institute for Astronomy, within the School of Physics, University of Sydney, between March 2007 and November 2010. The work presented in this thesis is, to the best of my knowledge and belief, original except as acknowledged in the text. Specific contributions by my collaborators are detailed below:

- I ran all the N-body simulations for the studies presented in Chapters 3 and 4. However, the algorithm for generating initial conditions was written by my associate supervisor, Chris Power.
- In Chapter 4, the ray-tracing code was written entirely by myself. Part of the design was based on the ray-bundle method (RBM), as noted in the text. The multiple lens-plane RBM results shown in Figures. 4.4 and 4.5 were generated by my collaborator, Paul Lasky.

I hereby declare that I have not submitted this material, either in full or in part, for a degree or diploma at this university or any other institution.

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Madhura Killedar

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Date

Included Publications

Some of the original research contained in this thesis is presented in the form of a journal article that has been published in a peer-reviewed journal. It consists of research conducted by myself, in consultation with my supervisor Geraint Lewis.

Ly α absorbers in motion: consequences of gravitational lensing for the cosmological redshift drift
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Chapter 1

Introduction

Modern cosmology has provided us with a well-defined picture of the Universe that successfully describes a huge range of observed phenomena. The Universe, or so it appears, is flat, homogeneous and isotropic on large scales; its expansion history has been mapped out over a wide range of redshifts thanks to surveys of ‘standard candle’ supernovae. The familiar luminous component, in the form of stars and gas, accounts for but a fraction of the total matter-energy budget; the bulk is in the form of a mysterious dark component; we dub this description the Lambda Cold Mark Matter (Λ CDM) model. While the nature of dark matter and dark energy remain elusive, several fundamental properties have been constrained. Dark matter dominates the mass in galaxy clusters and large-scale structure, and these structures leave an imprint on the images of background sources via the gravitational lensing effect. In addition, cosmic distance measures have a strong dependence on dark energy. Hence, current and upcoming weak lensing, strong lensing and quasar surveys will confront Λ CDM predictions, which are constructed with a combination of cosmological N-body simulations and a range of analytical and numerical tools. The present study investigates a selection of these predictions on a wide range of scales. I begin with a detailed overview of the aforementioned standard cosmological model, theory of structure formation and the gravitational lensing effect.

1.1 The Cosmological Model

The [standard Λ CDM cosmological model] continues to be an exquisite fit to the existing data.

– Komatsu et al. (2011)

In the following section, I outline the standard concordance cosmology framework that underlies the present set of studies. This includes fundamental principles of General Relativity and homogeneity, the preferred interpretation of redshift, and the components of cosmological fluid that eventually determine structure formation and cosmic expansion.

1.1.1 General Relativity

The General Theory of Relativity (GR) is the framework within which we interpret how and why objects move. Here, the force of gravity is the result of the curvature of space-time, which in turn is governed by the density distribution of various matter-energy components. The equivalence principle is central to GR; it states that within a freely-falling frame, a test body follows a trajectory that is consistent with the laws of special relativity and is independent of inertial mass. This trajectory is known as the geodesic.

GR proved successful in that it was able to explain the perihelion shift of Mercury, then an outstanding issue. In addition, the first observation of gravitational lensing by the Sun (Dyson et al., 1920) confirmed Einstein’s 1919 prediction of the magnitude of light deflection (see Section 1.4 and Appendix B.1), which was twice the Newtonian prediction. But the true success of GR has emerged in the intervening years with numerous experiments providing evidence to support the theory. One such piece of evidence is the laboratory-based observation of gravitational redshift (Pound & Rebka, 1960; Pound & Snider, 1964). Another is the time-delay and deflection of radio photons around the Sun, which supports GR at the $10^{-4} - 10^{-5}$ level (Shapiro et al., 2004; Bertotti et al., 2003); experiments such as this, carried out within the Solar System, test gravity only in the weak-field, low-velocity regime. In addition, long-term monitoring has revealed a gradual decrease in the orbital period of binary pulsars as energy is lost through the emission of gravitational waves (Taylor & Weisberg, 1989; Taylor, 1994; Stairs et al., 1998); consistency with the strong-field GR prediction is subject to the accuracy of the measured distance to the binary system. Although alternative gravity theories abound in the literature (e.g. Brans & Dicke, 1961; Buchdahl, 1970; Bekenstein, 2004; Dvali et al., 2000; Capozziello & Fang, 2002), in this thesis I work within the framework of GR.

1.1.2 Principles of Cosmology

The Cosmological Principle states that our Universe is homogeneous and isotropic everywhere (Milne, 1933). It is derived from the Copernican Principle, which is the assumption that we are not privileged observers. Isotropy and homogeneity on $\gtrsim 100$ Mpc scales has been observed in large galaxy redshift surveys, such as the 2-degree Field Galaxy Redshift Survey (2dFGRS; Peacock et al., 2001; Cole et al., 2005) and the Sloan Digital Sky Survey (SDSS; Seven-Year Data Release; Abazajian et al., 2009), as well as the temperature maps of the cosmic microwave background radiation (CMB) from the Wilkinson Microwave Anisotropy Probe (Seven-Year Data Release WMAP7; Jarosik et al., 2011). See Figure 1.1 for images from these observations. When these observed properties are combined with the Copernican Principle, the Cosmological Principle follows. The Friedmann-Lemaître-Robertson-Walker (FLRW) geometry encompasses this principle; when combined with knowledge of the densities and properties of the various matter-energy components that constitute the Universe, it describes its history and predicts its evolution and eventual fate. On smaller scales, however, there is much structure in the form of galaxies, clusters and superclusters, linked via vast filamentary structures. Modern cosmology also assumes that this smaller-scale inhomogeneity does not falsify the large-scale predictions made under FLRW geometry.

1.1.3 Cosmological Redshift and the Expansion of Space

The concept of a ‘fundamental observer’ who has no motion relative to the cosmological fluid—in other words, has zero *peculiar velocity*—is generally introduced when describing cosmic expansion. The fixed coordinates of such observers are called *comoving coordinates*. The time-dependent *scale factor*, $R(t)$, defines the conversion between the coordinate separation and spatial separation of (or proper distance between) two fundamental observers on a hypersurface of constant time; this parameter characterizes the (large-scale) dynamics of the Universe. The FLRW line-element is given by

$$ds^2 = -dt^2 + R^2(t) [d\chi^2 + S_k^2(\chi) (d\theta^2 + \sin^2\theta d\phi^2)], \quad (1.1)$$

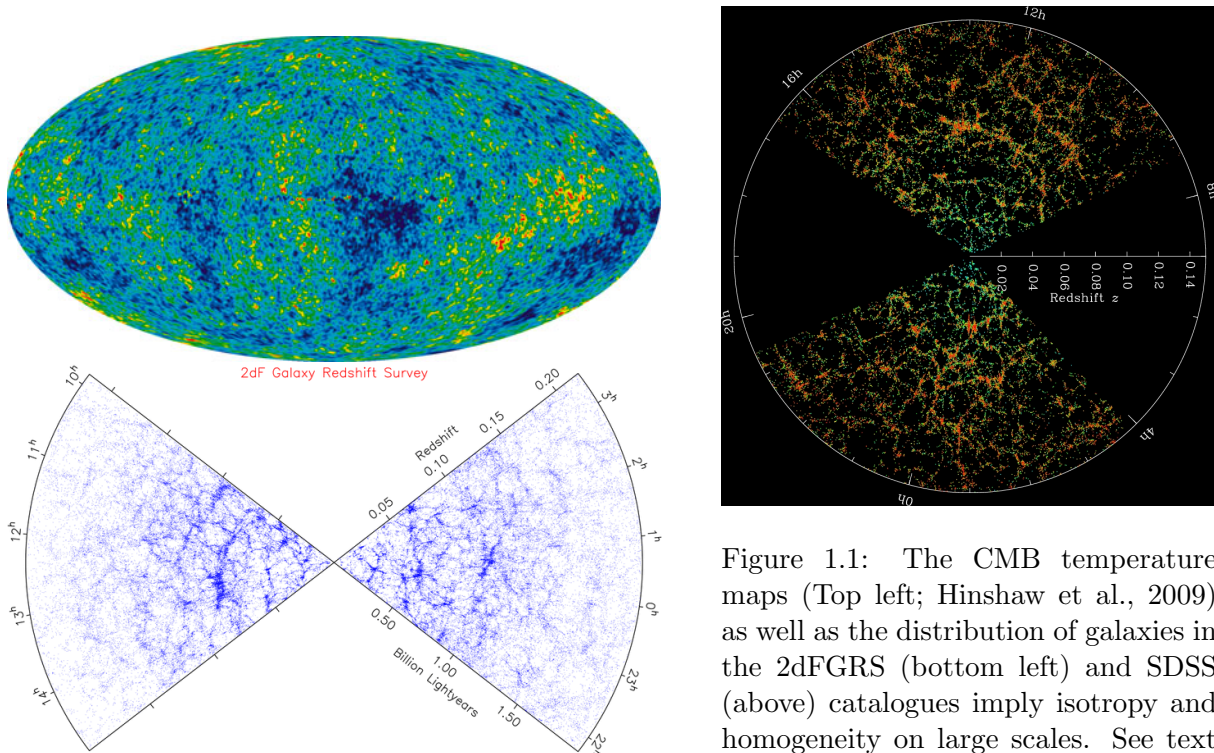


Figure 1.1: The CMB temperature maps (Top left; Hinshaw et al., 2009) as well as the distribution of galaxies in the 2dFGRS (bottom left) and SDSS (above) catalogues imply isotropy and homogeneity on large scales. See text for further references.

where χ , θ and ϕ are comoving coordinates. The topology-specific $S_k(\chi)$ is given by

$$S_k(\chi) = \begin{cases} \sin\chi & \text{if } k = 1 \\ \chi & \text{if } k = 0 \\ \sinh\chi & \text{if } k = -1 \end{cases}, \quad (1.2)$$

where the spatial curvature parameter $k=1/0/-1$ defines a closed/flat/open universe. Notice that the metric allows an evolving scale factor. The Friedmann (-Lemaître) equations are:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda c^2}{3}; \quad (1.3a)$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}; \text{ and} \quad (1.3b)$$

$$\dot{\rho}_i = -\frac{3\dot{R}}{R} \left(\rho + \frac{p}{c^2}\right). \quad (1.3c)$$

Λ , G and c denote the cosmological constant, the gravitational constant and the speed of light respectively. The first two of the Friedmann equations are the differential equations that determine the time evolution of the scale factor with respect to the pressure, p , and density, ρ , of the cosmological fluid and the relative densities of the various fluid components. The last is the continuity equation for the cosmological fluid and expresses energy-momentum conservation in the case of a flat space-time with gravity neglected.

The phenomenon of cosmological redshift is interpreted as being the decrease in the energy of a photon as a result of the increase in the scale factor over time. There are only a handful

of observations that can directly test if this interpretation is appropriate; these are: the time dilation effect on supernova light curves (Wilson, 1939), the redshift dependence of the surface brightness (SB) of galaxies (the Tolman SB test; Tolman, 1930, 1934; Hubble & Tolman, 1935), the redshift dependence of CMB temperature (e.g. Bahcall et al., 1973) and the change in the observed redshift of a single object over time (Sandage, 1962; McVittie, 1962; Loeb, 1998). Given this interpretation, the relationship between cosmological redshift, z , and the dimensionless scale factor, a , is:

$$a(t) \equiv \frac{R(t)}{R(t_0)} = \frac{\lambda}{\lambda_{obs}} \equiv \frac{1}{1+z}, \quad (1.4)$$

where t_0 is the present epoch and λ_{obs} is the observed wavelength of a photon with a rest wavelength λ . Redshift is thus observed as a shift in the wavelength of absorption or emission lines of well-known transitions in the spectra of extra-galactic objects and is often interpreted as a radial velocity. Hubble (1929) mapped the relationship between recession velocities and luminosity distances of galaxies within a few megaparsecs, finding a linear correlation. Although the scales probed were small and included large systematic errors from peculiar velocities, the first ‘Hubble diagram’ was, nevertheless, a catalyst for the acceptance the expanding-universe hypothesis. The evolution of the scale factor is encapsulated in the Hubble Parameter, $H(t)$, which is defined by:

$$H(t) = \frac{\dot{R}(t)}{R(t)}, \quad (1.5)$$

and is given by the first of the Friedmann equations (Eqns.1.3a). Observations of ‘standard candle’ type Ia supernovae (SNIa) have been essential in mapping out the functional form of $a(t)$ by essentially extending the diagram out to higher redshifts. Such observations reveal a late-time acceleration under the assumption of a metric theory of gravity (Riess et al., 1998; Perlmutter et al., 1999). This follows a period of deceleration at $z \gtrsim 0.5$ (Riess et al., 2004).

1.1.4 Concordance Cosmology

There are several matter-energy components of the cosmological fluid: radiation, curvature, pressureless matter and dark energy; each has a distinct equation of state $w = p/\rho$. Radiation dominated the early universe, but is now almost negligible. A fluid with $w < -1/3$, which dominates the current epoch, is required to explain the late time acceleration (see Section 1.1.3); a ‘dark energy’ with negative pressure is the proposed solution. If characterised as a Cosmological Constant, Λ , dark energy has an equation of state $w = -1$. Matter is predominantly in the form of a pressureless, non-relativistic ‘dark matter’ (CDM), which is discussed further in section 1.2; only a small fraction is baryonic. Together, these components determine the evolution and dynamics of the Universe, so there are a wealth of studies aimed at measuring the distribution of the energy budget of the Universe amongst its various constituents. The parameters of interest here are the densities of radiation, Ω_r , matter, Ω_M and dark energy, Ω_Λ , each relative to the critical density for closure: $\Omega_i \equiv \rho_i/\rho_c$ for each of the components denoted by the subscript i . The critical density is given by

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}. \quad (1.6)$$

Finally, the curvature density parameter is defined by $\Omega_k = 1 - \Omega_M - \Omega_\Lambda - \Omega_r$ in the absence of any other contributions to the cosmological fluid. An extra subscript 0 may be added to denote the densities at the current epoch.

Various complementary observational probes have been used to constrain combinations of these parameters; together they are able to break degeneracies and find a concordance in the values. For example, Big Bang Nucleosynthesis (BBN) finds constraints on the baryon fraction,

while large structures, such as galaxy clusters, are used to estimate the baryonic to dark matter ratio. The matter power spectrum measured in the CMB by the Cosmic Microwave Background Explorer (COBE) and WMAP along with BBN fit a total matter density of $\Omega_{M,0} \sim 0.33$, well below critical density, and yet CMB measurements also point to a flat curvature $k = 0$. A combination of CMB, baryonic acoustic oscillations (BAOs) and SNIa measurements (summarized in Figures 1.2, 1.3 and 1.4) find the best fit parameters to be $\Omega_{M,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$, $\Omega_{k,0} = 0$ and $\Omega_{r,0} = 0$. These and other parameter measurements can be found in Komatsu et al. (2011).

There are other cosmological parameters that describe the global dynamics of the Universe and structure formation within it. Several measurements are normalised against these parameters. One is σ_8 , the normalisation of the matter density power spectrum. The other is the Hubble Constant, H_0 , the current value of the Hubble Parameter (see Eqn. 1.5). To simplify scaling between different cosmological models, the quantity $h \equiv H_0/100 \text{ km s}^{-1} \text{ Mpc}^{-1}$ is often used. H_0 has been determined to better than 10 percent accuracy by various independent measurements. For example, the results from the Hubble Key Project find $H_0 = 72 \pm 8 \text{ km/s/Mpc}$ (Freedman et al., 2001). Refsdal (1964) proposed a method of measuring H_0 using measurements of the differential time-delay between multiple gravitationally lensed images of a variable background source (see Sec. 1.4.1). In practice, this method has proved challenging; the measured values of H_0 depends greatly on the choice of lens model (e.g. Falco et al., 1985; Keeton & Kochanek, 1997; Schechter et al., 1997) and assumed cosmology (e.g. Kundic et al., 1997; Koopmans & Fassnacht, 1999), leading to a large range of values quoted in the literature. Recent time-delay measurements of the gravitational lens B1608+656 yield a value of $H_0 = 70.6 \pm 3.1 \text{ km/s/Mpc}$; the lens model is based on substantial observational data and the matter and dark energy density are fixed to $\Omega_{M,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$ with the dark energy assumed to be in the form of a cosmological constant (Suyu et al., 2010).

1.2 Dark Matter

Is there more to a galaxy than meets the eye [...]?

– Faber & Gallagher (1979)

Within the standard Λ CDM model, most of the matter content of the Universe is in the form of a cold, collisionless, non-baryonic matter. The dark matter is primarily responsible for large-scale structure formation. It generates gravitational potential wells into which baryonic matter collects. As a result, the galaxies and galaxy clusters that are visible to us are all hosted by dark matter haloes that extend much farther than the luminous structure. Yet the nature of dark matter remains a mystery. Here, I review the observational evidence that has led to the success of this proposal and list some of its leading candidates.

1.2.1 The Evidence

The requirement for a large quantity of non-luminous matter was proposed when the velocity dispersions of Coma and Virgo galaxy clusters were shown to imply exceptionally large mass-to-light ratio for member galaxies (Smith, 1936; Zwicky, 1937). These studies acknowledged that the baryonic intergalactic matter (IGM) had not been accounted for, but subsequent measurements of X-ray emission deemed the ionized gas insufficient to bind the clusters (Holberg et al., 1973). Many years later, several independent studies of the rotation curves of nearby spiral galaxies agreed that the rotational velocities remained constant out to several tens of kiloparsecs (Roberts & Rots, 1973; Rubin et al., 1978; Bosma, 1978; Krumm & Salpeter, 1979).

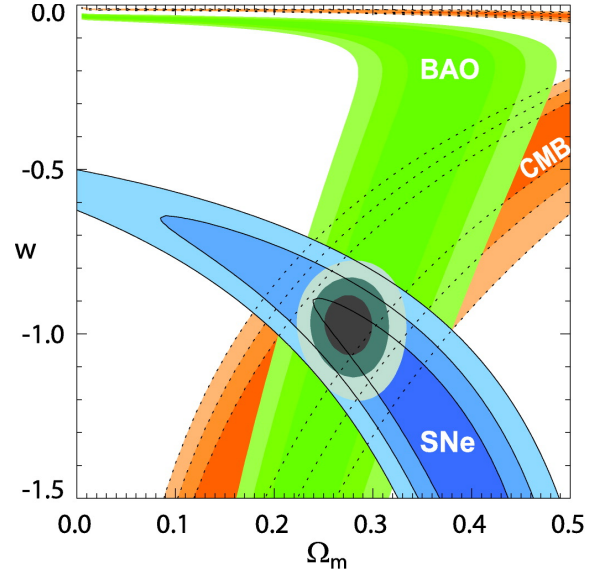
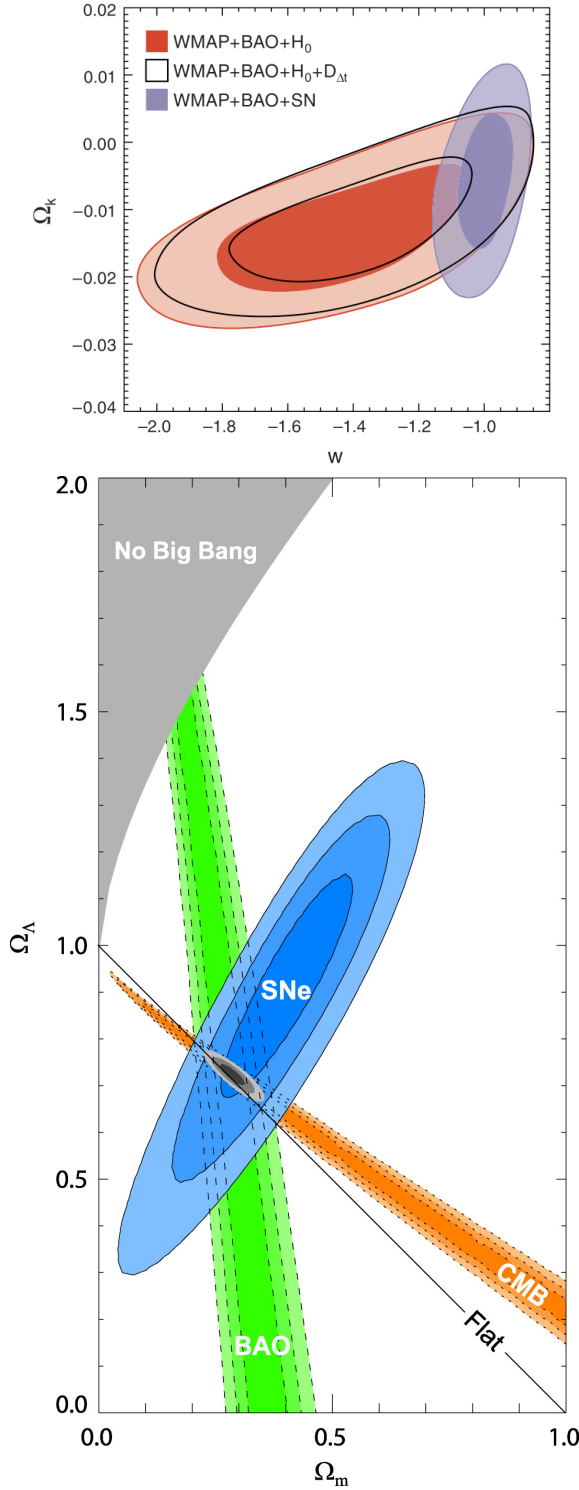


Figure 1.2: (Above): Contours at the 68.3%, 95.4% and 99.7% confidence levels on w and Ω_M for a flat universe. The plot shows the individual constraints from WMAP5, BAO and the Union SN set, as well as the combined constraints (filled gray contours, statistical errors only). Taken from Kowalski et al. (2008).

Figure 1.3: (Above Left): Joint two-dimensional marginalized constraint on w and Ω_k . The contours show the 68% and 95% CL from WMAP7+BAO+ H_0 (red), WMAP7+BAO+ H_0 + $D\Delta t$ (black) and WMAP7+BAO+SN (purple). Taken from Komatsu et al. (2011).

Figure 1.4: (Bottom Left): Contours at 68.3%, 95.4%, and 99.7% confidence level on Ω_Λ and Ω_M obtained from WMAP5, BAO and the Union SN set, as well as their combination (assuming $w = -1$). Taken from Kowalski et al. (2008).

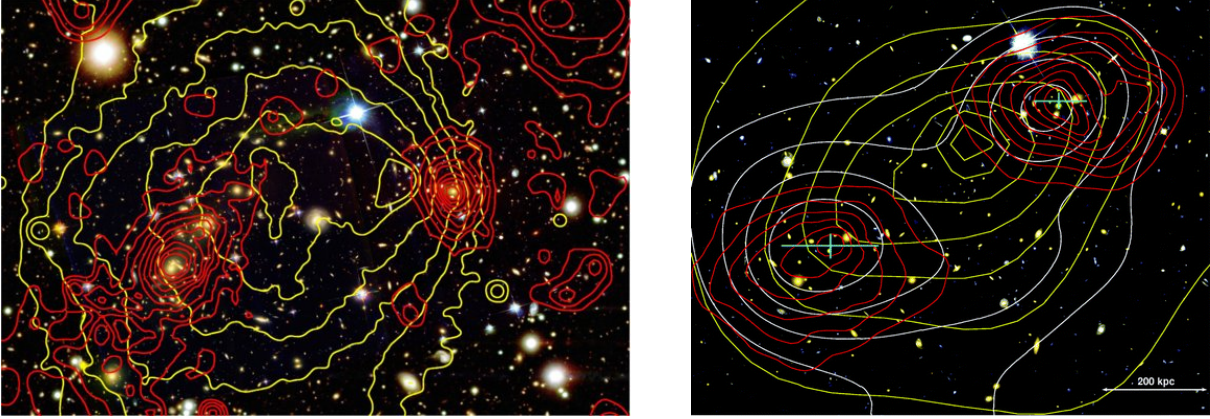


Figure 1.5: The “Bullet Cluster” 1E0657-558 (left; Bradač et al., 2009) and the MACS J0025.41222 (right; Bradač et al., 2008) reveal clusters in a post-collision state. The background galaxy images are optical colour composites. Overlaid in red contours is the surface mass density κ for a high redshift source from the combined weak and strong lensing mass reconstruction; X-ray brightness contours are overlaid in yellow. The separation between the gravitational mass peaks and X-ray peaks - the former coinciding with the peak in the galaxy distribution - is an important piece of evidence in favour of the existence of dark matter.

These flat rotation curves gave momentum to the suggestion that galaxies include a massive dark halo component (Einasto et al., 1974; Ostriker et al., 1974). Subsequent discoveries of gravitational lens galaxies (see Sec. 1.4) provided another measure of gravitational mass (Walsh et al., 1979; Weymann et al., 1980), which also implied large mass-to-light ratios (e.g. Adams & Boroson, 1979). The agreement between gravitational and dynamical mass provided support to the notion that luminous matter did not provide the full description of the dominant substance in objects on both galactic and cluster scales.

More recently, post-collision images of galaxy clusters that have merged along an axis in the plane of the sky exhibit a clear separation between the peaks in baryonic matter (via X-ray gas temperature) and the gravitational mass (via gravitational lensing) (Clowe et al., 2006; Bradač et al., 2008). In a well-known case of 1E 0657-558, nicknamed the Bullet Cluster (see Figure 1.5), this separation indicated that most of the gravitational mass was of a collisionless nature; the peak in the gravitational mass distribution matched with the peak in the (sparse) galaxy distribution, rather than the hot X-ray gas, the dominant baryonic component (Clowe et al., 2006). The results have been presented as direct evidence for the existence of dark matter.

1.2.2 Particle Candidates and Properties

Dark matter in the generic sense, i.e. matter that is not luminous, does not necessitate the introduction of an exotic particle. Several baryonic candidates have been proposed and considered as solutions to explain observed discrepancies; these include large Jupiter-sized planets, brown dwarfs, low mass stars ($10^{-3} < m_s/M_\odot < 0.1$) and low-mass black holes and stellar remnants, which are collectively referred to as massive astrophysical compact halo objects (MACHOs). MACHOs could, in principle, be detected via their microlensing effect on extragalactic sources (Press & Gunn, 1973; Paczynski, 1986; Griest, 1991). More exotic candidates include sterile massive τ -neutrinos (Dodelson & Widrow, 1994), which are massive enough to allow large-scale

structure formation, and axions (Peccei & Quinn, 1977), but the clear front-runner is the supersymmetric weakly interacting massive particle (WIMP). Laboratory experiments aimed at detecting specific candidates have constrained the particle cross sections and/or rest masses (e.g. Ahmed et al., 2009; Overduin & Wesson, 2004). Lensing studies of galaxy clusters complement the experimental approach and have constrained the self-interaction cross section of dark matter on large scales (e.g. Miralda-Escudé, 2002; Markevitch et al., 2004).

1.3 Galaxy Clusters

Galaxy clusters represent the place where astrophysics and cosmology meet

– Borgani et al. (2008)

Galaxy clusters are the largest known gravitationally bound objects in the Universe. As such, they serve as useful laboratories for studies of the growth of structure, gas physics, galaxy formation and cosmology. Given clusters emit across a broad range of wavelengths, we are provided with various observational probes for individual study, and the opportunity to perform numerous complementary surveys with which to formulate statistics such as the mass function, correlation functions and redshift evolution of cluster properties. Analytical models provide limited insight into the nonlinear growth of structure; numerical simulations are necessary for the inclusion of complex processes into our models. Indeed, synthetic clusters have indicated that galaxy clusters are far from being isolated bodies; they are overdensities that have resulted from gravitational collapse, grow through mergers with other galaxies, groups and clusters, display a wealth of substructure and continue to be fed by matter flowing through filamentary structures (Colberg et al., 1999). Neither are they identical nor self-similar; the variations encountered in their properties has all but destroyed the notion of a ‘*typical cluster*’. Some observed clusters are undergoing a merger, leading to double-peaked density distributions; in the latter stages of the merger, this dynamical event may manifest itself simply by a transient ellipticity in the mass distribution along the merger axis. However, with careful analysis of the large database of clusters collected from simulations, we may begin to develop meaningful interpretations of our observational findings.

1.3.1 Structure Formation

In the hierarchical picture of structure formation, small masses form via gravitational instability acting on primordial density fluctuations; they then merge with other collapsed structures and assemble into the massive virialised objects we see at the present epoch (Press & Schechter, 1974; White & Rees, 1978). The initial gravitational collisionless collapse is often described by the spherical infall model (see Sec. 1.3.2). The subsequent growth of cluster-scale haloes via accretion of smaller haloes, dubbed subhaloes, are demonstrated by N-body simulations; high mass resolution simulations show that a virialised halo at the current epoch can contain many hundreds of subhaloes (Moore et al., 1998; Ghigna et al., 2000). However, present day substructures are but remnants of the subhaloes that were accreted by the host and suffered from gravitational heating, dynamical friction and tidal stripping during the infall process (e.g. Ghigna et al., 1998). The highly pressurized intra-cluster medium (ICM) strips gas from these merging substructures, thereby reducing their mass (e.g. Dolag et al., 2009). The effect is also apparent from the comparison of galaxies in cluster environments to those in the field; the former have lower masses and smaller truncation radii than the latter for the same morphological type or luminosity (Brainerd et al., 1996; dell’Antonio & Tyson, 1996; Natarajan et al., 1998; Wilson et al., 2001; Natarajan et al., 2002). Only a fraction ($\lesssim 10\%$) of accreted subhaloes survive these

destructive processes (Gao et al., 2004), but the chance of survival is enhanced for subhaloes that are more concentrated (White & Rees, 1978).

1.3.2 Define “Mass of...”

The ‘total mass’ of a cluster is not a well defined quantity - not surprising since their host haloes have ill-defined boundaries - but there exist some useful working definitions. The need for a strict definition is encountered when analysing, say, the cluster mass function or scaling relations between different mass proxies. Problems arise when light is used as a tracer for mass and assumptions like hydrostatic equilibrium or spherical symmetry are applied. In this thesis, only a rough estimate of mass is required.

The collapse of a (spherical) halo is generally modelled by following the evolution of a spherical top-hat perturbation (Gunn & Gott, 1972); a spherical region with a uniform overdensity that exceeds some threshold initially evolves along with the background universe, but eventually overcomes the expansion, collapses and virializes. The so-called virial overdensity, Δ_c , is defined with respect to the critical density *at the cluster formation redshift*. The solution for a flat, radiation-free cosmology ($\Omega_k = 0$, $\Omega_r = 0$, $0.1 < \Omega_M(z) < 1$; Eke et al., 1996) has been fitted with 1% accuracy to a relatively simple function (Bryan & Norman, 1998):

$$\Delta_c(z) = 18\pi^2 + 82[\Omega_M(z) - 1] - 39[\Omega_M(z) - 1]^2. \quad (1.7)$$

Strictly, there is a cosmological dependence as well as variation among haloes depending on their individual formation history. However, a virial overdensity of $\Delta_c = 200$ is a convenient, physically motivated, and oft-used, choice. For further discussion on the choice of virial overdensity, see Lahav et al. (1991) and White (2002). The region of the cluster which is in virial equilibrium is contained in the virial radius, r_{vir} , and the virial mass, M_{vir} , is simply the total mass within this radius, i.e.

$$M_{\text{vir}} \equiv M(< r_{\text{vir}}) = \frac{4}{3}\pi\Delta_c\rho_c r_{\text{vir}}^3. \quad (1.8)$$

1.3.3 Density Profile

The density profiles of the dark matter haloes that host galaxy clusters are one of the formative tests of the Λ CDM model. The distribution and abundance of giant luminous arcs (a result of lensing by the cluster), weak lensing, X-ray emission and the velocity profiles of cluster members are distinct observables that are sensitive to the shape of the cluster on various scales. N-body simulations have been essential in providing the theoretical framework for this test. For many decades, the simple power law profile, $\rho(r) \propto r^{-\beta}$, with a flat core was sufficient for cluster studies using analytical models and observational fits to galaxy number counts (Gunn & Gott, 1972; Yahil, 1974). The power law function was eventually replaced by profiles that allowed the slope of the density profile to steepen with radius. With the advent of N-body simulations, theoretical studies of cluster-sized haloes have been able to move beyond the spherical approximation and introduce tidal fields, triaxiality, substructure and redshift evolution into the analysis; however, the choice of function that one should fit to the density profile remains open to debate.

Navarro et al. (1995) analysed simulated dark matter haloes in the CDM cosmology, finding that across four decades in mass (from galaxy to cluster sizes), and a range of cosmological parameters, the haloes are well described by a ‘universal’ mass profile. The two-parameter Navarro-Frenk-White (NFW) radial function is:

$$\rho(r) = \frac{\delta_c \rho_c}{(r/r_s)(1 + r/r_s)^2}. \quad (1.9)$$

Here, ρ_c is the critical density for closure (see Eqn. 1.6) at the halo redshift. $r_s = r_{200}/c_\Delta$ is the characteristic scale radius, which marks the transition in the slope of the profile. r_{200} is the radius within which the mean density of the halo is $200\rho_c$. c_Δ is the dimensionless concentration parameter. Finally, the characteristic overdensity is given by:

$$\delta_c = \frac{200}{3} \frac{c_\Delta^3}{\ln(1 + c_\Delta) - c_\Delta/(1 + c_\Delta)}. \quad (1.10)$$

The two parameters that describe the NFW profile are not independent but correlated, with lower mass haloes having higher concentrations (Cole & Lacey, 1996). The relationship appears to satisfy a power law so that $c_\Delta \propto M^{-\alpha}$ where $0.09 \lesssim \alpha \lesssim 0.13$, according to studies of dissipationless N-body simulations (e.g. Bullock et al., 2001; Dolag et al., 2004; Neto et al., 2007; Duffy et al., 2008), while $0.02 \lesssim \alpha \lesssim 0.28$ is determined by X-ray and lensing studies of observed clusters (e.g. Buote et al., 2007; Comerford & Natarajan, 2007). Duffy et al. (2008) find a redshift dependent fit for haloes over a range of masses $10^{11} < M/M_\odot < 10^{15}$ and redshifts between $z = 0$ and $z = 2$:

$$c_{200} = \frac{5.71}{(1+z)^{0.47}} \left(\frac{M_{200}}{2 \times 10^{12} h^{-1} M_\odot} \right)^{-0.097}. \quad (1.11)$$

Care must be taken when using best-fit values from this relation and making comparisons between observations and predictions; the slope and normalization of the concentration-mass relation has a dependence on the choice of cosmological parameters, particularly σ_8 , and scatter is induced by the inclusion of unrelaxed clusters in the sample (Macciò et al., 2008; Coe, 2010).

The exact values of the inner slope of the density profile and core radius of dark matter haloes have been the subject of much debate. While flat cores are deemed a poor choice for observed clusters, singularities are unrealistic. The NFW profile diverges in the centre and has presented us with the ‘cusp/core crisis’ in the modelling of galaxies (e.g. Borriello & Salucci, 2001; de Blok et al., 2001, 2003). It is unclear whether dynamical friction resulting from baryon-dark matter interaction will solve this problem (El-Zant et al., 2001; Li & Chen, 2009). Still, we have a model that is useful for a description of the structure outside the core of galaxies; on cluster scales the cuspy profile is still acceptable (e.g. Dahle et al., 2003). The singular NFW profile is flatter than the singular isothermal profile in the central regions (well within the scale radius), but at large radii, the density drops off more steeply than in the isothermal model. The profile may be extended to a more general form which allows for an extra parameter, β , defining the inner slope; it is given by:

$$\rho(r) = \frac{\delta_c \rho_c}{(r/r_s)^\beta (1 + r/r_s)^{3-\beta}}. \quad (1.12)$$

The analysis in Navarro et al. (1995) was conducted using somewhat low-resolution simulations run within the CDM cosmology, where $\Omega_M = 1$ and there is no cosmological constant; subsequent studies of high-resolution simulations have found a much steeper cusp across a large range of halo masses with $\beta \sim 1.5$ (Moore et al., 1998, 1999; Ghigna et al., 2000; Fukushige & Makino, 1997, 2001), although these were also conducted in the same cosmology. Studies in which the simulations were run in a Λ CDM universe have found a milder deviation from the NFW inner slope, albeit for galaxy-size haloes (Klypin et al., 2001; Power et al., 2003), and a possible mass dependence where the more massive cluster haloes more closely resemble the NFW form (Jing & Suto, 2000).

While the NFW profile is invoked as a model profile throughout this thesis, there are other successful functional forms that have described cluster halo mass profiles well. For example, Einasto (1965) used the following law to model the three-dimensional distribution of stars in

the Milky Way:

$$\rho(r) = \rho_{-2} \exp \left(-\frac{2}{\alpha} \left[\left(\frac{r}{r_{-2}} \right)^\alpha - 1 \right] \right). \quad (1.13)$$

Here, ρ_{-2} and r_{-2} denote the density and radius that mark the transition in the slope of the profile; α is the slope at large radii. The Einasto profile has been shown to be a good fit to cluster-sized Λ CDM haloes (Merritt et al., 2006; Gao et al., 2008). The aforementioned profiles have been fitted to haloes found in ‘dark-matter only’ simulations, but the inclusion of baryons has been shown to alter the profile somewhat, as discussed in Sec. 2.4.

Background galaxies that are gravitationally lensed into huge arcs provide a powerful observational test to complement the simulations (see Sec. 1.4.1). Since giant arc properties and statistics are sensitive to the chosen lens model, they serve as an important tool to constrain the true profile of the cluster lenses (Hammer, 1991; Wu & Hammer, 1993; Bartelmann, 1996). In particular, the number counts and number ratios of tangential and radial arcs are sensitive to the central density profile (Wu & Hammer, 1993; Molikawa & Hattori, 2001; Oguri et al., 2001). The statistics of wide-separation images are also sensitive to concentration (Keeton & Madau, 2001). With this knowledge, studies that fit models of the cluster lens to observed arcs have agreed that the NFW profile is a reasonable fit; at the very least, a finite core radius of 20-100 kpc is required (e.g. Wu & Hammer, 1993; Grossman & Saha, 1994; Molikawa et al., 1999; Comerford et al., 2006; Shu et al., 2008).

1.3.4 Triaxiality

An NFW density profile describes the overall shape of a cluster quite well, but the model is spherically symmetric, while both observed and simulated galaxy clusters exhibit triaxial shapes (e.g. Dubinski & Carlberg, 1991; Warren et al., 1992; Cole & Lacey, 1996; Jing & Suto, 2002; Comerford et al., 2006). Indeed, the spherical collapse model described in Sec. 1.3.2 has been extended to the Sheth-Tormen ellipsoidal collapse model, which better reproduces the simulated halo mass function and halo-mass bias (Sheth et al., 2001). In the physically-motivated, triaxial extension of the NFW profile, the radial distance r can be re-defined according to the following equation:

$$r^2 = c^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right), \quad (1.14)$$

where x , y and z represent the principal coordinate positions, and

$$s \equiv \frac{c}{a} \quad \text{and} \quad q \equiv \frac{b}{a} \quad (1.15)$$

are the axis ratios given that $a \geq b \geq c$. Together, the two axis ratios define the shape of a halo, but a single-value parametrization is in common use (see Sec. 3.3.3). Halo shapes can be characterised as either prolate ($a \gg b \approx c$) or oblate ($a \approx b \gg c$). Simulated samples of galaxy- and cluster-scale haloes exhibit more prolate shapes than oblate (see Fig. 1.6), but there exists a mass dependence with more massive haloes being more prolate (e.g. Frenk et al., 1988; Warren et al., 1992; Paz et al., 2006; Allgood et al., 2006; Bett et al., 2007).

Given a numerically simulated dark matter halo that has been discretized into N_{part} equal-mass particles, its axis ratios can be determined with the following procedure, which follows Dubinski & Carlberg (1991):

1. Compute and diagonalise the inertia shape tensor $I_{ij} \equiv \sum_{k=1}^{N_{\text{part}}} x_i x_j$.
2. Identify the principal axis as the dominant eigenvector of I_{ij} .

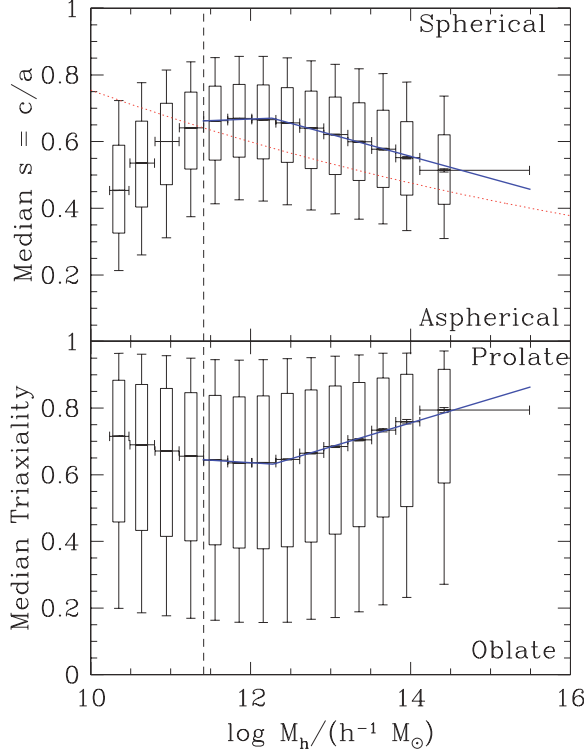


Figure 1.6: The median of the axial ratio s (see Eqn. 1.15; top panel) and the triaxiality parameter T_2 (see Eqn. 3.9; bottom panel). The low- N_{part} limit of 300 particles is marked with a dashed line, and data points have also been plotted below this limit. Error bars on the medians are plotted, but most are vanishingly small. The whisker bars are percentiles, at the equivalent of 1σ (68 per cent of haloes, boxes) and 2σ (95 per cent of haloes, bars); from the bottom to the top of each graph, these show where approximately 2.5, 16, 84 and 97.5 per cent of the haloes have s (or T_2) below these values. The thick blue lines show broken-line fits to the data. The dotted red line is the fit of Allgood et al. (2006) who used a different definition of a halo. Image taken from Figure 12 of Bett et al. (2007).

3. Calculate the eigenvalues of I_{ij} ($\lambda_1 > \lambda_2 > \lambda_3$).
4. Determine the axis ratios using

$$\frac{\lambda_3}{\lambda_1} = s^2 \quad (1.16)$$

and

$$\frac{\lambda_2}{\lambda_1} = q^2. \quad (1.17)$$

The departure from spherical symmetry has a large bearing on gravitational lensing predictions, since the assumption of spherical symmetry can lead to biased results. This is the motivation behind the study presented in Chapter 3.

1.4 Gravitational Lensing

The image distortions attributable to a lumpy universe can be thought of in two ways, either as an impediment to learning about the intrinsic properties of the distant objects, or as a way of learning about the stuff en route.

- Virginia Trimble, *The First Lenses* (2001)

One of the foremost predictions of GR (see Sec. 1.1.1) is the deflection of light by mass. In other words, mass and the associated gravitational potential field introduces a curvature in the space-time metric. The path along which photons propagate is described by the Geodesic Equations:

$$\frac{d^2 x^\alpha}{d\lambda^2} = -\Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda}. \quad (1.18)$$

Here $\Gamma_{\beta\gamma}^\alpha$ denotes the Christoffel symbols associated with the metric, x^i represents any of the four coordinates specified by the subscript i , and λ is the affine parameter. The deflection of photons results in the distortion of images; the so-called gravitational lensing effect manifests itself on different scales and is exploited for a range of astrophysical studies, some of which are investigated in this thesis. Here, I introduce the conventional notation for the components that describe the effect of gravitational lensing on the shape of a beam to first order. Each background source experiences lensing in the form of convergence, κ , and shear, γ . Convergence is the isotropic Ricci focusing of a beam due to enclosed matter, while shear is the tidal stretching of the beam along a particular axis; together, they serve to change the area of the source on the sky, resulting in its magnification or demagnification, due to the conservation of surface brightness. For a single thin lens, κ is the scaled surface density of the lens, and is related to the gravitational potential, ψ , via the Poisson Equation

$$\frac{1}{2}\nabla^2\psi = \kappa = \frac{\Sigma}{\Sigma_{\text{crit}}}, \quad (1.19)$$

where the critical surface density for gravitational lensing is given by

$$\Sigma_{\text{crit}} = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}}. \quad (1.20)$$

Thus lensing is characterised by D_s , D_d and D_{ds} , the angular diameter distances from the observer to the source, from the observer to the (thin) lens, and from the lens to the source respectively. Total shear can be written in complex notation; its dependency on the two orthogonal components are given by:

$$\gamma = \gamma_1 + i\gamma_2, \quad (1.21)$$

where the two components are linearly related to the second derivatives of the (projected) gravitational potential along two orthogonal directions by

$$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \quad \text{and} \quad \gamma_2 = \psi_{,12}. \quad (1.22)$$

The deformation of the beam is described as a mapping from the source plane to the image (observed) plane. The Jacobian, $\mathbf{A}$, of the lens mapping is a real and symmetric 2×2 matrix given by

$$\mathbf{A} \equiv \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}. \quad (1.23)$$

The flux magnification of an image is given by the inverse of the determinant of the Jacobian, so

$$\mu = \frac{1}{(1 - \kappa)^2 - |\gamma|^2}. \quad (1.24)$$

One may now consider the effect of lensing on the apparent position of the image of the source of light. From Eqn. 1.19, ψ , the two-dimensional effective lensing potential is given by

$$\psi(\mathbf{x}) = \frac{1}{\pi} \int_{\mathbb{R}^2} d^2x' \kappa(\mathbf{x}') \ln |\mathbf{x} - \mathbf{x}'|. \quad (1.25)$$

Here $\mathbf{x}$ is a dimensionless vector formed by scaling the image position, $\boldsymbol{\xi}$:

$$\mathbf{x} = \frac{\boldsymbol{\xi}}{\xi_0}. \quad (1.26)$$

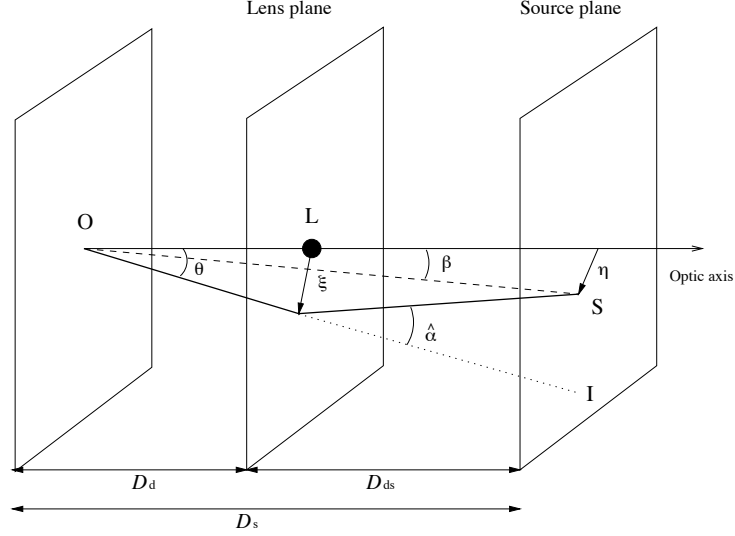


Figure 1.7: Diagram of the gravitational lensing effect for a single lens plane. The light ray propagates from the source S at the position η in the source plane to the observer O , passing the position ξ in the lens plane where it is deflected by an angle $\hat{\alpha}$. The angular position of the source S relative to the optical axis is denoted by β , and that of the image I relative to the optical axis is denoted by θ . D_d , D_{ds} and D_s denote the angular diameter distances from observer to lens, lens to source, and observer to source. Taken from Umetsu (2010).

The scaled deflection angle is the gradient of the lensing potential:

$$\boldsymbol{\alpha} = \nabla\psi \quad (1.27)$$

$$= \frac{1}{\pi} \int_{\mathbb{R}^2} d^2x' \kappa(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2}. \quad (1.28)$$

The gravitational lens equation is therefore given by

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta}), \quad (1.29)$$

where $\boldsymbol{\beta}$ is the angular source position and $\boldsymbol{\theta}$ is the angular position of the image on the sky. The deflection angle, $\hat{\boldsymbol{\alpha}}$, is related to its scaled counterpart by:

$$\hat{\boldsymbol{\alpha}} = \frac{\xi_0 D_s}{D_d D_{ds}} \boldsymbol{\alpha}. \quad (1.30)$$

Gravitational lensing results in the distortion of images of background sources by the intervening matter distribution; the effects fall under three categories: weak lensing, strong lensing and microlensing. Far from massive objects, images are distorted at the percent level, an effect known as *weak lensing*. Closer to foreground galaxies or the centres of clusters, we can see highly magnified images ($\mu \gg 1$), arc-like features and/or the same source imaged multiple times; this falls under the banner of *strong lensing*. The difference between large and subtle distortions is phenomenological rather than mathematical, although the surface density of the lens plane is often used as criteria for determining the need to work within a particular regime. *Microlensing*

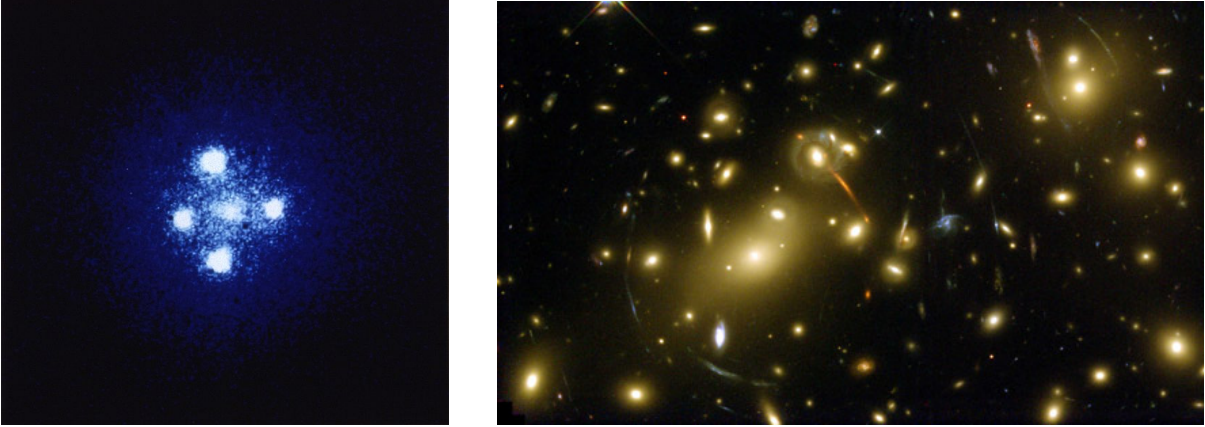


Figure 1.8: Examples of strong gravitational lensing features. *Left:* A background quasar is imaged multiple times by a galaxy lens forming the ‘Einstein Cross’ in Q2237+0305 (NASA, ESA and STScI). *Right:* Giant luminous arcs, the result of strongly lensed galaxies, are observed in the background to images of the massive galaxy cluster lens A2218 (NASA, ESA and STScI).

refers to variations in the intensities of point-like images due to changes in the observer-lens-source configuration, e.g. when a star within a galaxy lens moves near the image. This leads to changes in “strong lensing” features on microarcsecond, and therefore unresolved, scales.

Observations of lensed galaxies are often used to perform lens reconstructions. This method is sensitive to the total gravitational mass (mostly dark matter) rather than baryonic matter and so is unaffected by biases associated with luminous tracers. Gravitational lensing thus provides a clean probe of structure and a unique insight into gravity models and the nature of dark matter (see Sec. 1.2).

1.4.1 Strong Lensing

When the surface density of the lens is large ($\kappa > 1$) and/or the field varies across an image, the image is affected not only by shear but flexion. Flexion is the third-order effect in gravitational lensing and is effectively the gradient of the shear component (see Goldberg & Bacon, 2005). An image suffering flexion is stretched into a highly magnified giant arc; such giant arcs have been detected in the background to massive galaxy clusters (e.g. Lynds & Petrosian, 1986; Soucail et al., 1987; Luppino & Gioia, 1992; Fort et al., 1992). In addition, a single source can be lensed into multiple images, making Eqn. 1.29 a one-to-many mapping. Examples of these features are exhibited in Figure 1.8. The tendency for a lens to split a source into multiple image depends not only on the surface density of the lens, but on the chance alignment of the observer, lens and source. Given a lens mass distribution and a source redshift, it is possible to define a *caustic* (or source caustic): a set of boundaries which separate regions with image multiplicities which differ by two. It is also the locus of points in the source plane for which the Jacobian determinant of the lens mapping vanishes. Caustics map to *critical curves* in the image plane, which indicate the positions of highly magnified images (for a given source redshift). Figure 1.9 shows some typical caustic patterns for a highly elliptical lens.

Both tangential caustic and radial caustics are seen, which form tangential and radial images, respectively. Radial images are more likely to be hidden by the lens itself, but have been detected (Fort et al., 1992). When a source lies on a caustic, one observes merging images. If the

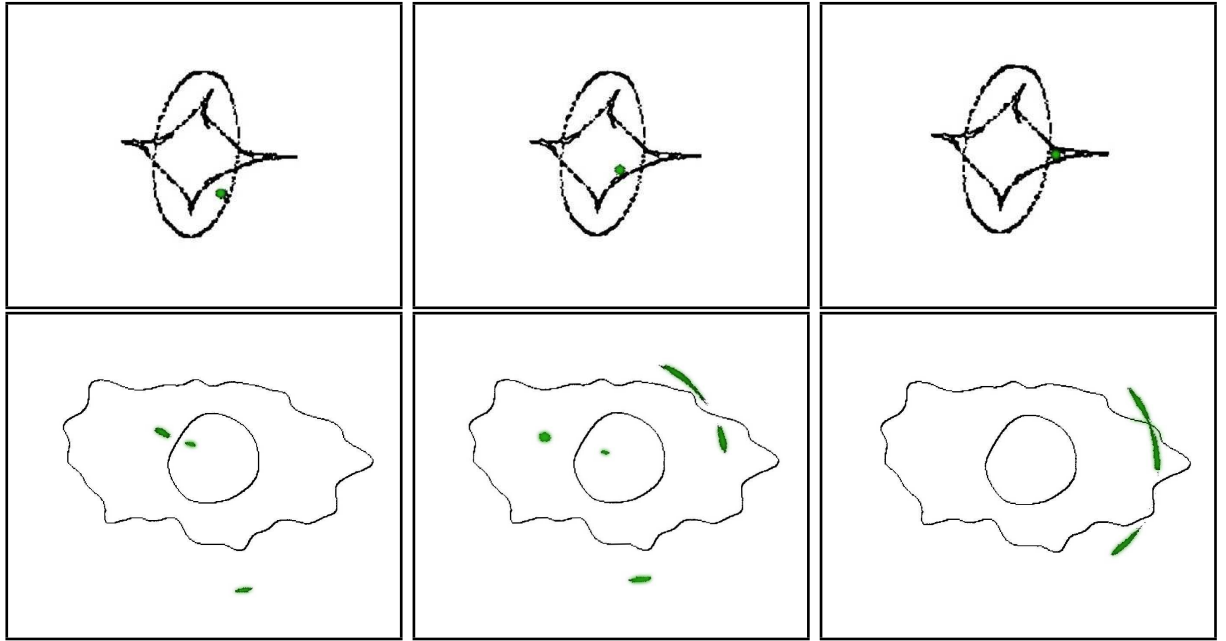


Figure 1.9: Diagrams of a source caustic (above) and the corresponding critical curve (below). A source is shown in green on the panels above, while the corresponding images are shown in green on the panels below. The panels correspond to: a source lying near a fold of a radial caustic (left); a source lying near a fold of a tangential caustic (middle); and a source lying near a cusp (right).

source is located near the fold caustic, two images are seen to merge, as depicted in the middle panels of Figure 1.9. On the rare occasion that the source lies near a cusp caustic, where two folds meet, three images merge simultaneously, as shown in the rightmost panels of Figure 1.9 (Blandford & Narayan, 1986; Grossman & Narayan, 1988); the longest arcs are produced in this case (Grossman & Narayan, 1988). The positions, shapes and magnifications of multiple images and giant luminous arcs have been used to constrain the mass in the central regions of the lens cluster (e.g. Takahashi & Chiba, 2001; Comerford et al., 2006). In addition, the abundance of these giant arcs are a powerful probe of cosmology with the ability to constrain structure formation parameters σ_8 , Ω_M and Ω_Λ (Bartelmann et al., 1998; Cooray, 1998; Takahashi & Chiba, 2001; Wambsganss et al., 2004). The lens mass distribution initially ascertained from lensing studies can then allow future searches for the more distant objects within the predetermined critical curves. This technique, exploiting lenses as natural gravitational telescopes, has gained momentum over the past few years with various searches for high redshift galaxies and supernovae behind galaxy clusters (e.g. Richard et al., 2006, 2008; Goobar et al., 2009; Bradač et al., 2009).

1.4.2 Weak Lensing

Weak lensing induces a distortion to the images of background galaxies at the percent level; theoretical and numerical studies predict a shear amplitude of 2 – 5 percent in the field on scales below 10 arcminutes (Jain & Seljak, 1997). In theory, measuring the induced ellipticity allows one to determine the gradient in the lensing potential, so weak lensing studies generally aim to map out the surface density of the lensing mass from the distortions of the background sources.

One of the major challenges in weak lensing is that the intrinsic ellipticity of *individual* galaxies is substantial and unknown to the observer, so the effects of the intervening large-scale mass distribution are only recovered by averaging over an ensemble of galaxies. A number density of 20 arcmin^{-2} is a typical depth for ground-based observations. A coherent distortion is the signature of lensing, assuming that the alignments of the background sources are not correlated. Cosmological weak lensing, which is often referred to as cosmic shear, was first detected at the beginning of the 21st century (Wittman et al., 2000; Bacon et al., 2000; Van Waerbeke et al., 2000) although the equivalent effect in the outskirts of cluster lenses had been detected a decade earlier (Tyson et al., 1990). Since then, weak lensing measurements from the Hubble Space Telescope Cosmic Evolution Survey (COSMOS; Schrabback et al., 2010), the Canada-France-Hawaii-Telescope Large Area Survey (CFHTLS; Hoekstra et al., 2006; Fu et al., 2008) and COMBO-17 (Brown et al., 2003) have begun to measure the projected two-dimensional matter power spectrum of the large-scale mass distribution and place constraints on Ω_M and σ_8 . In addition, dedicated surveys have been planned with the measurement of cosmic shear, and in particular, the determination of the nature of dark energy, as one of their main science drivers. These include the ground-based Panoramic Survey Telescope and Rapid Response System (Pan-STARRS)¹, the VST-Kilo-Degree Survey (KIDS)², the Dark Energy Survey (DES)³, the Large Synoptic Survey Telescope (LSST)⁴ as well as the space-based Euclid (Laureijs, 2009)⁵ and the Joint Dark Energy Mission (JDEM)⁶, which has recently been rebranded as the Wide-Field InfraRed Survey Telescope (WFIRST).

1.4.3 Magnification Bias

While various missions actively *search* for the weak lensing signal, large surveys for galaxies, quasars and supernovae suffer a two-fold *magnification bias* as a result of the same phenomenon (e.g. Wambsganss et al., 1997). When a source is magnified, its total surface area on the sky is increased; consequently, for a given area of sky observed, the region of the source plane being sampled is decreased. On the other hand, magnification will push some sources above the observational threshold in flux-limited surveys. These two effects have counteracting effects on source counts, but by no means cancel each other out. Also, for sources that are demagnified, the effect is reversed; indeed the majority of sources are de-magnified relative to the case of a perfectly homogeneous universe (see Sec. 4.3.1).

The net bias is dependent on intrinsic source properties, selection procedures and the probabilistic lensing effect. For example, the steeper the slope of the intrinsic source luminosity function, the greater the effect of magnification in increasing the source count. The extent of the magnification bias also increases with source redshift so surveys that push the limits of detection will be the most affected (Wyithe et al., 2011). Additionally, a survey may be designed such that regions close to intervening galaxies may be avoided (Dyer & Roeder, 1974), otherwise a correlation with foreground galaxies can provide evidence of the gravitational lensing bias. This is related to the fact that magnified objects are overrepresented in flux-limited samples (Turner et al., 1984), adding to the Malmquist bias. Such a correlation has been searched for within different quasar samples on a range of scales (e.g. Seldner & Peebles, 1979; Webster et al., 1988; Boyle et al., 1988; Bartelmann & Schneider, 1994; Myers et al., 2003; Gaztañaga, 2003; Scranton et al., 2005). Interpretations are complicated by the possibility that extinction by dust

¹<http://pan-starrs.ifa.hawaii.edu>

²<http://www.astro-wise.org/projects/KIDS/>

³<http://www.darkenergysurvey.org/>

⁴<http://www.lsst.org>

⁵<http://sci.esa.int/euclid>

⁶<http://jdem.lbl.gov/> and <http://jdem.gsfc.nasa.gov/>

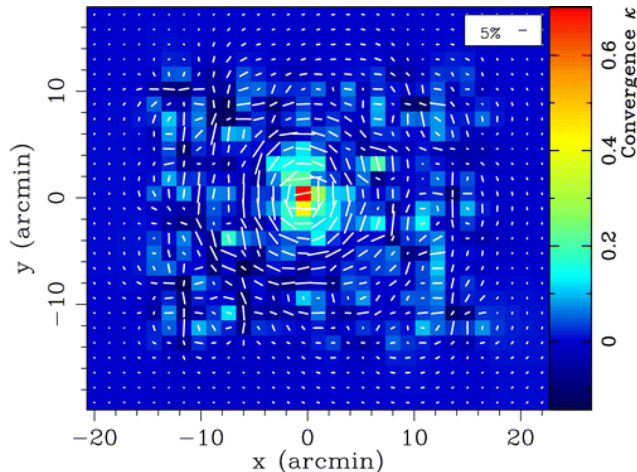


Figure 1.10: Projected mass distribution κ of galaxy cluster A1689 on a grid reconstructed from a combination of weak lensing and strong lensing data. Overlaid on the image is the reconstructed gravitational shear field. A stick with a length of 5% shear is indicated in the top right corner. Image taken from Umetsu & Broadhurst (2008).

present in foreground galaxies may lead to an *anti*-correlation, depending on the wavelength of the survey. In addition, the intrinsic source luminosity function and magnification probability distribution are unknown. An appropriate model for the latter may be incorporated into the analysis of source counts and this is precisely the subject of the study presented in Chapter 4 of this thesis.

1.4.4 Lens Reconstruction

The gravitational lens inversion problem is non-trivial, and there is considerable literature discussing the most appropriate, accurate and efficient way to reconstruct the lens and source profile from a given distribution of images, with or without multiple images, with or without redshifts and with or without a luminous lens. A distribution of many weakly lensed galaxy images can be used to reconstruct the surface density field, $\Sigma(\boldsymbol{\theta})$, given the quadrupole moments of the images, for which the expected value is proportional to the gravitational shear, γ (e.g. Kaiser & Squires, 1993; Bartelmann, 1995b; Kaiser, 1995). The shear field is related to the convergence field via the following relation

$$\kappa(\boldsymbol{\theta}) = \frac{1}{\pi} \int_{\mathbb{R}^2} d^2\theta' D^*(\boldsymbol{\theta} - \boldsymbol{\theta}') \gamma(\boldsymbol{\theta}'), \quad (1.31)$$

where $D(\boldsymbol{\theta})$ is the complex kernel given by

$$D(\boldsymbol{\theta}) = \frac{\theta_2^2 - \theta_1^2 - 2i\theta_1\theta_2}{|\boldsymbol{\theta}|^4}. \quad (1.32)$$

These methods are based on the assumption that the background galaxy orientations are statistically random, i.e. that the observing field is sufficiently large. The observed mean distortion of background galaxies does not give a measure of the shear, *per se*, but the *reduced* shear g , which is related to γ by

$$g = \frac{\gamma}{1 - \kappa}. \quad (1.33)$$

In the weak lensing limit ($\kappa \ll 1$), this reduces to the shear. The mass distribution recovered from cosmic shear studies is actually a projected mass density; thus cosmological results depend on predictions of the projected matter power spectrum. Furthermore, due to the large number of source galaxies required, accurate redshifts cannot be ascertained for all of them; a redshift

distribution is usually assumed. However, without redshift information for multiple source galaxies, the lens surface mass density cannot be uniquely determined as the expected image ellipticities and the position of the critical curves are invariant under the transformation

$$\kappa \rightarrow (1 - \lambda) + \lambda\kappa, \quad \gamma \rightarrow \lambda\gamma \quad (1.34)$$

for an arbitrary constant λ (Schneider & Seitz, 1995). This so-called *mass-sheet degeneracy* also affects strong lensing reconstructions since flux ratios also remain unchanged, even though individual flux magnifications are affected. When giant arcs or multiple images of the same source are identified, a whole new regime of lens reconstruction is opened up. Several approaches combine both strong and weak lensing data and break this degeneracy (e.g. Kneib et al., 2003; Bradač et al., 2004), mostly applied to cluster lensing. Methods that aim at reconstructing the lens and/or source of a strong lensing system, with or without the inclusion of weak lensing data, come in two flavours: ‘parametric’ or model-dependent (e.g. Kochanek, 1991; Kneib et al., 1996; Jullo et al., 2007) and ‘non-parametric’ or ‘grid-based’ (e.g. Wallington et al., 1996; Williams & Saha, 2000; Brewer & Lewis, 2006; Suyu et al., 2009). The former requires an assumed functional form for the lens profile, while the latter provides surface density maps that must be de-projected in order to make any comparisons with theoretical predictions. Recent works have seen methods that do not fall cleanly into these categories as they allow the number of parameters to vary with the number of observables (Coe et al., 2008; Jullo & Kneib, 2009).

1.5 Thesis Outline

In this thesis, I explore the theoretical predictions of gravitational lensing on a range of scales. I begin with an investigation of strong lensing on cluster scales. The triaxiality of galaxy clusters introduces a strong orientation dependence in the cross section for multiple imaging; in Chapter 3, I analyse a large sample of simulated clusters and correlate the shape of the cross section probability distribution with various cluster properties so that multiple imaging statistics may be interpreted appropriately for future structure formation studies. In Chapter 4, I make the first direct comparison between a ray-tracing method that numerically integrates the path of a photon and one which invokes the multiple lens-plane algorithm, to determine how this oft-used approximation affects the weak lensing predictions, in particular the magnification probability distribution for background sources. Gravitational lensing can itself be the source of bias in measurements that ignore its effects; in Chapter 5, I study the effect of foreground lenses on the prospects of the envisioned Redshift-Drift Experiment (also known as the Sandage-Loeb test). A discussion of my findings and proposed future studies are presented in Chapter 6.

Chapter 2

Simulating Structure

*But all hold course unalterably fixed;
They follow destinies foreplanned
- O World of Many Worlds, Wilfred Owen*

N-body simulations play a unique role in the study of nonlinear growth of cosmological structure, a regime in which analytic methods are limited, and have done so since the 1970's (e.g. Miyoshi & Kihara, 1975; Aarseth et al., 1979). Simulations that describe cosmological scales have been widely used to model clustering properties, among other statistical descriptions of the universe for a given cosmology. Subsequent comparisons with survey data and analytical models provide a test of the physics that drives structure formation. The basic methodology involves:

1. Discretizing the mass within a cosmological volume into ‘particles’ which represent some underlying fluid.
2. Generating initial conditions for the simulation at some early epoch based on some chosen cosmology.
3. Evolving the positions and velocities of these particles under gravity.
4. Capturing the results at specific epochs

The structures modelled by N-body simulations are the subject of analysis in Chapters 3 and 4. There are a number of issues surrounding the numerical method that must be taken into consideration when interpreting and determining the validity of results. An overview of these issues is presented in the following chapter, followed by a description of the N-body code and group-finding algorithms used in this thesis. Some of the discussion revolves around the appropriate modelling of cluster cores, as this is relevant for the work presented in Chapter 3.

2.1 Evolving Structure under Gravity

N-body simulations model the evolution of matter under the action of gravity by following individual particle trajectories. The post-Newtonian approximation is invoked to allow a reasonable method of dealing with inhomogeneities that perturb the FLRW metric. The perturbations to the metric can be small even when the density perturbations are large. As long as the velocities of the particles are much less than the speed of light, we may fix the expansion law based on the homogeneous mass distribution and describe the gravitational field of each particle by the Newtonian potential. The exact application of the gravitational force is complicated by the

discretisation of a smooth distribution of matter (addressed in the Sec. 2.2), but by and large the N-body problem is a computational challenge; the gravitational force on each particle as a result of all other particles must be calculated at each time-step. Taking a brute-force approach, the number of calculations grows as N_p^2 , where N_p is the total number of particles in the simulation. Several approaches to the N-body problem have been developed, most implementing a combination of the following two techniques. The first, the Barnes-Hut tree-code, divides the three-dimensional space into cubes (or cells) at each time-step, each of which may be recursively subdivided into 8 sub-cells, thus constructing a tree. To determine the force on each particle, the code descends through the hierarchy proceeding to the daughter cells if the ratio of the cell-size to the distance to the cell's centre-of-mass exceeds some fixed tolerance parameter θ (Barnes & Hut, 1986). The computation time grows as $N_p \log N_p$. The second scheme, the particle-mesh method (PM), interpolates each particle onto a regular grid, then uses Fourier techniques to solve the Poisson equation (e.g. Hockney & Eastwood, 1981; Klypin & Shandarin, 1983; Efsthathiou et al., 1985). The CPU time now scales as $N_g \log N_g$, where N_g is the number of grid cells; the spatial resolution, and consequently force accuracy, is limited by the mesh-spacing.

2.2 Discretisation

The particles whose positions and velocities are evolved in N-body simulations are essentially a Monte Carlo sampling of the underlying smooth mass distribution. Limitations are imposed for scales above the physical box size (L_{box}) as well as scales below the mean inter-particle separation ($\lambda_p = L_{\text{box}} / \sqrt[3]{N_p}$). Specifically, the smallest wavemode that can be represented in the initial power spectrum is the Nyquist scale $k \gtrsim k_{\text{Ny}}/2$, where $k_{\text{Ny}} = \pi/\lambda_p$ is the Nyquist wavenumber. The largest wavemode is defined by finite the box size. The smallest sensible box size is that for which the length of a side corresponds to the scale that has just begun to go non-linear at the final epoch of the simulation, usually $z=0$. Although the halo mass-function is affected by the truncation of the power spectrum, the halo shapes remain unaffected (Power & Knebe, 2006).

Two-body relaxation is arguably the most important discreteness effect in collisionless simulations; direct interactions between two point masses result in high accelerations when the separation vector, $\mathbf{r}$, is small. This effect eventually leads to the numerical depression of structure (Power et al., 2003). Hence, smoothing is required to describe the forces on the particles more accurately, particularly the forces from nearby particles. The most common method of smoothing is to replace the gravitational potential for each particle with that for a Plummer sphere of equal mass, thereby removing the singularity that appears at $r = 0$ for a point mass. The modified law that describes the force between two particles with masses m_i and m_j is:

$$\mathbf{F}_{ij}(\mathbf{r}) = Gm_i m_j \frac{\mathbf{r}}{(r^2 + \epsilon^2)^{3/2}}, \quad (2.1)$$

where ϵ is the softening length. When applying grid methods to calculate forces, the softening length may be interpreted as the width of the kernel that is convolved with the particle density field. Too large a length will bias the force calculations (Merritt, 1996) and result in numerically flattened halo cores (Moore et al., 1996). In practice, users of N-body simulations set a gravitational softening length at least an order of magnitude smaller than the global mean particle separation, and it may be allowed to vary with redshift. An ‘optimal’ value of the softening length may be judged based on the aims of the simulation analysis. For example, a choice that minimises force errors is dependent on the global mean particle separation and proportional to $\sim N_p^{-0.3}$ (Merritt, 1996; Dehnen, 2001); however, Rodionov & Sotnikova (2005) suggest a value that is a factor of 1.5-2 smaller than the mean particle separation *in the densest region that*

needs to be resolved; Power et al. (2003), meanwhile, suggest a value of $\epsilon \approx 4r_{\text{vir}}N_{\text{vir}}^{-0.5}$, where N_{vir} is the number of particles *within the virial radius* of the halo being analyzed.

Even after an N-body simulation has been completed, the analysis of the density distribution is subject to the effects of discrete sampling. While high density regions are well described by the particle positions, shot noise plagues the regions of low density. Algorithms such as Nearest Grid-Point (NGP), Cloud-In-Cell (CIC) and Triangular-Shaped Cloud (TSC) are popular methods of assigning mass to a regular grid (Hockney & Eastwood, 1988); further smoothing by a Gaussian or top-hat filter allows mass to be spread over a fixed scale. For studies undertaken in this thesis, a combination of CIC and Gaussian filtering are employed (see Sections 3.2.2, 4.2.4, 4.4.2, 3.2.3 and 3.6.2). The Delaunay Tessellation Field Estimator method (DTFE; Schaap & van de Weygaert, 2000) is a technique for reconstructing the density field from a discrete sampling; it has been shown to perform well when a wide range of densities are present, and has been gaining popularity.

2.3 Generating Initial Conditions

At early times, the growth of structure is linear and the matter power spectrum is linearly related to its primordial form via a transfer function (Bardeen et al., 1986). N-body simulations may begin at the epoch when linear evolution of matter is no longer valid for the scales represented in the simulation box. Up until that point, however, the linear matter power spectrum is sufficient to describe the evolution of matter distribution. Several codes, such as **CMBFAST** (Seljak & Zaldarriaga, 1996), **CAMB** (Lewis et al., 2000) and **CMBEASY** (Doran, 2005) can be used to determine the linear matter power spectrum / transfer function for a given set of cosmological parameters Ω_{M} , Ω_{Λ} , Ω_{k} , h and σ_8 .

The pre-initial distribution¹ of particles can come in two possible configurations. In one, particles are regularly distributed onto a grid; this forms a ‘lattice’. In the other, the particles are randomly distributed, then evolved gravitationally, but with the sign of gravity reversed, until the forces on each particle are balanced; this is referred to as a ‘glass’. To generate the initial conditions (ICs) for the simulation, the matter power spectrum is applied to the pre-initial particle configuration. This is usually done via the Zel’dovich approximation (Zel’dovich, 1970), which uses first-order Lagrangian perturbation theory. Particles are perturbed from their pre-initial position and the initial velocities are proportional to this perturbation.

The starting redshift, or redshift of the ICs, is chosen such that all modes sampled by the simulation box are linear up until that epoch (i.e. the Zel’dovich perturbation is small compared to the mean inter-particle spacing; Lukić et al., 2007) and at $z = 0$ all memory of the pre-initial distribution is lost (Crocce & Scoccimarro, 2006). If the chosen redshift is too low the formation of the smallest-scale structures is delayed (Jenkins et al., 2001), but if the starting redshift is too high, force errors are introduced (Lukić et al., 2007). With each particle assigned a position and velocity at some initial redshift, we are now ready to begin the N-body simulation.

2.4 Simulations without Baryons?

The simulations undertaken for the work presented in Chapters 3 and 4 incorporate collisionless particles only; that is to say, dark matter structure is evolved under gravity with the effect of baryons ignored. On large-scales, the distribution of gas follows the dark matter potential wells; the weak lensing effect is sensitive to large-scale structure, so the collisionless simulations

¹‘pre-initial’ is the label commonly given to the distribution of particles *before* the primordial power spectrum converts it into the ‘initial condition’ for the simulation.

will suffice. On galaxy-scales, gas cooling and feedback can alter the gravitational potential (e.g. Blumenthal et al., 1986), but the role of baryons in shaping the core of clusters is unclear. The temperature of the ICM is too hot to allow efficient cooling, so the baryonic matter must simply follow the dark matter potential and would not have a significant impact on its shape (e.g. Broadhurst & Barkana, 2008). Nevertheless, simulations that incorporate various baryonic processes, such as gravitational heating, gas cooling, star formation and AGN feedback, exhibit indirect consequences. For example, given sufficient mass and force resolution, dense galactic dark matter (sub)haloes survive in clusters (Moore et al., 1996; Klypin et al., 1999), but dynamical friction drives massive subhaloes towards the cluster core where they are more effectively tidally stripped of their mass (Tormen et al., 1998), allowing more mass to be deposited around the core than otherwise, therefore modifying the central cluster profile (Barkana & Loeb, 2010). Radiative cooling *increases* halo concentration, while AGN feedback has the opposite effect, but the efficacy of the first mechanism is dependent on the strength of the latter, which in turn is constrained by observed stellar fractions (Duffy et al., 2010). Similar results are found with regards to the cross-section for giant arcs, i.e. cooling and star formation serve to increase the strong lensing cross-section (Rozo et al., 2008), while AGN feedback moderates this effect (Mead et al., 2010). While the relevance to the study presented in Chapter 3 is acknowledged, the net result on the total density profile in a realistic application of all the aforementioned baryonic processes requires further investigation, so for the present work, collisionless simulations will suffice. The absence of gas also results in the exclusion of magnetic fields from analysis; but this is deemed appropriate as the energy density of large-scale magnetic fields is a few orders of magnitude below the kinetic and potential energies of the particles (Miyoshi & Kihara, 1975).

2.5 GADGET2

The cosmological simulations of structure formation are performed with the widely-used, publicly available parallel N-body code, GALaxies with Dark matter Gas intEracT (**GADGET2**; Springel, 2005). The collisionless dark matter is represented by N_p particles of equal mass within a box with periodic boundary conditions imposed. The mass of each particle is given by:

$$m_p = \frac{2.77 \times 10^{11}}{N_p} \Omega_{M,0} \left(\frac{L_{box}}{\text{Mpc}/h} \right)^3 h^{-1} M_{\odot}. \quad (2.2)$$

The gravity solving scheme is a particle-mesh tree code, a hybrid of the two aforementioned methods (see Sec. 2.1). The gravitational potential is replaced with the Plummer-sphere equivalent via the force softening length, ϵ , but the force between two particles separated by more than 2ϵ is unsoftened. ϵ expands with the scale factor, $a(t)$, for high redshift, but is capped at a fixed physical length, so after a certain epoch it remains constant. That is to say, the simulations begin with a redshift-dependent gravitational force-softening length of $\epsilon = \epsilon_{co}/(1+z)$ kpc/h (Plummer-equivalent). After a redshift of $z \lesssim z_t$, the softening length is capped at $\epsilon = \epsilon_{phys}$ kpc/h. Therefore $z_t = \epsilon_{co}/\epsilon_{phys} - 1$. Note that Newtonian gravity is applied with GR incorporated by scaling the simulation box, and the physical distance between the particles, according to the evolution of the scale factor. **GADGET2** employs an adaptive time-stepping scheme in which particle acceleration limits the chosen step-size.

2.6 Identifying Clusters

The predictive power of N-body simulations is limited by how one draws parallels between simulated and observed structure. The work presented in Chapter 3 requires the identification

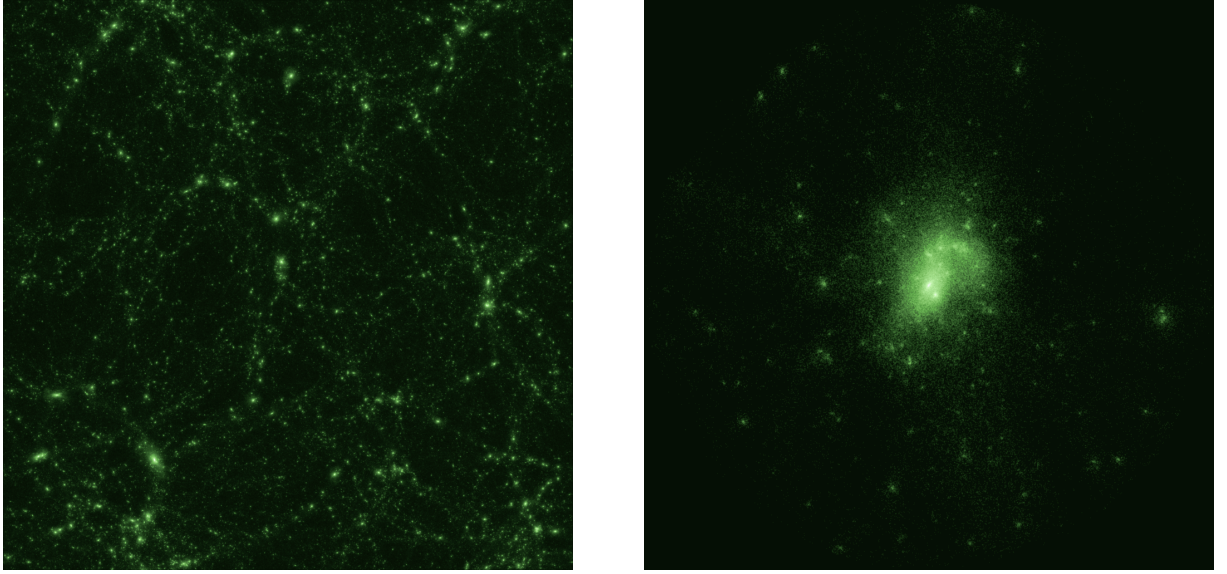


Figure 2.1: *Left:* The large-scale structure distribution in a $z=0$ snapshot from a **GADGET2** simulation. The box is 100 Mpc/h across. *Right:* One of the simulated cluster-sized haloes and all the mass within 3 virial radii from the halo centre. Lighter colours represent denser regions. The mass distributions have been artificially smoothed over 10 kpc/h and 6 kpc/h scales respectively.

of the dark matter haloes that host galaxy clusters as well as substructure within a larger-scale object, i.e subhaloes within a host/parent halo. The difficulty with the former lies in quantitatively defining the fairly intuitive idea of haloes that are nestled within a vast cosmic web; with the latter, problems arise in the separation of subhaloes from the host in which they are embedded. There exist a wide range of ‘group-finding’ algorithms that perform these tasks; each takes a slightly different approach and therefore is sensitive to slightly different forms of structure. These include: Spherical Overdensity (SO; Press & Schechter, 1974), **SKID** (Stadel, 2001), **SUBFIND** (Springel et al., 2001), **AMIGA/MLAPM Halo Finder (AHF/MHF)** (Gill et al., 2004), **HOP** (Eisenstein & Hut, 1998), **DENMAX** (Gelb & Bertschinger, 1994), **ISODEN** (D. W. Pfitzner & J. K. Salmon, 1996), **PSB** (Kim & Park, 2006), **VOronoi BOund Zones (VOBOZ)** (Neyrinck et al., 2005) and **Bound Density Maximum (BDM)** (Klypin et al., 1999). Each algorithm may implement one or more of the following procedures:

- Link particles within close proximity
- Locate the peaks in the density field
- Remove unbound particles by comparing their kinetic and potential energies
- Select a group of particles that surpass a minimum mean overdensity
- Track mass accretion through merger trees.

Reference	Cosmology ($\Omega_{M,0}$, $\Omega_{\Lambda,0}$, h , σ_8)	L_{Box} (Mpc/h)	m_p ($M_{\odot}/h$)	z_i	ϵ_{co} (kpc/h)	ϵ_{phys} (kpc/h)
Sec. 3.4	0.27, 0.73, 0.71, 0.9	100	4.5×10^9	39	16	5
Sec. 3.5	0.27, 0.73, 0.71, 0.9	256	8.0×10^{10}	39	50	20
Sec. 4.4.1	0.27, 0.73, 0.71, 0.9	50	6.3×10^8	39	16	16
Sec. 4.4.5	0.27, 0.73, 0.71, 0.9	50	6.3×10^8	39	16	5

Table 2.1: Database of simulations analyzed throughout this thesis. Parameters are described in the text.

2.6.1 AHF

The publicly available MPI parallel code AMIGA’s Halo Finder (AHF) identifies overdensities using a subset of the aforementioned procedures. This particular group-finder is used to analyse simulations and identify the haloes for the studies presented in Sections 3.4 and 3.5. A brief overview is given here, but for further details, the reader is referred to Gill et al. (2004), Knollmann & Knebe (2009) and the User Manual².

An adaptive mesh-refinement scheme constructs a grid hierarchy based on particle density; the scheme locates prospective halo centres as density peaks and forms the hierarchical tree structure from which substructure can be classified. For all subhaloes, particles within a sphere with a radius equal to half the distance to the host halo are collected for tentative inclusion. Unbound particles are iteratively removed, by comparing the particle velocities to the local escape velocity. The extent of each halo, its virial radius, is defined by the radius at which the average enclosed density drops below the virial overdensity (see Sec. 1.3.2). Alternatively, the virial radius may be defined as the radius at which the density profile begins to rise, either because a neighbouring halo is encountered or background density of the host halo rises above that of the embedded subhalo.

Subhaloes are haloes that lie within the virial radius of a more massive host. The particles that make up a subhalo are also included in the parent halo. **MergerTree**, one of the bundled software packages within AHF, can be employed to identify subhaloes through the analysis of AHF output. The identification of particles common to larger and smaller subhaloes between different snapshots of the same simulation allows one to construct a merger tree; while this is the main intention of the **MergerTree**, it is not the application required for the present study. Instead, it is used to exclude subhaloes from cluster samples.

2.7 Database of Simulations

Throughout this thesis, several simulations have been run and analyzed for a range of studies. In Figure 2.1 I provide a database of the relevant physical and technical parameters of each of the simulations carried out. Although the individual simulations are presented in each chapter where they are analyzed, the reader may refer to this database for an overview. The cosmological parameters are based on the 1-year WMAP results (Spergel et al., 2003), which allow for consistent comparison with earlier works in the literature.

²<http://popia.ft.uam.es/AMIGA/page1/page1.html>

Chapter 3

Strong Lensing by Triaxial Galaxy Clusters

Assume a spherical cow.

– *Anonymous*

Galaxy clusters are the most powerful known natural gravitational telescopes. Optical images of their centres reveal spectacular effects of strong lensing as distant galaxies are imaged multiple times and distorted into huge arcs, such as those shown in Figure 1.8. Cluster lenses also produce larger separation images which can constrain the density profiles of dark matter haloes (Maoz et al., 1997; Keeton & Madau, 2001). Numerous instances of galaxy sources that are multiply imaged by cluster lenses have been discovered (e.g. Kneib et al., 1996; Broadhurst et al., 2000, 2005; Sharon et al., 2005), but quasars multiply-imaged by clusters are much rarer; at the time of writing, only two such lensing systems have been discovered, both from the quasar sample of the Sloan Digital Sky Survey (SDSS; Inada et al., 2003, 2006). The low number of large-separation quasar lens systems is generally deemed consistent with the Λ CDM model, with some caveats. The number of predicted multiply-imaged quasars increases with σ_8 , yet rather high values of $\sigma_8 \approx 0.9$ seem to be preferred over $\sigma_8 \approx 0.7$ (Li et al., 2007; Hennawi et al., 2007a). Oguri et al. (2004) use the observed number of lenses to constrain a rather high value for the normalisation ($\sigma_8 = 1_{-0.2}^{+0.4}$) given a steep inner density profile for a cluster ($\beta = 1.5$), noting that σ_8 could only be lowered by steepening the profile even further. One of the major goals of the next generation of cosmological surveys, such as the highly anticipated wide-field synoptic surveys LSST and Pan-STARRS, is to search for variable sources. They are expected to detect $\sim 10^8$ quasars of which roughly 300 will be multiply-imaged by clusters (Hennawi et al., 2007a). The statistics of multiple images have been shown to be highly dependent on $\Omega_{\Lambda,0}$ via the dependence on angular diameter distances (Kochanek, 1996; Maoz et al., 1997; Premadi et al., 2001). However, the triaxiality of clusters can increase the expected abundance of multiply-imaged quasars, compared to the spherical prediction (Oguri & Keeton, 2004). It is essential, therefore, that the extent to which cluster properties and nearby structure impact on the observed statistics of multiply-imaged sources is incorporated into the Λ CDM prediction.

The population of clusters that exhibit strong lensing are a specific subset of the general population of clusters. They sample the high mass end of the cluster mass function (Comerford & Natarajan, 2007) and have larger concentrations for a given mass (Fedeli et al., 2007; Hennawi et al., 2007b; Oguri & Blandford, 2009). They are no more triaxial than the general cluster population (Hennawi et al., 2007b; Meneghetti et al., 2010), however, their major axes are preferentially aligned with the line of sight (Miralda-Escude, 1993; Hennawi et al., 2007b; Oguri & Blandford, 2009; Meneghetti et al., 2010). Although the preferential orientation can boost strong lensing to a greater extent for more triaxial clusters, this effect is apparently balanced

by the decrease in the probability of strong lensing when the line of sight is misaligned with the major axis (Wu & Hammer, 1993). Member galaxies appear to have a minimal effect of strong lensing efficiency (Flores et al., 2000; Meneghetti et al., 2000) although their efficacy is dependent on the ellipticity of the cluster (Meneghetti et al., 2007). The role of the cD galaxy, located at the cluster core, is moderated by other factors; it has the ability to increase the giant-arc cross section by 50 percent (Meneghetti et al., 2003a), with an efficacy that is dependent on the size/mass/Einstein radius of the cluster lens, but the enhancement of the multiple imaging cross section is small compared to the effect of the chosen model profile (Wyithe et al., 2001).

When analysing individual triaxial cluster lenses, adopting a spherical model for the de-projection of the two-dimensional lensing data leads to an underestimate of the giant arc abundance (Bartelmann & Weiss, 1994; Bartelmann et al., 1995) and overestimates of the radial to tangential arc ratio (Molikawa et al., 1999), the cluster mass (Bartelmann, 1995a; Oguri et al., 2005) and concentration (Meneghetti et al. 2010). In addition, since strong lensing is sensitive to the orientation of cluster lenses, individual clusters are able to exhibit a range of strong lensing cross sections depending on the viewing angle (Dalal et al., 2004; Ho & White, 2005). In this chapter, I investigate how the probability distribution of the strong lensing cross section is dependent on a range of cluster properties. The strong-lensing phenomenon will be characterized not by the formation of giant-arcs, which has been given much attention in the literature, but by multiple imaging. To this end, I construct the Multiple Imaging Cross Section Probability Distribution Function, the $\sigma_{\text{MI}}\text{PDF}$, from a sample of simulated clusters by analyzing numerous projection directions. The role played by associated filamentary structure is investigated.

3.1 Synthetic Clusters

As a result of the subtleties that affect strong lensing features, analytical models fail to describe the predicted arc statistics accurately and one must turn to N-body simulations to synthesize clusters within the chosen cosmology (Meneghetti et al., 2003b). However, in order to disentangle the effects of halo shape and substructure, simple model lenses are invaluable. Here I discuss the use of simulations to provide a ‘simulated cluster’ sample.

3.1.1 Simulated Clusters

Predictions based on cluster lenses are less susceptible to the effects of non-gravitational physics (see Sec. 2.4) than galaxy lenses; the N-body simulations presented in this chapter are all collision-less. The dark matter distribution is discretised into N_p^3 particles distributed within a periodic box with co-moving length of L_{box} ; the initial displacements are in a ‘glass’ configuration. One may be somewhat concerned that the properties of the simulated haloes could be affected by the choice of numerical parameters. So we ask the question:

“How many particles should be contained within the virial radius?”

The answer to this is dependent on the properties of the halo that one is interested in. As Power et al. (2003) demonstrate, resolving regions closer to the centre of a halo requires more particles. For example, the inner mass profile down to 3 percent of the virial radius can be trusted only if the virial radius contains at least 5×10^4 particles. However, when Jing & Suto (2002) model cluster haloes in a cosmological simulation, they find that replacing high resolution simulated clusters, which have 10^6 particles within the virial radius, with clusters that sample only 10^4 of these particles has a negligible effect on the measured triaxiality and decreases the concentration by only 8 percent.

Strong lensing is a function of cluster redshift (Bartelmann et al., 1995). Early studies found strong lensing clusters to lie typically around $z_L \approx 0.3$ (Wu & Hammer, 1993), however, the optimal value is increased for higher source redshifts (Li et al., 2005). Recent high-redshift cluster surveys, such as the Las Campanas Distant Cluster Survey (LCDCS; Zaritsky & Gonzalez, 2003), the MAssive Cluster Survey (MACS; Ebeling et al., 2001) and the Red-sequence Cluster Survey (RCS; Gladders et al., 2003) have provided samples of high redshift $0.3 < z < 0.7$ strong lenses (Horesh et al., 2010; Zitrin et al., 2011). In the present work, I select clusters from simulation snapshots at $z_L \approx 0.3$ to investigate. At this redshift, the angular diameter distance to the lens is $D_d \approx 650$ Mpc/h, so one arcsecond corresponds to a physical scale of ~ 3 kpc/h.

The Extent of the Lens

Gravitational lensing is the result of all mass that lies along the line of sight. While that mass may be dominated by a single primary lens, the projection of large scale structure can contribute to the strong arc abundance (Hennawi et al., 2007b). Indeed, previous studies have shown that the triaxial major axis of cluster haloes are somewhat preferentially aligned with external filamentary structure (Bailin & Steinmetz, 2005; Altay et al., 2006; Zhang et al., 2009) and lie parallel to the surface of nearby voids (Patiri et al., 2006; Brunino et al., 2007), and conversely, the minor axes lie perpendicular to the ‘host’ wall or filament (Aragón-Calvo et al., 2007) and are directed radially away from nearby voids (Patiri et al., 2006; Brunino et al., 2007). one must appreciate that these filaments may play an important role in skewing the σ_{MI} PDF at the high optical depth end. Hence, the mass within a few virial radii of the cluster centre may be projected onto the lens plane that is constructed for each orientation. The ‘lens’ would then contain not just the cluster, including substructure, but correlated large-scale structure as well. Thus, the lenses constructed from simulated clusters include all particles for which $r < \zeta_{\text{cut}}$, where r is the distance of the particle from the cluster centre. The results are repeated for a range of values for ζ_{cut} so that one can attempt to disentangle the contribution of various components of the lens (see Sec. 3.5).

3.2 Measuring the Cross Section for Strong Lensing

Ways of characterizing the cross section for strong lensing by a galaxy cluster lens vary across the literature; a range of quantifiers have been identified and are listed below:

- images with an arc length above a threshold which ranges from 5 to 10 arcseconds (Bartelmann et al., 1995)
- images with an arc width above a threshold of up to 3 arcseconds (Bartelmann et al., 1995)
- images with a curvature radius (relative to deflection angle at the image) above a threshold of up to 10 (Bartelmann et al., 1995)
- images with an arc length-to-width ratio above a threshold which ranges from 5 to 15 (Bartelmann et al., 1995; Bartelmann, 1995a; Oguri et al., 2003; Puchwein & Hilbert, 2009)
- images with magnifications above a threshold of 10 (Hilbert et al., 2007)
- probability of a source producing at least one image with an arc length-to-width ratio above a threshold which ranges from 5 to 15 (e.g. Flores et al., 2000; Fedeli et al., 2006, 2007; Meneghetti et al., 2010; Fedeli et al., 2010)

- probability of a source producing multiple images, at least one of which have a magnification above a threshold ranging from 5 to 25 (Wambsganss et al., 2004)
- probability of a source producing multiple images with separations above a threshold ranging from 10 to 30 arcseconds (Hennawi et al., 2007a)
- the Einstein radius above a threshold of up to 40 arcseconds (Oguri & Blandford, 2009)

Note that the cross sections defined in the image plane and those defined in the source plane are fundamentally different. The former are proportional to the probability of observing images with particular properties, and the latter are proportional to the probability of observing single-source *systems* with a particular property.

Most of the aforementioned strong lensing characterizations are aimed at comparisons with observations of strongly lensed *galaxies*, which are extended sources and will have highly distorted images, in addition to large magnifications. Measurements of the cross section for the production of giant arcs are complicated by the intrinsic ellipticities of the source galaxies. The present work quantifies strong lensing by the cross section for sources to produce multiple images, and as such is also able to provide an insight into the statistics of cluster-lensed *quasars*, which are point sources and whose images are not distorted. This is achieved by modelling the source caustic; the area of the caustic is directly proportional to the probability that an infinitesimal source at that redshift will be multiply imaged. Note that the cross-sections would be reduced for finite source sizes since a source that overlaps with the caustic will form merging, unresolved images; cross sections for multiply-imaged galaxies must include a range of source sizes. Ultimately, multiple imaging predictions formulated for comparison with a specific survey strategy must also take into account source colours, the point spread function (PSF) of the detector/s and filter sensitivity, however, this is outside the scope of this thesis. Since a large number of projection directions are to be studied, special consideration is given to formulating a computationally efficient algorithm for measuring the cross section for multiple imaging of an infinitesimal source, which is hereafter denoted σ_{MI} .

3.2.1 Source Redshift

Strong lensing efficiency is a steep function of source redshift (Wambsganss et al., 2004; Dalal et al., 2004), as the critical density for lensing (see Eqn. 1.20) has a strong dependence on this value, so although the focus here is on the assumptions placed on the cluster lens, a careful choice of source redshift is essential. The results here are presented for source redshifts of $z_s = 1$ and/or $z_s = 2$, a choice which is motivated by the range of known strongly lensed sources (e.g. Gilmore & Natarajan, 2009; Bayliss et al., 2011). The angular diameter distance to a source at redshift $z_s = 1$ is $D_s \approx 1180$ Mpc/h, while for $z_s = 2$ it is $D_s \approx 1240$ Mpc/h. Given a lens redshift of $z_L \approx 0.3$, the lens-source angular diameter distances are $D_{ds} \approx 760$ Mpc/h for $z_s = 1$ and $D_{ds} \approx 960$ Mpc/h for $z_s = 2$.

3.2.2 Drawing the Critical Curve

A large number of orientations are to be analyzed for each given cluster in the sample; the source caustic must be determined for each orientation of each cluster. To achieve this, the critical curve (or the caustic in the image plane) must be determined first, then mapped to the source plane (see Sec. 3.2.4). The caustic is primarily dependent on mass in the projected core of the cluster. Therefore, for a given cluster and orientation, the simulation particles belonging to the lens are projected onto one of two planes perpendicular to the line of sight, and distributed on a grid using a Cloud-in-Cell algorithm. One of these planes is a low resolution 512×512

mesh plane with a co-moving length to $2\zeta_{cut}$. It contains mass projected at sufficient projected distance from the core such that it plays a minor role in shearing a source behind the cluster centre. The other plane, containing the mass projected near the centre, has a higher spatial resolution (approximately 0.2''-0.3'' for the clusters studied in this work). This 512×512 mesh ‘core’ plane is also used to model the critical curve, so it must be large enough to encompass the critical curve. The size of the critical curve will generally scale with the (square root of the) projected mass within the core region of the cluster, but the projected mass will vary for each orientation; for computational efficiency, a choice of grid-size dependent on some cluster property, but reasonable for all orientations is preferred. Since the projected mass will be dominated by the unprojected mass within the cluster core, a rough (over)estimate for the radius of the critical curve, θ_{cc} , is made by the Einstein radius for a Schwarzschild lens (see Eqn. B.3) with the same mass as that contained within a fixed fraction of the virial radius; after some trial and error, this fraction is chosen to be one fifth:

$$\theta_{cc} = \sqrt{\frac{D_{ds}}{D_d D_s} \frac{4GM(< 0.2r_{vir})}{c^2}}. \quad (3.1)$$

The size of both planes are therefore different for each cluster and dependent on source redshift via the angular diameter distances. In addition, the particles attributed to each grid are only chosen if their projected positions, θ_p , lie well within the grid scale in order to ensure that no mass bleeds out during smoothing (see Sec. 3.2.3). For example, a particle will be placed on the core plane only if it satisfies $\theta_p \leq 0.85\theta_{cc}$. Otherwise, it may be eligible for inclusion in the low-resolution plane. Convergence tests showed that doubling the resolution for either plane did not significantly alter the results.

The convergence, $\kappa(\boldsymbol{\theta})$, is mapped across the plane by scaling the mass in the grid according to Eqn. 1.19. The shear, $\gamma(\boldsymbol{\theta})$, is mapped across each of the planes using Fourier techniques, following Wambsganss (1990). Firstly the two shear components are evaluated by convolving the convergence field with the appropriate kernel (see Eqns. 3.2 and 3.3). Then they are combined according to Eqn. 3.4, which follows from Eqn. 1.21.

$$\gamma_1(\boldsymbol{\theta}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{-(\theta_1 - \theta'_1)^2 + (\theta_2 - \theta'_2)^2}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^4} \kappa(\boldsymbol{\theta}') d^2\boldsymbol{\theta}' \quad (3.2)$$

$$\gamma_2(\boldsymbol{\theta}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{-2(\theta_1 - \theta'_1)(\theta_2 - \theta'_2)}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^4} \kappa(\boldsymbol{\theta}') d^2\boldsymbol{\theta}' \quad (3.3)$$

$$\gamma(\boldsymbol{\theta}) = \sqrt{\gamma_1^2(\boldsymbol{\theta}) + \gamma_2^2(\boldsymbol{\theta})} \quad (3.4)$$

The shear in the ‘core’ plane, at this stage, only includes the effect of central mass. In the central region of the large-scale low resolution plane, there is some minor shearing as a result of cluster triaxiality, substructure and external structure. This is added to the shear determined in the ‘core’ plane via bilinear interpolation. The determinant of the lens mapping Jacobian can then be mapped within the ‘core’ plane via Equation 1.24. The critical curve is defined as the locus of points at which the determinant of the Jacobian vanishes, such that the magnification diverges (see Section 1.4.1).

3.2.3 Smoothing

The projected mass distribution is smoothed with a Gaussian filter to reduce the effects of shot noise that result from discretizing the underlying mass distribution. Regions of different densities have different smoothing requirements, so three distinct regions are identified for

density-dependent smoothing: the cluster centre, the outer core and the outer cluster + external structure. The latter corresponds to mass that has been projected onto the low-resolution lens-plane (see Sec. 3.2.2) and is *always* smoothed with a Gaussian filter of size σ_{ext} for any given projection. The mass projected onto the high-resolution core is distributed onto one of two grids, each with the same resolution as the parent grid. One grid contains the mass that is projected to a cluster-centric radial distance of $\theta/\theta_{cc} < 0.25$, and is smoothed with Gaussian filter of size σ_{inner} . The rest of the mass is placed on the other grid and smoothed with Gaussian filter of size σ_{outer} . Each grid is zero-padded before smoothing (since the mass distribution is not periodic) then the smoothed density fields added together again.

The smoothing length, σ_R , for a given region (denoted by the subscript R) is empirically chosen by applying a linear relationship with the mean projected interparticle spacing averaged over that region, λ_R :

$$\sigma_R = 0.3\lambda_R, \quad (3.5)$$

where the interparticle spacing is given by

$$\lambda_R = \frac{\sqrt{A_R}}{N_R^{1/3}}. \quad (3.6)$$

Here A_R denotes the surface area of the region and N_R denotes the number of simulation particles that would be projected onto that region. Further discussion on the ratio of smoothing length to the mean projected interparticle spacing is presented in Sec. 3.6.2. Note that the filters used are each truncated so that they are circular and do not involve computations that result in underflows; however, they are still normalised.

3.2.4 The Cross Section for Multiple Imaging

The multiple imaging cross section is defined as the area of the region bounded by the source caustic (see Figure 3.1). The critical curve must therefore be mapped to the source plane. The lens mapping, given in Eqn. 1.29, is determined by the deflection angle across the lens plane. Thus, the deflection angle vector is mapped across the lens plane by convolving the surface density with an appropriate kernel given in Eqn. 1.28. Then the grid containing the critical curve is scanned horizontally and vertically; when a point on the critical curve is encountered, it is mapped to the 512×512 mesh source plane, thus allowing one to ‘draw’ the caustic. Difficulties may arise as the number of points on the critical curve can be fewer than the number of points required to form a closed boundary at the source, particularly when mapping the radial caustic. This ‘closed boundary requirement’ is necessary for the measurement of the cross-section. In addition, the source plane has an angular size of $\sim 2/3$ of the lens plane so that the maximum possible detail may be recovered, while still ensuring that the caustic is entirely contained; this aggravates the problem at hand. So immediately after each critical point in the lens-plane has been mapped, a check is made against the four neighbouring cells that have already been scanned (out of a total eight neighbouring cells). Any of the four neighbouring cells that also lie on the critical curve are identified. If their corresponding caustic points are separated from the most recently mapped caustic point by more than a pixel, these are linearly interpolated to produce extra caustic points, thereby closing any gaps. In addition, the source caustic is smoothed by a 2×2 -pixel top-hat filter to thicken each line. Each cell in the source-grid that forms this caustic is ‘tagged’ as such. The bounded area is then measured by applying the following computationally efficient algorithm.

1. Begin at any corner of the grid.

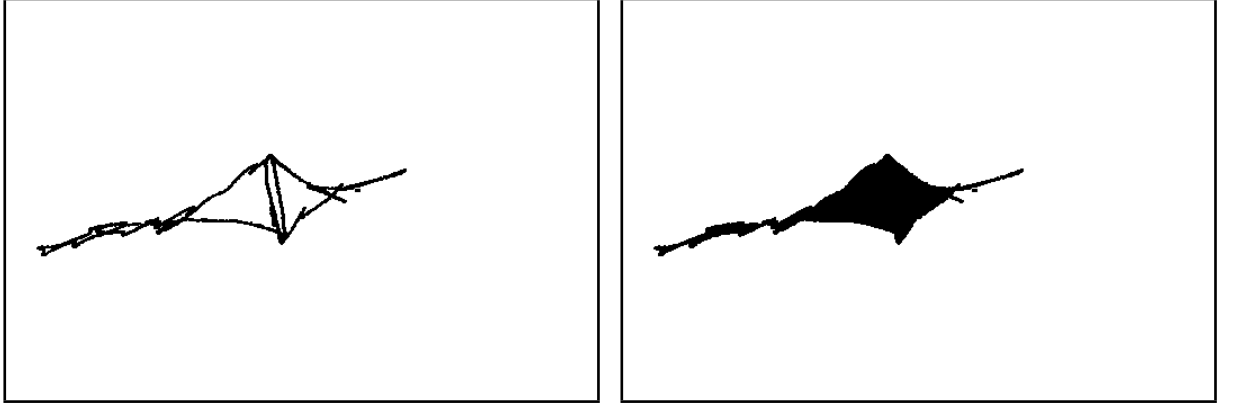


Figure 3.1: An example of a source caustic (left) and the black region identified as the ‘multiple-imaging area’ used to calculate the multiple imaging cross section (right). The $z=2$ source planes are $\sim 49''$ across and have a pixel resolution of $\sim 0.1''$. This caustic has a cross-sectional area of $\sigma_{\text{MI}} \approx 46 \text{ arcsecs}^2$.

2. Identify the four neighbour cells that share an edge with the current cell.
3. Check each of these neighbouring cells to see if they have been ‘tagged’ as either lying on the caustic boundary, or as having been checked already. If *not*, then tag them as being *outside* the multiple-imaging region, tag them as having been checked and add them to a ‘check-neighbour-list’.
4. Go to the next cell on the ‘check-neighbour-list’, making that the current cell. Then repeat steps 2 and 3. When there are no more cells left on the list, subtract the number of cells tagged as lying outside the multiple-imaging region from the total number of cells in the source-grid. The result is proportional to the required cross-section.

Note that this algorithm makes the assumption that the caustic does not touch the edges of the source grid, i.e. it is fully encompassed by the source plane. Also all cells that lie on the boundary are not included within the multiple imaging region.

3.3 Characterizing Cluster Properties

Some of the present study is an investigation into efficacy of cluster properties on the probability distribution of σ_{MI} . The practical definitions of these properties are described below.

3.3.1 Mass

Each of the haloes identified by AHF has a virial mass fitted using spherical overdensities relative to background density. Since one does not know the triaxiality of the halo in advance, halo finding routines use spherical shells to determine the radius which encloses the required overdensity. Although a triaxial halo should be fitted with equally triaxial overdensities, for a rough categorizing of clusters by their masses a spherical fit is sufficient.

3.3.2 Concentration (a.k.a. Peakiness)

The concentration parameter has been introduced in the context of the popular NFW profile (see Sec. 1.3.3); this property is defined with respect to a choice of overdensity and scale. An alternative is to define a parameter $c_f \equiv r_{vir}/r(fM_{vir})$ which is the ratio of the virial radius and the radius that contains a specified fraction, f , of the virial mass (Colín et al., 2000; Knebe et al., 2002; Power & Knebe, 2006). This is often useful when the clusters are not easily fit to an NFW profile. In the present work, for computational efficiency, a rough estimate of ‘peakiness’ for a given halo is determined by taking the ratio of the virial mass to the mass enclosed within a fixed fraction of the virial radius. Our ‘concentration parameter’ is therefore defined as

$$c_f = \frac{M(< fr_{vir})}{M_{vir}}. \quad (3.7)$$

In this work, I adopt $f = 1/5$.

3.3.3 Triaxiality

Triaxiality is intuitively a measure of the extent to which a smooth axisymmetric profile deviates from spherical symmetry. The triaxial shape is defined via the axis ratios q and s (see Sec. 1.3.4), however, several authors have introduced a single parameter to describe an intrinsically two-parameter effect in a suitable manner. For example, one may compare the longest and shortest axis as a measure of asphericity:

$$T_1 \equiv \frac{a}{c}. \quad (3.8)$$

On the other hand, one may also characterise the shape with the parameter

$$T_2 \equiv \frac{a^2 - b^2}{a^2 - c^2} = \frac{1 - q^2}{1 - s^2}, \quad (3.9)$$

so that $T_2 = 1$ represents a maximally prolate shape and a halo for which $T_2 = 0$ is maximally oblate (Franx et al., 1991; Warren et al., 1992; Knebe et al., 2002). In addition, Macciò et al. (2007) adopt

$$T_3 \equiv \frac{b + c}{2a} = \frac{1}{2}(q + s), \quad (3.10)$$

which can distinguish between spherical and prolate haloes. The AHF catalogues include the axis ratios q and s from which any of these parameters may be constructed.

3.4 Preliminary Study: An Individual Halo

Here I present a detailed study of a single cluster-mass halo. The cluster studied here is taken from a cosmological simulation carried out with the parallel Tree-PM SPH code **GADGET2** (Springel, 2005). I adopt a Λ CDM cosmology, with the following values for the cosmological parameters: $\Omega_{M,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$, $h=0.71$, $\sigma_8 = 0.9$; the density normalization is higher than the current standard, but is the same as that adopted by Dalal et al. (2004) so this allows for consistent comparison. The 256^3 -particle simulation had a box-size of $L_{\text{box}} = 100$ Mpc/h giving it a mass resolution of $m_p = 4.5 \times 10^9 h^{-1} M_\odot$. The simulation was started at an initial redshift of $z_i = 39$ until the present epoch, but the cluster is the most massive halo identified by AHF from the $z = 0.3$ snapshot. The softening length applied began at $\epsilon_{co} = 16$ kpc/h, capped at $\epsilon_{phys} = 5$ kpc/h. This cluster halo has a mass of $M \approx 8.5 \times 10^{14} h^{-1} M_\odot$ containing $\sim 190,000$ particles, making it one of the most massive haloes one may expect to find in such

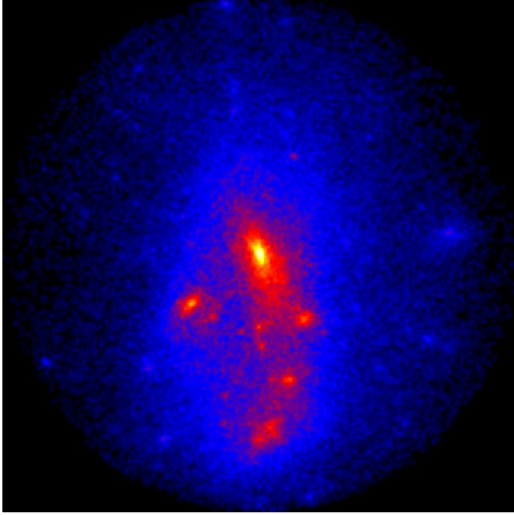


Figure 3.2: A simulated cluster at $z = 0.3$ and all mass contained within one virial radius of its centre. The mass is projected such that the triaxial major axis of the cluster is aligned vertically. See Sec. 3.4 for details.

simulations. It has a virial radius of 2 Mpc/h measured with an overdensity of 200 *relative to the mean matter density*¹. It is quite a prolate cluster ($q = b/a = 0.7$; $s = c/a = 0.5$).

The lensing features were analysed after the lens was projected along 200 random lines of sight. I address the choice of number of projections in Sec. 3.6.1. Mass projected outside the ‘core region’ was smoothed with a Gaussian filter of size $\sigma_{ext} = 6$ kpc/h. Figure 3.3 presents examples of the critical curves and caustics produced by this $z=0.3$ cluster lens for a source redshift of $z=2$, in order of decreasing strong lensing cross section. The same lens can produce vastly different caustic structure when viewed from different angles. The largest multiple imaging cross sections are generally the result of a large radial caustic compared to the tangential caustic. A large range of cross-sections for multiple imaging are produced with different orientations, with values ranging from < 10 arcsecs² to > 140 arcsecs². The total mass projected into the core (in the vicinity of, but not necessarily within, the critical curve) ranges from 10 to 20% of the total virial mass of the cluster. Previous studies found variations in cross-sections albeit those associated with the formation of giant arcs; Dalal et al. (2004) found values ranging from 0 to over 1500 (kpc/h)² in the source plane, which corresponds to 0 – 45 arcsecs² at a source redshift of $z_s = 1$. The cross section is fairly small ($\sigma_{MI} < 40$ arcsecs²) when the line of sight makes a large angle ($\theta > 50^\circ$) with the triaxial major axis as seen in Figure 3.4. Conversely, if the lens is viewed *along* this axis ($\theta \lesssim 40^\circ$), the cross section can become quite large ($\sigma_{MI} > 70$ arcsecs²), a result that is consistent with the findings of Meneghetti et al. (2010) in that strong lensing clusters are preferentially aligned with the major axis. Figure 3.4 exhibits much scatter at small angles; this may indicate that the role of substructure and/or external structure is non-negligible.

The cross section probability distribution, shown in Figure 3.5, is highly skewed towards large cross sections, as a result of these special orientations. The mass distribution in the cluster can be interpreted as a smooth cluster-scale halo, with galaxy-scale haloes embedded within their host. The triaxiality of the cluster-scale halo alone could be expected to provide the orientation bias that leads to the skewed distribution, but if the member galaxies of the cluster lie preferentially along the line of sight, then they may also have a similar effect. Indeed, Dalal et al. (2004) warned about the effects of shot-noise, particularly with regards to the projection of substructure.

¹The AHF group-finding algorithm uses a overdensity, Δ_M , relative to the mean matter density rather than the critical density. The virial radius is therefore given by: $r_{vir}^3 = 3M_{vir}/[4\pi\Delta_M\Omega_{M,0}\rho_{c,0}(1+z_L)^3]$.

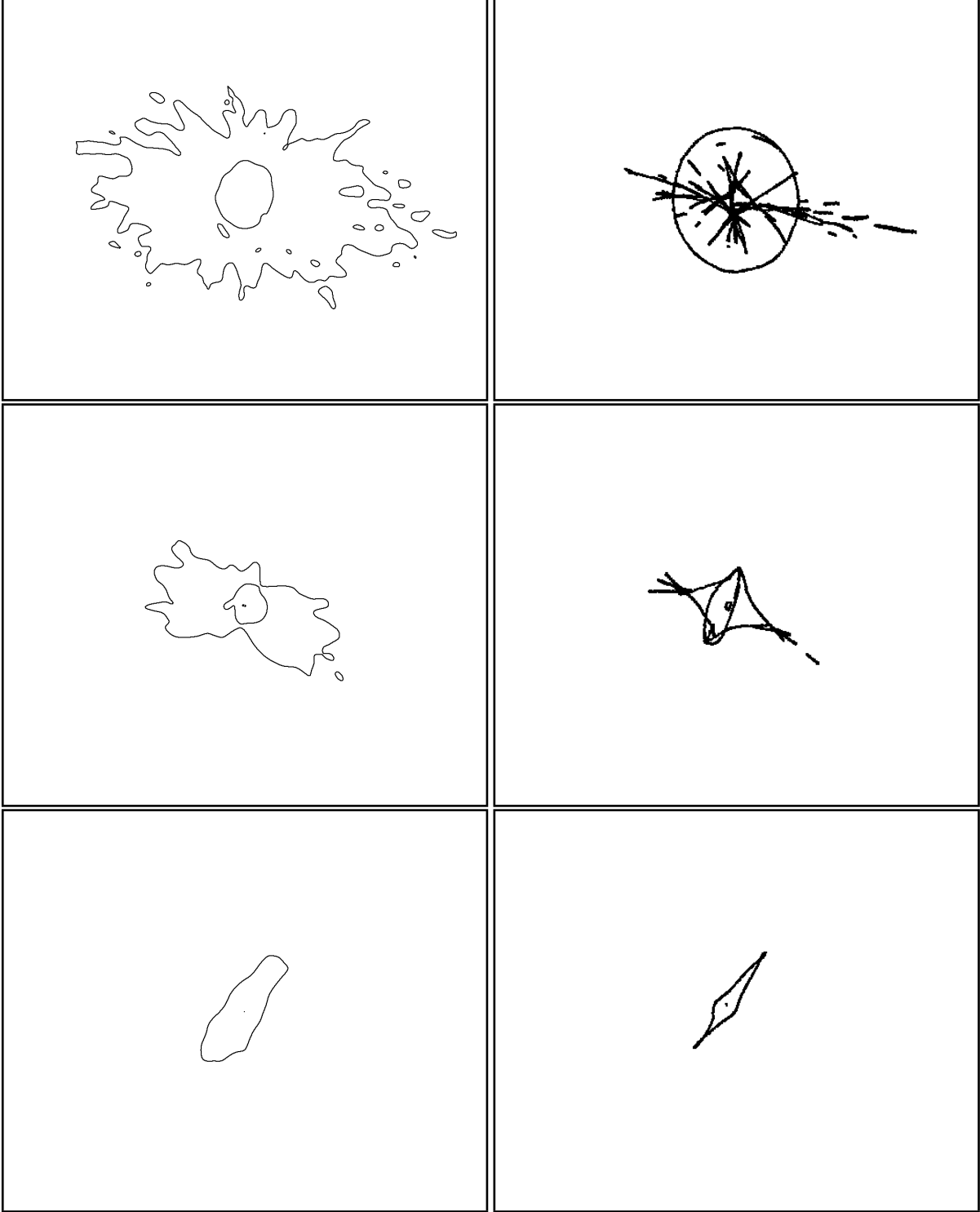


Figure 3.3: Various image and source caustic structures produced by the *same* large $8 \times 10^{14} M_{\odot}$ mass halo. On the left are the critical curves at $z=0.3$, each drawn on lens planes $\sim 105''$ across. On the right are the corresponding source caustics at $z=2$, each drawn on source planes $\sim 49''$ across. The orientation angles are, from top to bottom, 19° , 31° and 67° relative to the triaxial major axis. The resulting cross sections for multiple imaging are (top to bottom, in arcsecs^2) 146, 32, and 11.

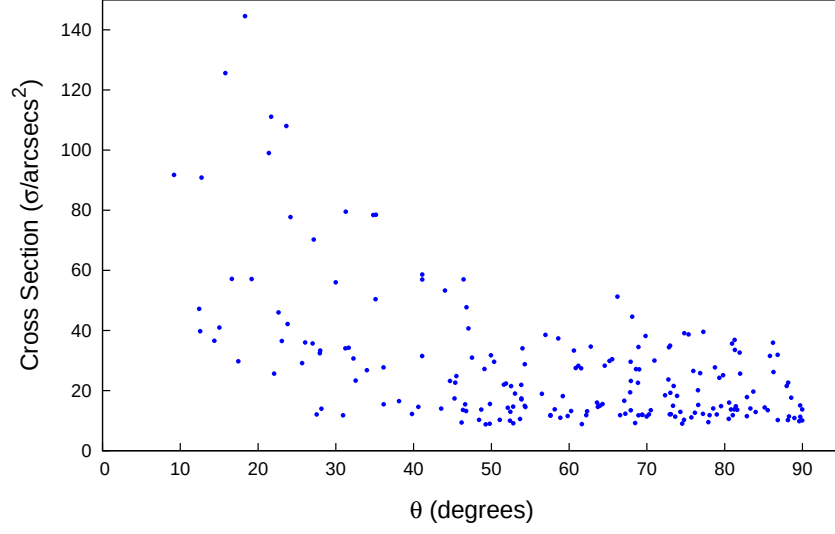


Figure 3.4: The cross section for multiple imaging, σ_{MI} , as a function of viewing angle, θ . θ is measured relative to the triaxial major axis. A total of 200 orientations are sampled and the source plane is at redshift $z = 2$.

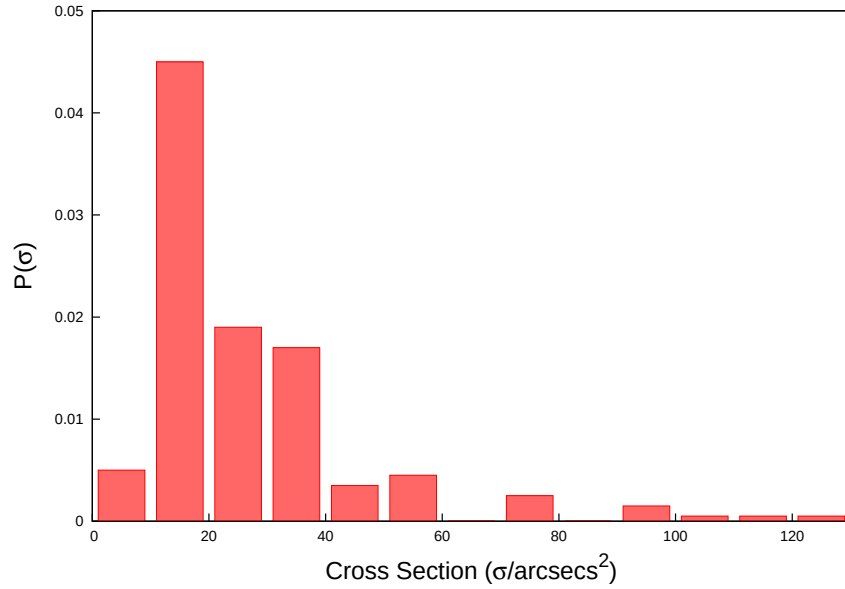


Figure 3.5: The probability distribution function for the cross section for multiple imaging, σ_{MI} , for a $8 \times 10^{14} M_{\odot}$ mass halo. A total of 200 orientations are sampled.

Virial Mass ($10^{14}M_{\odot}$)	Peakiness (Mpc/h)	$s = c/a$	$q = b/a$
12	0.13	0.75	0.79
12	0.13	0.61	0.72
11	0.16	0.62	0.88
11	0.11	0.68	0.88
9.6	0.16	0.69	0.77
9.6	0.19	0.64	0.76
9.2	0.14	0.59	0.78

Table 3.1: Properties of the seven simulated clusters presented in Sec. 3.5, listed in decreasing order of mass.

3.5 Associated Mass Outside the Virial Radius

While analytical models of cluster lenses have a finite extent, real lenses include the effect of all mass along the line of sight. In this section, I present the results of an analysis of the dependence of the σ_{MI} PDF on the inclusion or exclusion of external filamentary structure, for a range of source redshifts. This study uses massive haloes identified in a set of N-body simulations, with moderate box-size. The cosmological simulations each follow 256^3 particles of mass $m_p = 8 \times 10^{10} M_{\odot}$ in a cubic region of co-moving side $L_{\text{box}} = 256$ Mpc/h (assuming periodic boundary conditions) from an initial redshift of $z_i = 39$ until the present epoch using the parallel Tree-PM code **GADGET2** (Springel, 2005). The force softening length was chosen to begin at 50 kpc/h co-moving, but was capped at $\epsilon_{\text{phys}} = 20$ kpc/h. I adopt a Λ CDM cosmology, with the following values for the cosmological parameters: $\Omega_{\text{M},0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$, $h=0.71$, $\sigma_8 = 0.9$; these are the same values as those adopted by Puchwein & Hilbert (2009). The clusters are taken from the $z = 0.3$ snapshots.

The minimum number of particles required for a reasonable resolution of the halo core is 10^4 (see Sec. 3.1.1), corresponding to a halo mass of $M > 8 \times 10^{14} M_{\odot}$. However, the peak and spread in the cross-section distribution, based on a handful of orientations in one of the simulations, was found to be a steep function of cluster mass within the range $8 \times 10^{14} < M/M_{\odot} < 9.5 \times 10^{14}$, so only clusters with masses above $M/M_{\odot} > 9 \times 10^{14}$ were considered in this study. Of the four simulations run for this purpose, one contained four haloes, another contained three haloes and two contained none above this mass cutoff. The most massive cluster halo had a mass of $M/M_{\odot} = 2 \times 10^{15}$. Details of their properties are listed in Table 3.1. Note that these clusters like that studied in Sec. 3.4 have virial mass and radii determined using a virial overdensity defined relative to the mean matter density.

Figure 3.6 shows a single projection of each of the seven cluster-haloes. The projection directions were chosen such that the triaxial major axis would be aligned vertically, but the rotational angle was chosen at random. The figures show all mass within five virial radii of the cluster centres. The projected mass distribution seen is consistent with the findings of previous works: that this external mass is preferentially aligned with the major axis (see Sec. 3.1.1).

For each of the total of seven haloes included in the study, 150 orientations are analysed to produce their respective σ_{MI} PDFs. In all orientations of all clusters, the mass projected outside the ‘core region’ of each cluster was smoothed with a Gaussian filter of size $\sigma_{\text{ext}} = 10$ kpc/h. The analysis is repeated for source redshifts $z_s = 1.0$ and $z_s = 2.0$ and for $\zeta_{\text{cut}} = 2$ Mpc/h and

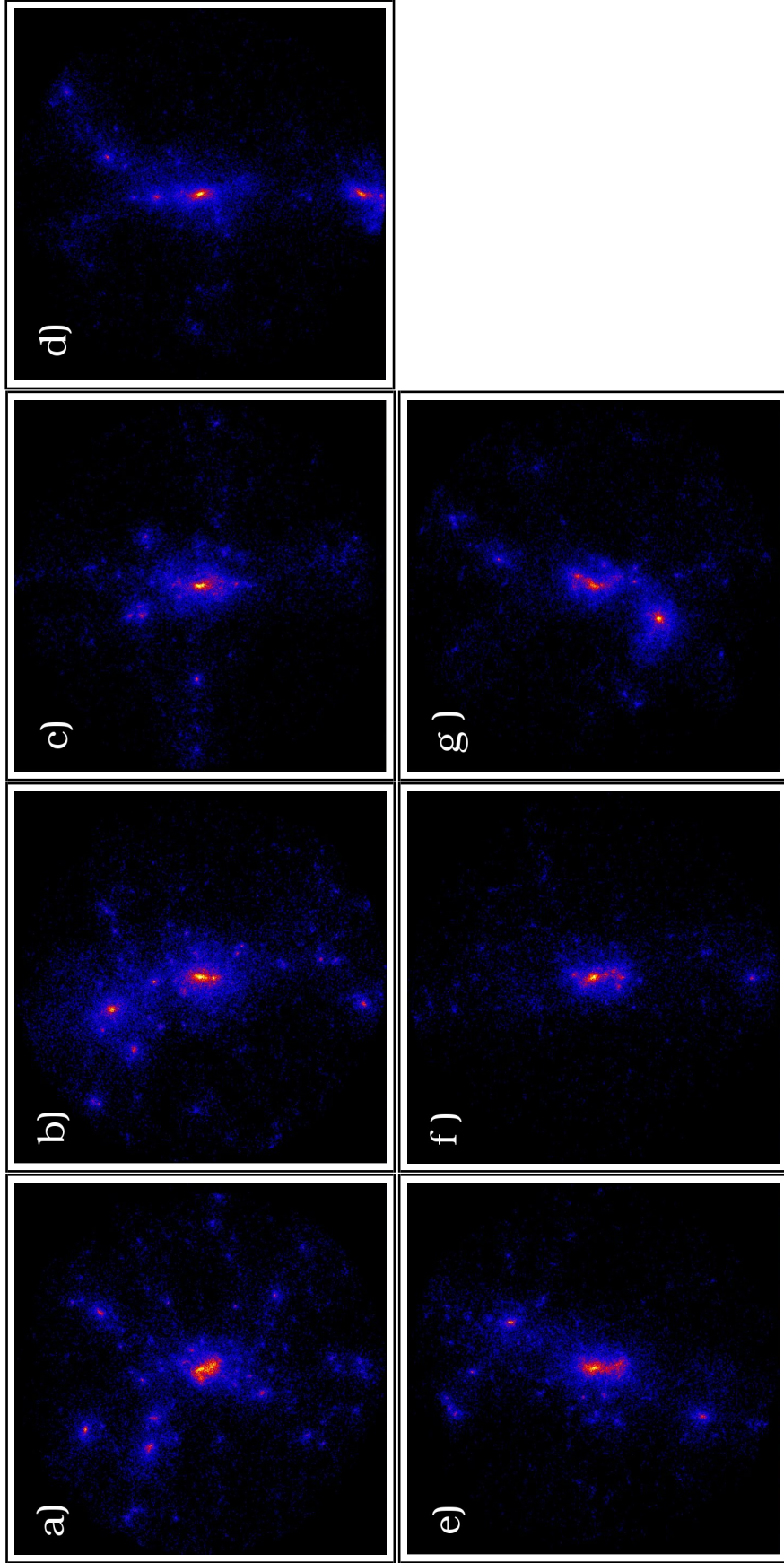


Figure 3.6: Each of the seven clusters analysed in Sec. 3.5. Here, they are projected with major axes aligned vertically along the page. Note that the projection is not unique. The mass within 5 virial radii of the cluster centre is included in each picture.

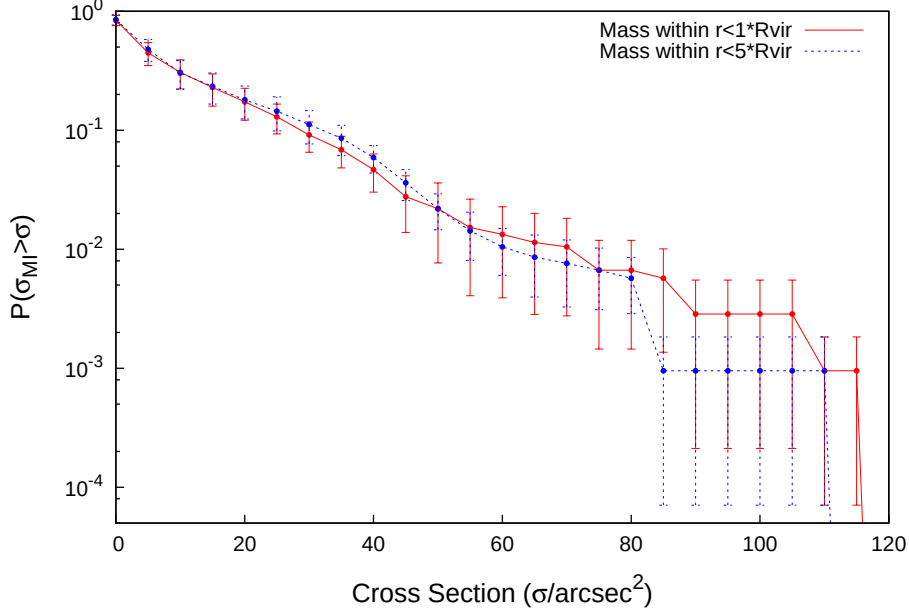


Figure 3.7: The cumulative probability distribution function for the cross section for multiple imaging (σ_{MI}) averaging the results for seven haloes with masses $9 \times 10^{14} < M/M_{\odot} < 2 \times 10^{15}$. The source plane is at redshift $z = 1$. 150 lines of sight were sampled for each halo. The error-bars represent a 1σ spread across all haloes.

$\zeta_{\text{cut}} = 10$ Mpc/h. The virial radii are approximately 2 Mpc/h for each cluster, so this allows a comparison of lens masses that do or do not include mass outside the virial radii. The σ_{MI} PDFs for all seven haloes are stacked and presented in Figures. 3.7 and 3.8.

The σ_{MI} PDFs are not affected by the inclusion of mass outside the virial radius. Even the filamentary structures, which are supposedly preferentially aligned with the major axes of these clusters do not have any significant effect on the σ_{MI} PDFs. One would expect that the small cross sections, which generally arise from lines of sight misaligned with the major axis, would not be affected by the presence of such filaments. However, the lines of sight that are aligned with the major axis lead to large cross sections that could be further enhanced by the elongated structures, as they would be projected onto the lens plane, but surprisingly, no such effect is seen. At lower source redshifts (see Figure 3.7), the inclusion of external mass appears to *reduce* the probability of large strong lensing cross sections ($\sigma_{\text{MI}} > 80$ arcsec²). This indicates that these strong lensing cases are so rare that the PDF suffers from small number statistics. Wambsganss et al. (2005) and Hilbert et al. (2007) have previously found that for a fraction of images that exhibit strong lensing characteristics, a single lens-plane (one of the planes from their multiple lens-plane method) was not sufficient to explain the strong lensing effect. Both studies found a strong redshift dependence; the requirement for a second lens plane to account for the strong lensing grew stronger with increasing source redshift. At a source redshift of $z_s = 1 - 2$, the fraction of affected images was 10 – 20% depending on the strong lensing criterion. Their lens plane thicknesses were on the order of 100 Mpc/h, so the relevant secondary lensing mass was unassociated with the primary lens. Puchwein & Hilbert (2009) studied the formation of giant

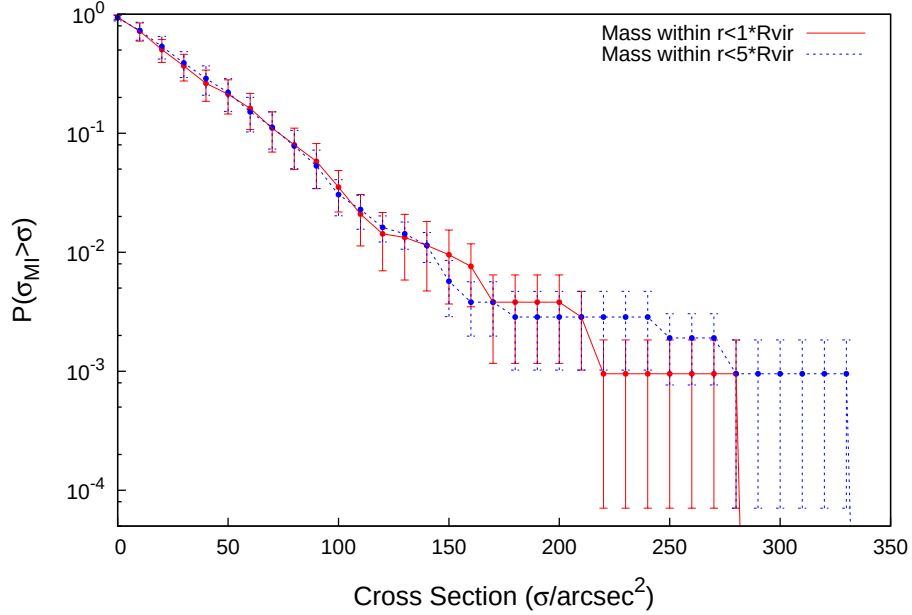


Figure 3.8: The cumulative probability distribution function for the cross section for multiple imaging (σ_{MI}) averaging the results for seven haloes with masses $9 \times 10^{14} < M/M_{\odot} < 2 \times 10^{15}$. The source plane is at redshift $z = 2$. 150 lines of sight were sampled for each halo. The error-bars represent a 1σ spread across all haloes.

arcs by eight cluster lenses in a similar mass range and redshift to those studied here. They found that the mass along the line of sight, more than 4 co-moving Mpc/h from the cluster, had no effect on the strong-lensing cross-sections across a wide range of source redshifts ($1 \lesssim z \lesssim 6$). Indeed, uncorrelated structure that lie along the line of sight were more likely to increase the strong lensing cross section than any correlated structure. In the present study, I include, at most, mass up to ~ 10 Mpc/h away, as this is more likely to be aligned with the cluster major axis. Puchwein & Hilbert (2009) also study lower mass clusters and find that they are more affected by line-of-sight structures; comparison to other mass ranges is outside the scope of this thesis. My findings, regarding multiple imaging, are consistent with those of Wambsganss et al. (2005), Hilbert et al. (2007) and Puchwein & Hilbert (2009), which are concerned with the formation of giant arcs. This is not unexpected, since both methods of characterising strong lensing are dependent on the size of the source caustic; however, multiple imaging is dependent on the area of the caustic, while the formation of giant arcs require a large perimeter for the tangential caustic (the radial caustic produces less distinguishable images). More elliptical lenses would lead to more elliptical caustics, and a closer equivalence the two measures, and vice versa. Ho & White (2005) noticed a strong orientation dependence on the giant arc cross section for one of their simulated clusters; while they attribute the strong lensing efficiency to a combination of neighbouring structure (apparently galaxy-scale) and filaments, my results suggest that the shape of the virialized mass is the dominant factor in determining the efficient orientations.

A related, but somewhat separate question is whether the orientation dependence is enhanced by mass associated with the lens, but outside the virial radius. To answer this question, the same

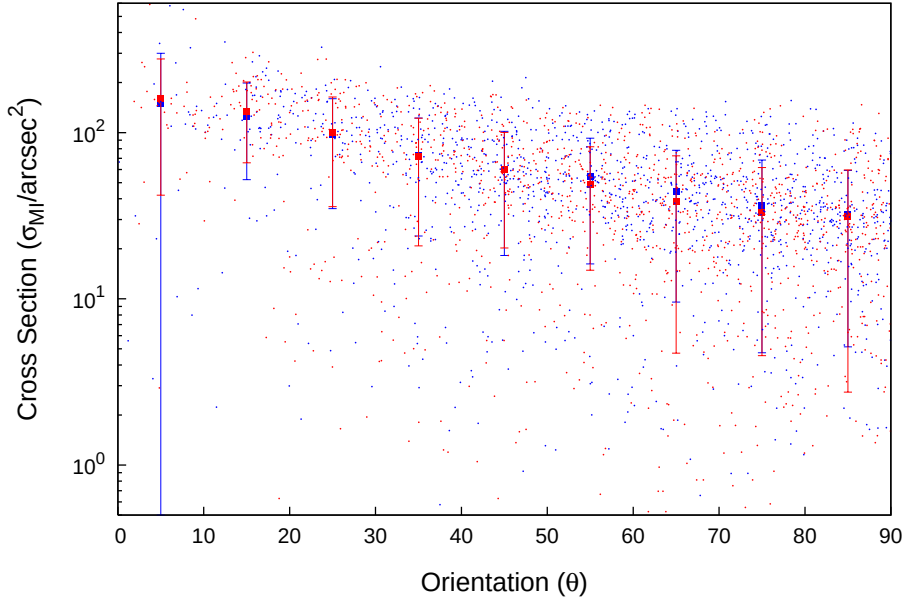


Figure 3.9: The cross section for multiple imaging (σ_{MI}) as a function of viewing angle, θ . θ is measured relative to the triaxial major axis. Each of the seven haloes are sampled with 200 orientations, and the results are combined here. Individual lines of sight are plotted as points while the mean cross-section of lines of sight uniformly binned by orientation are plotted as the larger squares, with 1σ error-bars. The lensing mass includes all mass within one virial radius (red) and 10 virial radii (blue) of the cluster centres. The source plane is at redshift $z = 2$.

seven clusters are analyzed again, with 200 orientations for each and *all 1400 orientations* and the resulting cross sections are presented as individual points in Figure 3.9. The orientations are also combined in bins of 10° in θ , the angle relative to the triaxial major axis. A clear orientation dependence is seen: lines of sight that are more aligned with the major axis lead to larger cross sections for multiple imaging. However, when we compare the results for lenses which include only the virial mass and lenses where mass out to 10 virial radii (20 Mpc/h) have also been projected, the *orientation dependence is not enhanced*.

3.6 Numerical Parameters

3.6.1 Number of Projections Analysed

The skewness in the σ_{MI} PDF makes it clear that a large number of projections are required to provide a full understanding of the range of strong lensing cross sections that can be produced by the same cluster. Throughout this work 150-200 orientations have been used to produce the relevant cumulative or differential probability distribution function. In this section, I confirm that this sample size is sufficient. The analysis of the simulated cluster discussed in Sec. 3.4 was repeated with three independent tests using 50 orientations each, three independent tests using 200 orientations, and one test using 450 orientations. The resulting cumulative distributions,

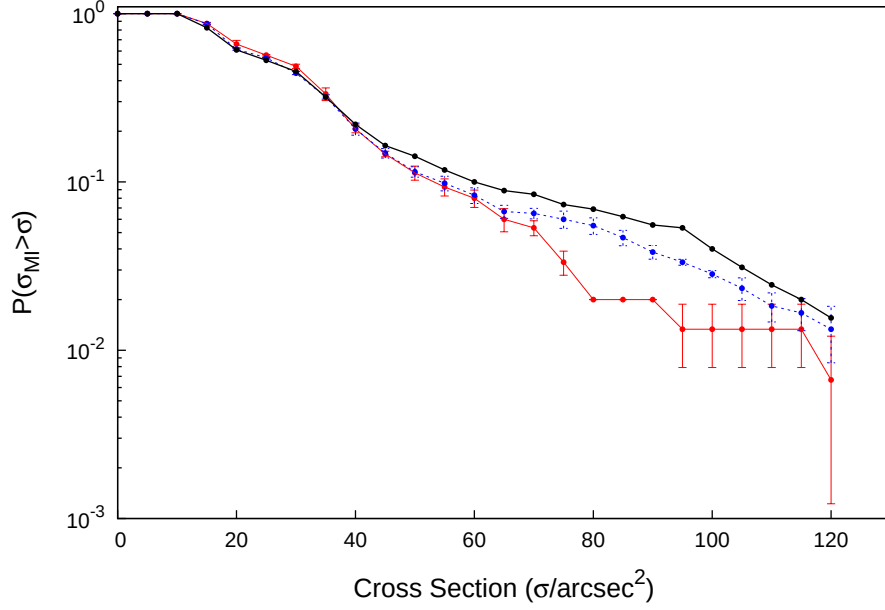


Figure 3.10: The cumulative probability distribution function for the cross section for multiple imaging (σ_{MI}). The results of analysing a single simulated cluster are shown for independent tests of 50 orientations (red), 200 orientations (blue) and 450 orientations (black). The source plane is at redshift $z = 2$. The error-bars represent a 1σ spread across each of the tests.

shown in Figure 3.10 show that 50 orientations will not sufficiently sample the rare orientations that results in large cross sections ($\sigma_{\text{MI}} > 60 \text{ arcsec}^2$). In fact, the small error bars for $80 < \sigma_{\text{MI}} < 90$ reflect that fact that out of each 50-orientation sample, no orientations led to $80 < \sigma_{\text{MI}} < 90$ and only one orientation resulted in a cross section of $90 < \sigma_{\text{MI}}$. However, 450 orientations provide only a minor change in the measured distribution compared to 200 orientations. Note that each point is not independent since these are cumulative distributions.

3.6.2 Smoothing Length

As described in Sec. 3.2.3, the projected lensing mass distribution is smoothed with a Gaussian filter in order to reduce the effects of shot-noise. Different regions are smoothed over different scales, but the smoothing length for a given region is chosen according to a criterion based on particle surface density in that region; the size of the filter, σ , is fixed to a fraction of the mean projected inter-particle separation, λ . The choice of the ratio σ/λ has the potential to affect the measure multiple imaging cross section. In an extreme case, if the smoothing length were to exceed the mean inter-particle separation, the simulated small-scale structure would be lost. Therefore, a range of choices for this ratio, all less than unity, was applied to the analysis of the simulated cluster that was presented in Sec. 3.4; the resulting cumulative distributions of σ_{MI} are presented in Figure 3.11. Smaller smoothing lengths produce grainy caustics and σ_{MI} is overestimated. As the smoothing length is increased to $\sigma/\lambda > 0.3$, the cumulative distribution converges at smaller cross sections ($\sigma_{\text{MI}} < 40 \text{ arcsrec}^2$), but the behaviour at larger cross sections is erratic. What is unclear is whether larger smoothing lengths are more appropriate or if

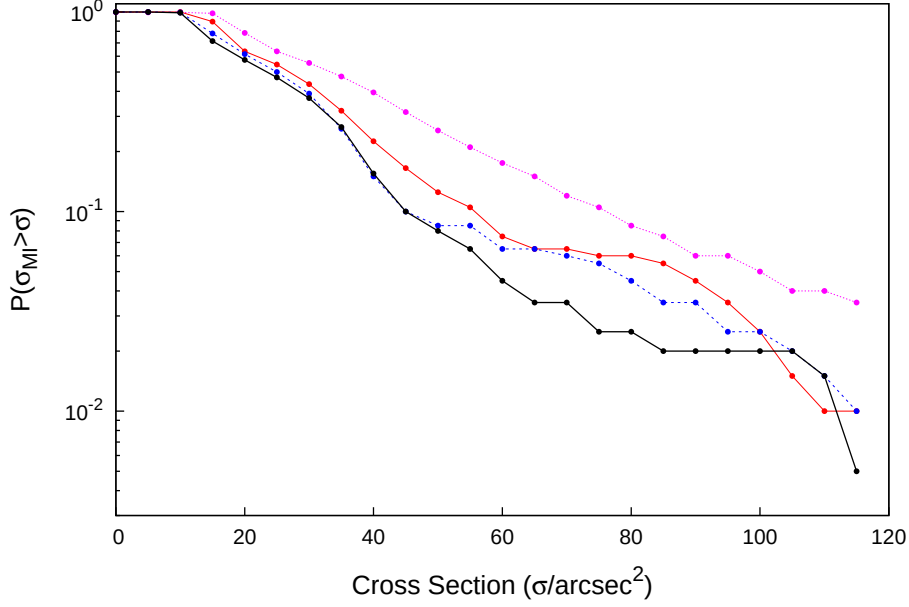


Figure 3.11: The cumulative probability distribution function for the cross section for multiple imaging (σ_{MI}). The results of analysing 200 orientations of a single simulated cluster are shown for various smoothing lengths: $\sigma/\lambda = 0.2$ (pink), $\sigma/\lambda = 0.3$ (red), $\sigma/\lambda = 0.35$ (blue) and $\sigma/\lambda = 0.4$ (black). The source plane is at redshift $z = 2$.

realistically simulated small-scale structure is being washed out, producing unphysical profiles. I conclude that smoothing length must at least exceed $\sigma/\lambda > 0.2$.

3.7 Summary

The large mass simulated haloes I have studied have revealed a number of important results regarding their ability to gravitationally lens a background source into multiple images. Firstly, the orientation dependence observed in previous works has been confirmed: the closer the line of sight is aligned with the triaxial major axis, the more likely a source will be multiply-imaged. Thus, samples of clusters chosen for the existence of multiple-imaging systems will be biased towards stronger line-of-sight alignments. Secondly, this orientation dependence is not enhanced by the projection of mass up to 10 virial radii or 20 Mpc/h from cluster centres. This is despite the mass being preferentially aligned to the triaxial major axis. Neither is the probability distribution for the cross section for multiple imaging affected by the projection of mass out to 10 Mpc/h. This is the most surprising result and implies that analytic models that describe the mass distribution of the primary lens will be sufficient for accurate predictions regarding the frequency of multiple imaging.

Chapter 4

Weak Lensing via Ray Tracing

Bloody hindsight, thought Archie. It's always 20/20.

– Archibald Jones, White Teeth, Zadie Smith

The distance-redshift relation, a fundamental property used to constrain cosmological parameters, suffers from scatter as a result of gravitational lensing, although the effect on the constraints remain unclear (e.g. Holz & Linder, 2005; Jönsson et al., 2006; Okamura & Futamase, 2009). Individual high-redshift sources, such as SNe, gamma ray bursts (GRBs) and merging supermassive black holes (MSBs), suffer from lensing-induced errors in distance and/or luminosity estimates (Babul & Lee, 1991). This was expected from the early work of Gunn (1967) who studied light propagation through an inhomogeneous universe and found that the mean brightness remained unchanged for extended sources, but the effect was worse for smaller and higher-redshift sources. Efforts are now underway to provide an appropriate method of correcting the observed brightnesses for magnification (e.g. Gunnarsson et al., 2006; Jönsson et al., 2009; Hilbert et al., 2010).

In addition, as described in Sec. 1.4.3, the probability distribution for magnification by gravitational lensing is required to make correction to predicted source counts for large surveys of high-redshift sources. Furthermore, time-delay studies that aim to constrain the Hubble Constant use the probability distribution for convergence to understand the errors on the measured H_0 due to the unknown external mass sheet (Suyu et al., 2010).

Studies of the statistical properties of large-scale weak lensing have predominantly made use of the multiple lens-plane algorithm (see Sec. 4.1.4), and while they regularly perform convergence tests, none have made a direct comparison to the approach that integrates the null geodesic. In this chapter, we aim to fill this gap, so that the full effect of large-scale projected masses may be determined.

4.1 Standard Ray Tracing Procedures

The propagation of light through an inhomogeneous universe results in the distortion of the cross-section of the beam. Although the Universe has significant small-scale inhomogeneities, the FLRW metric (Eqn. 1.1) is generally adopted to describe the average lensing effect, mostly due to a lack of reasonable alternatives. A simple integration of the optical scalar equations (see Sec. 4.1.1) would be possible in the case of a known metric, but that is the crux of the problem - the local metric for an inhomogeneous universe has no general (analytic) solution. Crude model universes incorporate inhomogeneities and describe their effects on light propagation (e.g. Dyer & Roeder, 1972; Linder, 1988; Futamase & Sasaki, 1989). An approach that can truly account for non-linear structure formation involves two major steps: a) modelling the mass distribution between observer and source and b) tracing the path a photon backwards from

the observer to the source and thus determining how the source was lensed. Numerous weak lensing studies have involved the application of ray tracing techniques to mass distributions modelled by N-body simulations (e.g. Jaroszynski et al., 1990; Wambsganss et al., 1998). Large slices of matter are projected onto single planes to form cosmological-scale lenses which allow computationally efficient calculations. Yet the cosmological scale mass distribution contains not only compact haloes, but filamentary structures as well which run the risk of being projected and overestimating the resulting magnification. What has been missing thus far is a one-to-one comparison of results that take a multiple-lens approach and a direct numerical integration of the null geodesic. In this chapter, I describe the results of this comparison, particularly with regards to the magnification probability distribution, which is expected to affect future surveys of standard candles.

4.1.1 The Optical Scalar Equations

An infinitesimal beam, which may be seen as a congruence of light rays that propagate together, has a cross-sectional size and shape that is observer-independent. The beam is anchored by the central ray, which has the photon four-momentum k^α . One can define a complex null unit vector e^α , which is orthogonal to both the observer's world line and the direction of propagation such that they satisfy

$$k^\alpha k_\alpha = k^\alpha e_\alpha = e^\alpha e_\alpha = 0 \quad e^\alpha e_\alpha^* = 1. \quad (4.1)$$

The *optical scalars* θ and σ represent the expansion and shear of the beam respectively. The non-linearly coupled optical scalar equations describe light propagation in general, curved spacetimes (Sachs, 1961):

$$\dot{\theta} + \theta^2 + |\sigma|^2 = -\frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta \quad (4.2)$$

and

$$\dot{\sigma} + 2\theta\sigma = -\frac{1}{2}R_{\alpha\beta\gamma\delta}e^{\star\alpha}k^\beta e^{\star\gamma}k^\delta, \quad (4.3)$$

where an overdot denotes differentiation with respect to the affine parameter, λ , and $\star$ denotes the complex conjugate. Here, $R_{\alpha\beta}$ is the Ricci tensor and $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor. Due to the orthogonality given in Eqns. 4.1, it is the Weyl tensor component of the curvature tensor that drives the evolution of the shear term, while the Ricci curvature term drives the evolution of convergence. By integrating these transport equations, it is possible to calculate the effect of each term on the cross section the beam.

4.1.2 The Dyer-Roeder Equation

When an infinitesimal beam manages to traverse the large-scale structure without encountering any *large* perturbations, and is not affected by shear, the evolution of cross-section is described by the (generalised) Dyer-Roeder equation (Dyer & Roeder, 1972, 1973, 1974):

$$(1+z)(\Omega_M z + 1)\frac{d^2 D}{dz^2} + \left(\frac{7}{2}\Omega_M z + \frac{\Omega_M}{2} + 3\right)\frac{dD}{dz} + \frac{3}{2}\tilde{\alpha}\Omega_M D = 0, \quad (4.4)$$

where $0 < \tilde{\alpha} < 1$ is the mass fraction of the universe that is smoothly distributed. Thus, $\tilde{\alpha} = 0$ corresponds to a light beam that contains no matter, while $\tilde{\alpha} = 1$ corresponds to a perfectly homogeneous universe. $D(z)$ is the angular diameter distance to redshift z in units of the Hubble distance c/H_0 . The angular diameter distance to a given redshift will strictly depend on the direction of observation, as each line of sight encounters a slightly different manifestation of the inhomogeneity. For a fixed matter density, the value of $D(z)$ found by solving the above

differential equation for $\tilde{\alpha} = 0$, hereafter denoted $D(\tilde{\alpha} = 0; z)$, defines the maximal value. Seitz et al. (1994) applied the optical scalar equations to an infinitesimal beam transported through a generalized inhomogeneous universe that is FLRW ‘on average’ to re-derive the maximal angular diameter distance.

4.1.3 The Born Approximation

The *Born approximation* is often employed in weak lensing analysis; here the gravitational potential along the actual geodesic is replaced with the potential found along the unperturbed path. The convergence, κ , is taken to be the integral along the radial path from the observer in the direction of the image; an effective surface gravitational potential is implied since the 3-dimensional mass distribution is projected along the line of sight.

$$\kappa(\chi) = \int_0^\chi W(\chi') \delta(\chi') d\chi', \quad (4.5)$$

where δ is the ‘relative perturbation in matter density’ and

$$W(\chi) = \frac{3}{2} \Omega_M H_0^2 g(\chi) (1+z) \quad (4.6)$$

is a weight function and

$$g(\chi) = r(\chi) \int_\chi^\infty d\chi' n(\chi') \frac{r(\chi' - \chi)}{r(\chi')}, \quad (4.7)$$

where $n(\chi)$ is the normalised radial distribution of source galaxies. Multiple deflections are decoupled from one another, since the result of one lensing event has no bearing on the position of the light ray if and when it encounters another lens. A lens-lens coupling term is sometimes implemented. Lee & Paczynski (1990) concluded that the magnification of a single-image sources were dominated by the matter within a straight light beam, implying that the Born approximation is sufficient. Similarly Vale & White (2003) find that measures of statistical properties using the Born approximation are in good agreement with semianalytic predictions. Over larger and larger distances, the Born approximation becomes less and less accurate; several studies have found the requirement for corrections in the construction of the shear power spectrum and higher order bispectrum (e.g. Van Waerbeke et al., 2001; Shapiro & Cooray, 2006).

4.1.4 Multiple Lens Planes and the Thin Lens Approximation

The majority of numerical weak lensing studies adopt the multiple lens-plane approach. The multiple lens-plane method requires the entire lens mass distribution to be divided up into contiguous boxes, usually of equal co-moving depth; each box is then projected onto a plane perpendicular to the line-of-sight and usually at the centre of the box. In backward ray-tracing methods, light rays are propagated (backward) from the observer to the source with deflections *only* occurring in successive lens planes. The deflection at a given plane is the result of matter within that plane only. The formalism for the resulting deflection has been written down in the multiple lens-plane equation, having been derived and developed by Blandford & Narayan (1986), Schneider & Weiss (1988) and Jaroszynski et al. (1990). It is reproduced here as the expression for the source position after deflection by N lens-planes:

$$\boldsymbol{\eta} = \frac{D_{os}}{D_{o1}} \boldsymbol{\xi}_1 - \sum_{i=1}^N D_{is} \hat{\boldsymbol{\alpha}}_i(\boldsymbol{\xi}_i), \quad (4.8)$$

where η is as in Eqn. 1.29 and ξ_i is the position of the ray in the i^{th} lens-plane, therefore ξ_1 is the position of the image. D_{os} , D_{o1} and D_{is} are the angular diameter distances from the observer to the source, from the observer to the first lens-plane, and from the i^{th} lens to the source respectively. The most accurate and/or efficient way to measure the deflection angle for

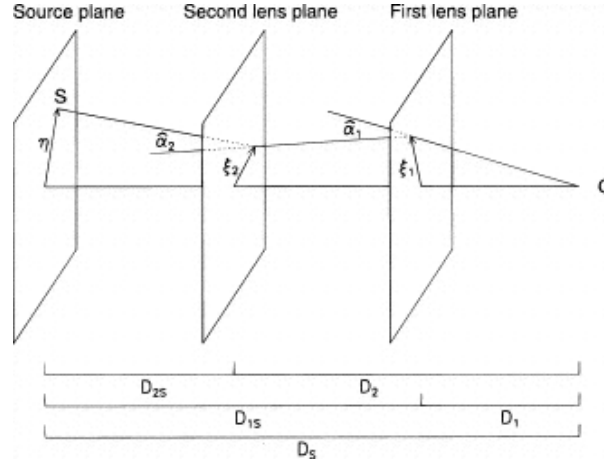


Figure 4.1: Diagram of the gravitational lensing effect for multiple lens-planes. The light ray propagates from the source S to the observer O , via a series of lens-planes (two shown here) where it is deflected by an angle $\hat{\alpha}_i$ at the i^{th} plane. The angular diameter distances are defined in the text. Taken from Premadi et al. (1998).

a light ray at each lens-plane is open to interpretation. A brute-force approach would be to sum up the effect of each point mass in the lens plane, perhaps out to a limiting impact parameter, as done in Fluke et al. (1999). Hierarchical tree-codes group together distant lenses into cells, with the number in each group increasing with distance from the light ray (see Section 2.1 for more details); the brute-force approach is applied to the nearest lenses only. Compared to the pure brute-force approach, there is a large speed-up with minimal loss of accuracy. Two-dimensional tree-codes were developed by Wambsganss (1990, 1999) and have been implemented by the methods presented in Wambsganss et al. (1998). Fourier methods require the mass distribution to be assigned to a grid after which the calculation of the gravitational field or deflection angle is done in Fourier space, usually by way of a fast Fourier transform (FFT); this alternative is also in wide use (Premadi et al., 1998; Jain et al., 2000) and is the approach employed in the present work although in three dimensions.

4.1.5 Backward Ray Shooting

In the majority of raytracing algorithms that take the multiple lens plane approach, light rays are propagated backwards from the observer in a tightly packed grid until they reach the source plane. Upon reaching the source redshift, they are ‘collected’ within a source grid. The number of rays that land in a source pixel is proportional to the magnification for that pixel position; therefore, the spatial resolution of the magnification map is limited by the size of the source grid. Errors are introduced by the finite size of the initial grid of rays and the finite number of rays that are traced. This method was introduced by Kayser et al. (1986) for microlensing studies and indeed microlensing studies have inspired the development of this approach (Schneider & Weiss, 1988). However, it was the work of Wambsganss (1990) upon which most theoretical

ray-tracing techniques have been based; the application of tree-methods to measure the deflection angle at each lens-plane have allowed efficient algorithms. Early ray-shooting algorithms were applied to a random distribution of point masses, which therefore did not incorporate the intricacies of clustering properties within various cosmological models (Schneider & Weiss, 1988; Paczynski & Wambsganss, 1989; Lee & Paczynski, 1990). Later studies coupled the multiple lens-plane approach with mass distributions taken from N-body simulations to study the effect and magnitude of gravitational lensing (Jaroszynski et al., 1990). Premadi et al. (1998) developed a similar algorithm but their model of the mass distribution includes both the “background matter” modelled by N-body simulations along with galaxies placed within the overdensities using semi-analytic methods and modelled as nonsingular isothermal spheres. Many more studies that employ ray-tracing methods have since been used to make shear maps (Jain et al., 2000) and weak lensing maps of the CMB (Carbone et al., 2008). More specifically, many studies have used the technique to measure the non-Gaussianity of the shear and convergence field (Jain & Van Waerbeke, 2000) and model the magnification probability distribution (see Section 4.3.1). This latter aim is also the focus of the present work, although an alternative three-dimensional method is developed for the purpose.

4.1.6 Three-Dimensional Approaches

Some three-dimensional approaches have been developed that allowed for comparison to the usual two-dimensional multiple lens-plane approaches. For example, Couchman et al. (1999) developed a three-dimensional algorithm for modelling weak gravitational lensing, comparing the two- and three- dimensional shear, they found that the projection of 100 Mpc/h scales led to errors of up to 9% depending on the redshift of the lens box. Vale & White (2003) perform three-dimensional ray-tracing by computing the deflection angle for rays many times along their path, advancing them in step-sizes of L_{Box}/N_g along the line of sight. In their study, they used a relatively number of grid points along the line-of-sight $N_g = 32$, such that the step-size was approximately 10 Mpc/h, but boxes of half that size were used for numerical tests. They found that their convergence maps were unchanged to 0.1%. White & Hu (2000) perform simulations of structure formation while simultaneously propagating photons through said simulation; they are able thus to produce convergence fields from the light rays they follow.

4.2 Full 3D Ray Tracing

The ray-tracing method developed in this thesis incorporates the design of the ray-bundle method without the multiple lens plane approach. The cosmological mass distribution is constructed using outputs of N-body simulations at appropriate epochs so as to allow the simulation boxes to be stacked end-to-end, as in RBM, but instead of collapsing each box into a 2D plane, the light rays are traced through the full extent of the box by numerically integrating the null geodesic. I am therefore able to quantitatively test the thin lens approximation.

4.2.1 Integrating the Null Geodesic

Few studies have taken on the task of numerically integrating the null geodesic (Eqns. 1.18) from observer to source. The earliest of these assumed metrics that were approximated via a simplified model for inhomogeneities and derived distance-redshift relations that were compared to the Dyer-Roeder approximations (Futamase & Sasaki, 1989; Watanabe & Tomita, 1990; Kasai et al., 1990). The latter, Kasai et al. (1990), drew attention to the spread in angular diameter distances given a small enough beam-size and the fact that the average value was consistently

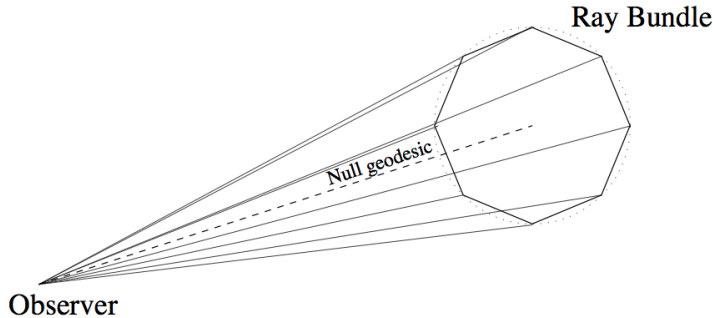


Figure 4.2: Schematic of the Ray-Bundle method in which a congruence of $N_{ray} = 8$ rays (solid lines) about the null geodesic (dashed line) is propagated from observer to source. Taken from Fluke (1999)

lower than the homogeneous FLRW estimate, or equivalently the Dyer-Roeder value with $\tilde{\alpha} = 1$. Tomita (1998) numerically integrated the null geodesic through N-body simulations albeit at low-resolution ($N_p = 32^3$, $L_{box} = 30 - 50$ Mpc/h), finding that various cosmologies exhibited differences in angular diameter distances.

4.2.2 The Ray Bundle Method

The Ray Bundle Method (RBM) was developed by Fluke et al. (1999), and see also Fluke (1999). Here, the multiple lens-plane method is used to propagate light rays from the observer to a given source plane, where the output of an N-body simulation code used to model the cosmological mass distribution. However, instead of employing a grid-based technique for calculating magnifications across the source plane, the RBM models each individual line of sight as an ‘infinitesimal bundle’ of light rays around a central ray and follows this bundle back to the source plane. An initially circular image (or more precisely, regular polygonal) is distorted by the time it reaches the source plane as a result of convergence and shear. These quantities can be determined for this specific line of sight as the image-source association is maintained. The RBM is very well suited to the weak lensing limit and provides a computationally efficient alternative to grid-based ray-shooting methods; however, it underestimates the total magnification in cases where multiple imaging is expected as only one image contributes to the magnification of each source. For this reason, rays that fall within a minimum distance of a lens (remembering that a distribution of point masses describes the lensing mass) are excluded from the analysis; these are likely to be highly de-magnified images from a set of multiple images. This affects the high magnification probabilities deduced, but does not significantly affect the weak lensing analysis (Fluke et al., 2002). The RBM avoids the danger of artificial shear due to a source pixel collecting light rays that have passed near the edge of a shooting grid.

Accuracy is limited not by the size of a grid, but by its two parameters: the number of rays in a bundle and the angular size of each image. Fluke et al. (1999) found that 8 rays was the “best compromise between accuracy and speed”; sufficient to reproduce the accuracy of a ray-shooting analysis that requires 100 image pixels per source pixel. The image radius had negligible effect on the minimum distance that a ray had to keep from a lens in order to achieve the required accuracy.

The ray-tracing method developed in this thesis uses the RBM design of a ray-bundle but does away with the multiple lens-plane treatment of the lensing mass. Instead, the evolution of the cross section of the bundle is determined by integrating the null geodesic through a set of N-body simulations.

4.2.3 Weak Field Metric

I assume the inhomogeneities present in large scale structure are small enough to be represented as a perturbation to the background FLRW metric, resulting in the expanding weak field metric; the line element is described by

$$ds^2 = -[1 + 2\psi(t, \mathbf{x})]dt^2 + a^2(t)[1 - 2\psi(t, \mathbf{x})]dx_i^2, \quad (4.9)$$

where i denotes the three spatial dimensions. The gravitational potential, which is defined with respect to the local perturbation from a smooth background (see Section 4.2.4), can be decoupled:

$$\psi(t, \mathbf{x}) = \frac{\phi(\mathbf{x})}{a(t)} \quad (4.10)$$

such that ψ is defined in physical units and ϕ in co-moving. The non-zero Christoffel symbols for this metric are then:

$$\begin{aligned} \Gamma_{tt}^x &= \frac{1}{a^3}\phi_{,x} & \Gamma_{tt}^y &= \frac{1}{a^3}\phi_{,y} \\ \Gamma_{tt}^z &= \frac{1}{a^3}\phi_{,z} & \Gamma_{tx}^x &= \Gamma_{ty}^y = \Gamma_{tz}^z = \frac{\dot{a}}{a^2}(a + \phi) \\ \Gamma_{tx}^t &= \Gamma_{yy}^x = \Gamma_{zz}^x = \frac{1}{a}\phi_{,x} & \Gamma_{xx}^x &= \Gamma_{xy}^y = \Gamma_{xz}^z = -\frac{1}{a}\phi_{,x} \\ \Gamma_{ty}^t &= \Gamma_{xx}^y = \Gamma_{zz}^y = \frac{1}{a}\phi_{,y} & \Gamma_{yy}^y &= \Gamma_{yx}^x = \Gamma_{yz}^z = -\frac{1}{a}\phi_{,y} \\ \Gamma_{tz}^t &= \Gamma_{xx}^z = \Gamma_{yy}^z = \frac{1}{a}\phi_{,z} & \Gamma_{zz}^z &= \Gamma_{zx}^x = \Gamma_{zy}^y = -\frac{1}{a}\phi_{,z} \\ \Gamma_{tt}^t &= -\frac{\dot{a}}{a^2}\phi & \Gamma_{xx}^t &= \Gamma_{yy}^t = \Gamma_{zz}^t = \dot{a}(a + \phi) \end{aligned} \quad (4.11)$$

The Christoffel symbols for this metric are dependent, not only on the gradients of the gravitational potential ϕ , but ϕ *itself*. However, the weak field condition requires that the magnitude of the perturbations satisfy $\phi \ll a$. The Geodesic Equations - the second order differential equations for the four coordinates - are then constructed.

4.2.4 The Gravitational Field

The perturbation field:

$$\delta(t, \mathbf{x}) \equiv \frac{\rho(t, \mathbf{x}) - \bar{\rho}(t)}{\bar{\rho}(t)} \quad (4.12)$$

relates the local density, $\rho(t, \mathbf{x})$, to the mean matter density, $\bar{\rho}(t)$, which is given by

$$\bar{\rho}(t) = \Omega_M \rho_c(t). \quad (4.13)$$

The perturbation is related to the gravitational potential $\psi(t, \mathbf{x})$, by

$$\psi(t, \mathbf{x}) = -\frac{4\pi G}{c^2} \bar{\rho}(t) \int_{\mathbb{R}^3} d\mathbf{x}'^3 \frac{\delta(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \quad (4.14)$$

The derivatives of the gravitational potential are given by

$$\frac{d\psi(t, \mathbf{x})}{d\mathbf{x}_j} = \frac{4\pi G}{c^2} \bar{\rho}(t) \int_{\mathbb{R}^3} d\mathbf{x}'^3 \delta(t, \mathbf{x}) \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \quad (4.15)$$

Then, one can relate the numerically determined co-moving potential to the above quantity via Eqn. 4.10 and derivatives in a similar fashion:

$$\frac{d\psi(t, \mathbf{x})}{d\mathbf{x}_j} = \frac{\phi(\mathbf{x})}{d\mathbf{x}_j} (1 + z(t)). \quad (4.16)$$

Constructing the grids that represent the perturbation field and its derivative requires a few steps. Firstly, if the lensing mass is discretized into particles, then the mass within a cube of co-moving side length L_{box} is assigned to the nodes of a regular grid using the well-known Cloud-in-Cell algorithm in three dimensions; the mean matter density is then subtracted off. A Fast Fourier Transform is applied to convolve the mass distribution with the appropriate kernels to determine the gravitational field and its derivatives according to Eqns. 4.14 and 4.15, using the popular software package ‘Fastest Fourier Transform in the West’ (FFTW¹). Depending on the lens, a periodic mass distribution may be implied, or not. If not, the density grid is zero-padded before performing the FFT convolution. At many stages during the integration, the local values of the field and its derivatives needs to be evaluated, so an interpolation scheme was written for this purpose. Here I only describe the scheme for the evaluation of $\phi(x_c, y_c, z_c)$ given the gridded values $\phi(x, y, z)$, but the same method is applied to evaluate the derivatives of the potential. The scheme is not strictly tri-cubic interpolation, but a three-dimensional version of the bicubic spline. Instead, it initially performs a one-dimensional spline to compute the derivative with respect to the z-axis; i.e. it fits a spline to find $d\phi(x, y, z)/dz$ across the entire grid. This is only done once for each lensing mass distribution. Each time the local values need to be found, the scheme identifies the gridded values that include n_{int} points on either side of the current location for each dimension, i.e. a $(2n_{int})^3$ mesh grid. Then, the scheme then performs the following steps:

1. The gridded derivative is interpolated across the chosen dimension, in this case the z-dimension, to find $\phi(x, y, z_c)$, i.e. the value with the current z-coordinate of the photon specified, but the other coordinates corresponding to grid points. $(2n_{int})^2$ interpolations are required.
2. Another set of $2n_{int}$ one-dimensional splines are performed, this time across the x-axis, to find $d\phi(x, y, z_c)/dx$.
3. $2n_{int}$ interpolations along the x-axis determine $\phi(x_c, y, z_c)$.
4. A single one-dimensional spline across the y-axis is used to find $d\phi(x_c, y, z_c)/dy$
5. Finally, a single interpolation along the y-axis allows one to evaluate $\phi(x_c, y_c, z_c)$

4.2.5 Evolution of the Scale Factor

For certain cosmologies, the Friedmann Equations (Eqns. 1.3a) can be solved to find the specific time dependence of the scale factor; for a spatially flat, radiation-free Lemaitre model ($\Omega_k = 0$, $\Omega_M + \Omega_\Lambda = 1$), with $\Omega_\Lambda > 0$, this dependence is given by :

$$a(t) = \left[\frac{\Omega_{M,0}}{\Omega_{\Lambda,0}} \sinh^2 \left(\frac{3}{2} \sqrt{\Omega_{\Lambda,0}} H_0 (t_0 - t) \right) \right]^{1/3}, \quad (4.17)$$

¹<http://www.fftw.org/>

where t_0 is given by

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_{\Lambda,0}}} \sinh^{-1} \frac{\Omega_{\Lambda,0}}{\Omega_{M,0}}. \quad (4.18)$$

Equations 4.17 and 4.18 were derived from Eqn. 15.36 in Hobson et al. (2006), the solution for a spatially flat, matter-only Lemaitre model, but with t used to denote *lookback time* instead.

As the photon traverses a single lens box, the scale factor will evolve (i.e. decrease) although the co-moving scale of the structure does not change appreciably over this time-scale. The value of the scale factor is required to evaluate the Christoffel symbols, so I evolve the lookback time for the photon as well as the spatial coordinates and use this to determine the scale factor at each position throughout the box.

4.2.6 Numerical Integration

The four second-order differential equations given in Eqn. 1.18 are reduced to eight coupled first-order differential equations, which are integrated with the affine parameter λ as the dependent variable. The Christoffel symbols are those presented in Eqn. 4.11. A classical fourth order Runge-Kutta integration scheme with fixed step-size was written and used to perform the integration. The subtlety is that it is the z -coordinate, rather than the affine parameter, that determines the boundaries of the integration. The exact evolution of the affine parameter will be different for each ray-bundle, and is not known in advance. Therefore, the integral is repeated in small steps of λ , $\Delta\lambda_{\text{RK}}$, until the z -coordinate reaches the value required at the source plane $D_C(z_s)$. The step-size is chosen such that the estimated resultant step in z is a fixed fraction, f , of the Fourier-grid resolution:

$$\Delta\lambda_{\text{RK}} = \frac{f_{\text{RK}} L_{\text{box}}}{n_{\text{FFT}} \dot{z}_0}, \quad (4.19)$$

where the overdot denotes the derivative with respect to λ and n_{FFT} denotes the number of points along one side of the Fourier-grid (see Sec. 4.2.4). If the ray overshoots the source plane, a simple linear interpolation, using the current and previous positions, is used to find the final source position. The use of a fixed step-size is deemed appropriate as the values of the Christoffel symbols are interpolated from gridded values, as described in Sec. 4.2.4. There would be little information gain from step-sizes much smaller than the grid resolution. Nevertheless, the effect of the choice of step-size is analysed, along with other numerical parameters, with results presented in Sections 4.5.1 and 4.5.2.

4.2.7 The Magnification Matrix

This method propagates eight rays and a central ‘anchor’ ray from observer to source. Initially, the eight rays are ‘regularly’ distributed around the anchor, which represents the null geodesic; they form an octagonal image, but by the time they reach the source, the shape they create has been distorted. From the positions of each rays, the Jacobian of the lens mapping, also known as the magnification matrix, can be determined. The method, described below, was used in Fluke et al. (1999), but is presented here for completeness.

For each ray, i , in a bundle, the separation vector connecting it to the anchor ray is denoted $\mathbf{x}^i$ in the image plane and $\mathbf{y}^i$ in the source plane. These vectors satisfy

$$\mathbf{y}^i = \mathbf{A} \mathbf{x}^i \quad (4.20)$$

under the assumption that the deformation is small. Then, for each possible pair of rays in the bundle, ${}^8C_2 = 28$ in total, labelled with the superscripts i and j , Eqn. 4.20 provides four equations:

$$y_1^i = A_{11}x_1^i + A_{12}x_2^i \quad (4.21)$$

$$y_2^i = A_{21}x_1^i + A_{22}x_2^i \quad (4.22)$$

$$y_1^j = A_{11}x_1^j + A_{12}x_2^j \quad (4.23)$$

$$y_2^j = A_{21}x_1^j + A_{22}x_2^j \quad (4.24)$$

which are solved simultaneously to determine the lensing matrix, $\mathbf{A}$, applicable to that pair of rays. Averaging over each pair, we find the ‘average’ magnification matrix. That the average of the 28 matrices provides a reasonable approximation to the magnification matrix for the null geodesic is non-trivial. The assumption is that the matter perturbations do not vary appreciably over the width of the beam, which is why smaller bundle-sizes are preferred. Having determined the ‘average’ magnification matrix for that ray-bundle, it can be seen from Eqn. 1.23 that the lensing quantities of interest can be found. The convergence is given by applying the relation

$$\kappa = 1 - \frac{1}{2}Tr(\mathbf{A}) \quad (4.25)$$

and the magnification for the single line of sight, as in Eqn. 1.24, is related to the determinant of the Jacobian by

$$\mu = (det\mathbf{A})^{-1}, \quad (4.26)$$

and therefore the shear is determined by combining Eqns. 1.24, 4.25 and 4.26:

$$\gamma^2 = (1 - \kappa)^2 - \mu^{-1} \quad (4.27)$$

$$= \frac{1}{4}Tr(\mathbf{A}) - det\mathbf{A} \quad (4.28)$$

4.3 Weak Lensing Statistics

4.3.1 The Magnification Probability Distribution

In an inhomogeneous universe, each line of sight probes a slightly different integrated mass and is sheared differently. The result is that for a given redshift z , a source is magnified (or demagnified) by some amount μ , *relative to the case of a perfectly homogeneous universe on all scales*. Overdense regions result in magnification and vice versa. The magnification probability distribution, or μ PDF, and indeed the statistical distribution of optical quantities such as convergence and shear, have been quantifiers of the large-scale lensing effect in many studies (Holz & Wald, 1998; Premadi et al., 2001; Das & Ostriker, 2006). $p(\mu, z)d\mu$ is the probability that a source at redshift z would be magnified by μ^* , where $\mu < \mu^* < \mu + d\mu$. It satisfies

$$\int_0^\infty p(\mu, z)d\mu = 1. \quad (4.29)$$

When the μ PDF is convolved with intrinsic luminosity distributions for a standard candle, the result describes the observed spread in magnitudes. Flux conservation (Weinberg, 1976) demands that $p(\mu, z)$ satisfies

$$\langle \mu \rangle \equiv \int_0^\infty \mu p(\mu, z)d\mu = 1. \quad (4.30)$$

If only truly weak lensing events are accounted for, the distribution is Gaussian; however, the complete distribution is highly skewed towards high magnifications. There exists a minimum magnification, indeed a minimum convergence, which corresponds to a line of sight that encounters no matter between observer and source; this is also referred to as an empty beam (see Sec. 4.3.2). The spread in this distribution increases with source redshift (Babul & Lee, 1991).

The high magnification ‘tail’ exhibits a power law trend $p(\mu) \propto \mu^{-3}$, which results in a formally divergent standard deviation. The high magnification region suffers from low number statistics and since it represents the effects of strong lensing, cannot be probed by ray-tracing methods such as RBM. On the other hand, the intermediate magnification region is fairly dependent on the cosmological model and thus provides an opportunity to test the relevant cosmological parameters (Wambsganss et al., 1998). Consider now the statistical parameters that describe the full μ PDF. The mode is below unity, which reflects the fact that most lines of sight probe underdense regions while most of the mass is locked up in overdensities that are rarely encountered (Refsdal, 1970). The median magnification value is a robust estimator and a powerful cosmological discriminator (Wambsganss et al., 1997).

4.3.2 Full Beam vs. Empty Beam

Magnification is the relative increase in flux as a result of gravitational lensing. There is a subtlety here. One may compare the flux received from the source to the flux received if the universe was entirely homogeneous, in which case the magnification is referred to as the full-beam magnification, μ_{fb} . This is the magnification referred to in Sec. 4.3.1. The so-called empty-beam magnification, μ_{eb} , is defined relative to the case where all the matter in the universe is locked up in compact objects, and all lines of sight are empty. As this is the case of minimal magnification (no convergence and assumed negligible shear), then μ_{eb} is always greater than unity. To derive the relationship between the two magnifications, we must compare the solid angles subtended by the source: Ω_{fb} in the full-beam scenario, Ω_{eb} in the empty-beam scenario and Ω_{img} the solid angle of the image observed. Given the conservation of surface brightness, the magnifications are defined by:

$$\mu_{fb} = \frac{\Omega_{img}}{\Omega_{fb}} \quad \text{and} \quad \mu_{eb} = \frac{\Omega_{img}}{\Omega_{eb}}. \quad (4.31)$$

The solid angles are related to the physical area of the source via the angular diameter distances (see Appendix A):

$$\Omega_{fb}(z) = \frac{A_{src}}{(1+z)^2 D_{fb}(z)} \quad \text{and} \quad \Omega_{eb}(z) = \frac{A_{src}}{(1+z)^2 D_{eb}(z)}. \quad (4.32)$$

Combining Equations 4.31 and 4.32, one is easily able to convert between the two:

$$\frac{\mu_{fb}}{\mu_{eb}} = \frac{D_{fb}^2(z)}{D_{eb}^2(z)}, \quad (4.33)$$

The angular diameter distance in the full-beam case, $D_{fb}(z)$, is equivalent to the Dyer-Roeder distance $D(\tilde{\alpha} = 1; z)$ which is also the FLRW solution presented in the Appendix by Eqn. A.6 with $z_1 = 0$. The empty-beam angular diameter distance, $D_{eb}(z)$, is found by solving the Dyer-Roeder equation for $\tilde{\alpha} = 0$:

$$D_{eb}(z) \equiv D(\tilde{\alpha} = 0; z) = c \int_0^z \frac{1}{(1+z')^2 H(z')} dz' \quad (4.34)$$

In the numerical ray-tracing method presented here, the method described in Sec 4.2.7 determines the full-beam magnification.

4.3.3 Numerically Modelling the PDFs

Using ray-tracing procedures to construct a magnification probability distribution is effectively equivalent to calculating the angular diameter distance in different directions out to the same

co-moving distance. The distribution of angular diameter distances for a given source redshift has been investigated by Hadrovic & Binney (1997). In the extremal case of a large beam-size, and therefore, large separations between rays, the angular diameter distances would be the same in all directions and the ‘distribution’ would be a single peak at $D_{fb}(z)$, the average FLRW value, implying homogeneity on those scales. However, if the beam-size is small enough, then the inhomogeneities induce a distribution.

Each ray-bundle probes a single light of sight. More specifically, it describes the lensing that has affected a single image of a fixed size on the sky. The number of bundles that exhibit a magnification of μ are thus proportional to the probability that an *image* is magnified by μ . However, the statistical quantity of interest is the magnification probability distribution for *sources*. Therefore the number counts that are used to produce the magnification probability histograms presented in Sec. 4.4.4 are weighted by the area covered by the beam at the source plane, or equivalently, the inverse of the magnification. Similarly the mean and standard deviation of these properties are also weighted.

4.4 Ray-Tracing Predictions

In order to construct the predicted probability distributions for the weak lensing statistics numerically, numerous null geodesics are integrated through a cosmological mass distribution that is generated with the aid of N-body simulations. The results are compared to those constructed with the original RBM (Fluke et al., 1999).

4.4.1 The N-body Simulation

The cosmological simulations are carried out with the parallel Tree-PM SPH code **GADGET2**. I adopt a Λ CDM cosmology, with the following values for the cosmological parameters: $\Omega_{M,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$, $h=0.71$, $\sigma_8 = 0.9$. The dark matter distribution is discretised into 256^3 particles distributed within a periodic box with co-moving length of $L_{\text{box}} = 50 \text{ Mpc}/h$, resulting in a mass resolution of $m_p = 6.3 \times 10^8 M_\odot$; the initial displacements are in a ‘glass’ configuration. The simulations begin at an initial redshift of $z_i = 39$, with a redshift-dependent gravitational force softening length of $\epsilon_{co} = 16 \text{ kpc}/h$ (Plummer-equivalent).

The space between an observer and source is divided up into individual regions, each modelled by a snapshot of a cosmological simulation at an appropriate redshift. The snapshot cadence, Δz , is chosen such that the light travel time corresponds to the length of the boxes. The line-of-sight integrated co-moving distance between an observer and source at $z = z_s$ is presented in Appendix A, but reproduced here:

$$D_c = c \int_0^{z_s} \frac{dz}{H(z)}. \quad (4.35)$$

Since this cannot be easily evaluated in general, we instead assume that the differences between the redshift of adjacent boxes are small, and thus:

$$L_{\text{box}} = \frac{c\Delta z}{H(z)}, \quad (4.36)$$

which is easily rearranged to determine the snapshot cadence. A total of 28 boxes are required to fill the space between an observer and a source at $z \approx 0.5$.

Note that the scale factor at the position of a light photon is required to integrate the null geodesic. This scale factor, is *not* identified with the ‘average’ redshift of the current snapshot being traced through; instead it is derived from the lookback time (see Eqn. 4.2.5) and evolves

during the integration through the box. That is to say, it is assumed that in the time it takes for a photon to traverse a box, the co-moving scale of the structure remains constant, however the physical scale length changes. The numerically derived value of the scale factor at each box interface is compared to the redshift of the appropriate snapshot using Eqn. 1.4 as a sanity check.

4.4.2 Sampling Region

In order to avoid repeated structure, a number of precautions were taken. Nine independent simulations were run, resulting in nine separate realisations. The snapshot for each required redshift was chosen randomly from among the nine realisations. The chosen box was then randomly translated and rotated 90° about any/all of the axes. From each 50 Mpc/h box I selected the lensing mass formed by particles within a prism 6.25 Mpc/h across the sky and 50 Mpc/h along the line of sight. This is split into 8 cubes each of volume $V = 6.25^3 (\text{Mpc/h})^3 \approx 244 (\text{Mpc/h})^3$. Each of these has its mass placed on a 128^3 point grid, using the Cloud-In-Cell algorithm. I apply an FFT to these boxes with zero-padding so that the density field (and the derivatives of the gravitational potential) due to structure *within the box only* is determined on a scale of 49 kpc/h. Note here that the simulation is run on a larger box such that the large-scale modes are included in the formation of structure, but I only use a smaller portion of the box for the lensing. The null geodesics of the eight rays and one anchor of each ray-bundle are numerically integrated using the methods described in Sec. 4.2.3 – 4.2.6. Lines of sight that fall too close to the edge of the boxes will be artificially sheared, so assuming that most of the lensing occurs due to mass within 0.5 Mpc/h, I only send out ‘beams’ that will fall within the central $5.25 \times 5.25 (\text{Mpc/h})^2$ region at the source redshift. For a source redshift of $z=0.5$, with an angular diameter distance of approximately 900 Mpc/h, this means that the area of sky sampled is about $0.33^\circ \times 0.33^\circ$ or 5×10^{-6} of the entire sky. To increase the sampling area, this entire procedure is used to trace the paths of 10 000 bundles, then repeated for another 10 000 bundles using another lensing mass distribution sampled from the same set of simulations..

4.4.3 Memory requirements

As small sections of the simulated boxes are used for the sampling, the assumption of periodicity is dangerous. If so, the coincidental presence of a large cluster would be accompanied by many large clusters in the vicinity, in other words, the power on small (6 Mpc/h) scales would be artificially increased. It is safer to assume that matter outside this region, atleast 1 Mpc/h from any given light bundle, plays a negligible role in lensing. So zero-padding is required to model the gravitational potential. This is when the computational challenge is most apparent. Each Fourier-grid ‘stores’ $(2 \times n_{\text{FFT}})^3$ double precision numbers and four grids are required, one of the potential and three for its derivatives. All the individual rays of each bundle are ray-traced through one box; their positions and directions are stored; then the field values are no longer needed. So at any one time, the bulk of the memory usage corresponds to four sets of three-dimensional Fourier-grids. In order to double the spatial resolution, the lensing mass would need to be placed on a grid with $n_{\text{FFT}} = 256$, and $\sim 22\text{GB}$ of RAM would be required during the FFT, although only an eighth of this is actually required for the ray-tracing. This presents a challenge to the construction of a ray-tracing prediction that requires statistically large numbers of lines of sight to be probed. Higher source redshifts will also require more simulation boxes to be stacked for this purpose; the computational time increases linearly with the number of simulation boxes required. Both requirements place increasing pressure on time-allocation on large-memory nodes; this is the kind of constraint that prevents codes such as this

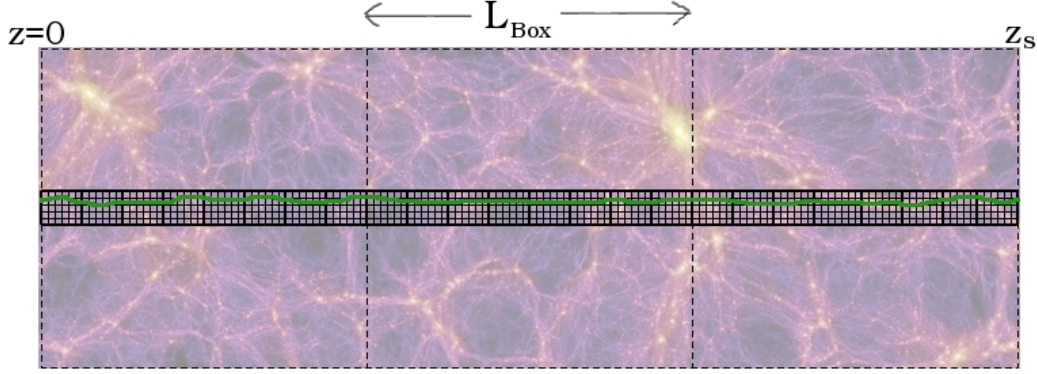


Figure 4.3: A two-dimensional schematic diagram of the ray-tracing method. Large simulation snapshots (dashed lines; three shown here) of length L_{box} fill up the space from the observer, $z = 0$, to the source plane at z_s . Small cubes (thick lines) from within these boxes form a prism of lensing mass. The mass in each cube is distributed onto Fourier-grids (thin lines) before calculating the gravitational potential and derivatives in order to numerically integrate the path (green) of a photon.

from being widely used. Nevertheless, it is necessary to determine if the predictions based on the widely-used multiple lens-plane method exhibit any biases.

4.4.4 Comparing Probability Distributions

The predicted probability distributions of magnification and shear are constructed by ray-tracing 2×10^4 bundles through the three-dimensional mass distribution described above. The results, shown in Figures 4.4 and 4.5 are compared to the probability distributions constructed with the multiple lens-plane RBM using 5×10^4 bundles. However, not all ray-bundles are included in the histograms. If a ray-bundle were to suffer from neither convergence nor shear, it would obtain the minimal magnification, $\mu_{eb,min} = 1$, or equivalently, $\mu_{fb,min} = D_{fb}^2(z_s)/D_{eb}^2(z_s)$. For a source redshift of $z = 0.5$, this minimum magnification is ≈ 0.965 . Those bundles that produce magnifications below this minimum value represent the demagnified components of a multiple-image system, a possible consequence of strong lensing. These will highly underestimate the total magnification of the associated source. On the other hand, if a magnified image is traced back to the source plane, but belongs to a multiple-image system, it will not be identified as such. Although it will also underestimate the total magnification, it will generally do so by a negligible amount.

The ray-tracing method developed in this thesis does not produce any bundles that must be excluded. In addition, larger magnifications and shears are not produced, as shown in Figures 4.4 and 4.5. Both of these are likely to be consequences of the poor spatial resolution of the Fourier grid. The results show that structure on scales of 50 kpc/h and above that are able to produce strong or even ‘moderate’ lensing are rare. In addition, the slope of the differential magnification probability distribution is found to be much steeper than for RBM, even in the regime of weak lensing. It is this result that is most surprising, but there are two aspects of the ray-tracing codes that can produce this discrepancy: the use of lens-planes and method by which the deflection of the light photon is calculated.

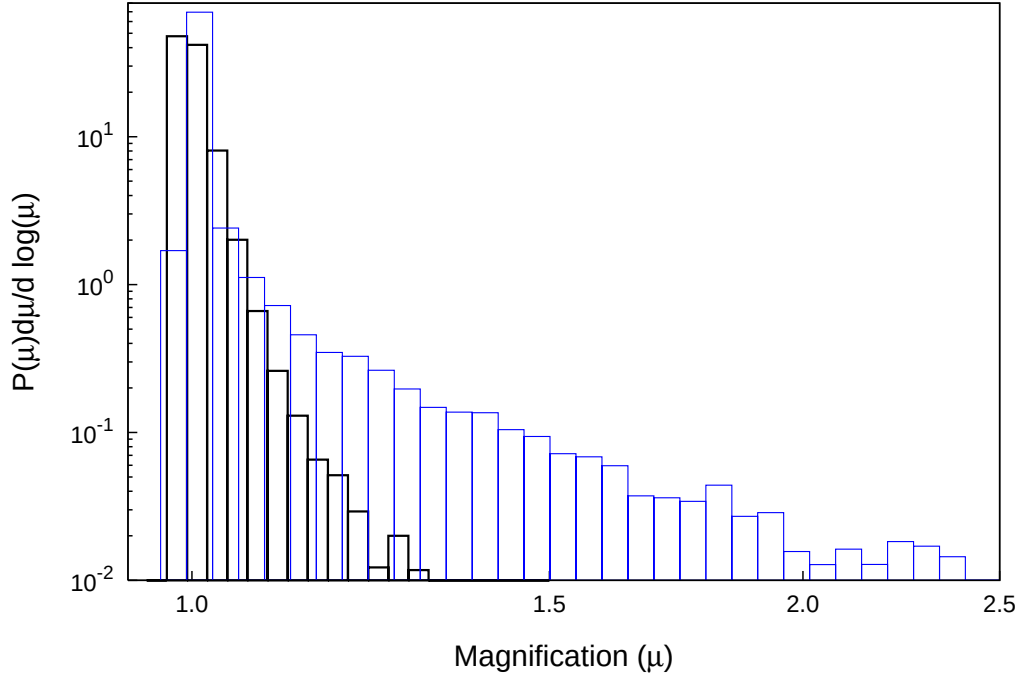


Figure 4.4: The magnification probability distribution for a source at redshift $z = 0.5$ as predicted by the multiple lens-plane RBM sampling 5×10^4 lines of sight (blue) and by the three-dimensional equivalent sampling 2×10^4 lines of sight (black).

The RBM method treats the lensing mass distribution as compact objects, even though the particles in the N-body simulation represents an underlying smooth dark matter distribution. The consequence of shot-noise can be to produce artificial caustics. The present work avoids this by applying Fourier methods. While the aim of the direct comparison between RBM and the method presented here is to quantify the effect of the multiple lens-plane approach, it is also possible that the difference in the treatment of the mass leads to the discrepancy.

4.4.5 Scale of Structure

The scale of structure that can be probed is limited by the spatial resolution of the Fourier grid. It is also limited by the force-softening length applied in the simulations (see Sec. 2.2). Another set of five simulations were run with a smaller force-softening length in order to test the effect of force-softening. The ray-tracing described above were repeated on these simulations. The results are shown in Figure 4.6. No significant difference is seen when comparing the results of ray-tracing through simulations that probe different scales. However, it is recognised that the differences in the simulated structure will primarily be on $\sim 10kpc/h$ scales; the limit imposed by the Fourier-grid resolution will dominate over the effect of force-softening.

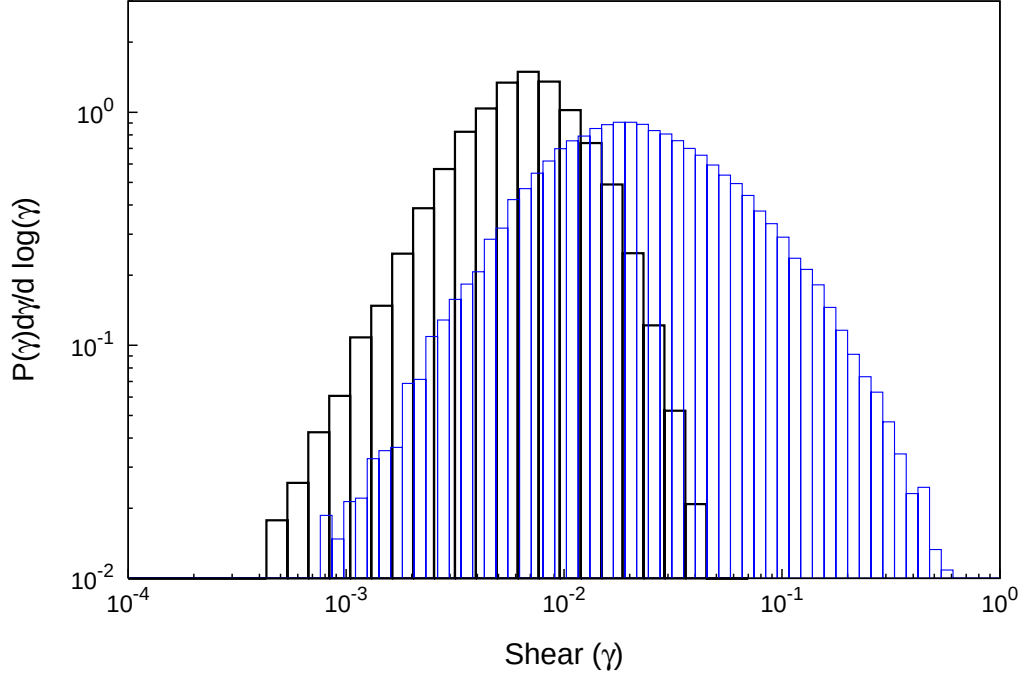


Figure 4.5: The shear probability distribution for a source at redshift $z = 0.5$ as predicted by the multiple lens-plane RBM sampling 5×10^4 lines of sight (blue) and by the three-dimensional equivalent sampling 2×10^4 lines of sight (black).

4.5 Parameter Dependencies

The rest of this chapter is devoted to verifying the accuracy of the ray-tracing code and determining its parameter dependencies. I perform ray-tracing over lensing mass distributions for which the analytic solution is known. Firstly, tests conducted on a homogeneous universe for which the perturbation is zero everywhere, i.e. $\phi(t, \mathbf{x}) = 0$, recover the known angular diameter distance, $D_A(z_s)$ (see Eqn. A.6 with $z_l = 0$).

Below I present results of ray-tracing in the vicinity of simple lens. Both of the lens models used are fairly compact, *thin lenses* if you wish, and so the thin lens approximation applied in the analytic solution is suitable. The analytic solutions should be read as magnification for the *image* at that location, rather than the total magnification of the source. This is an important distinction. The ray-bundle approach only follows the congruence corresponding to a single image, and therefore cannot account for the total magnification of multiply imaged sources; this is one of the reasons why it should not be used to model strong lensing. Yet for these simple lenses, we are still able to test our results against the known magnification of a single image for an otherwise strongly lensed source.

4.5.1 Schwarzschild lens

The Schwarzschild lens, a single density peak, is the simplest lens case one may study. The analytic solution for the magnification for an image at any location is well known and is presented

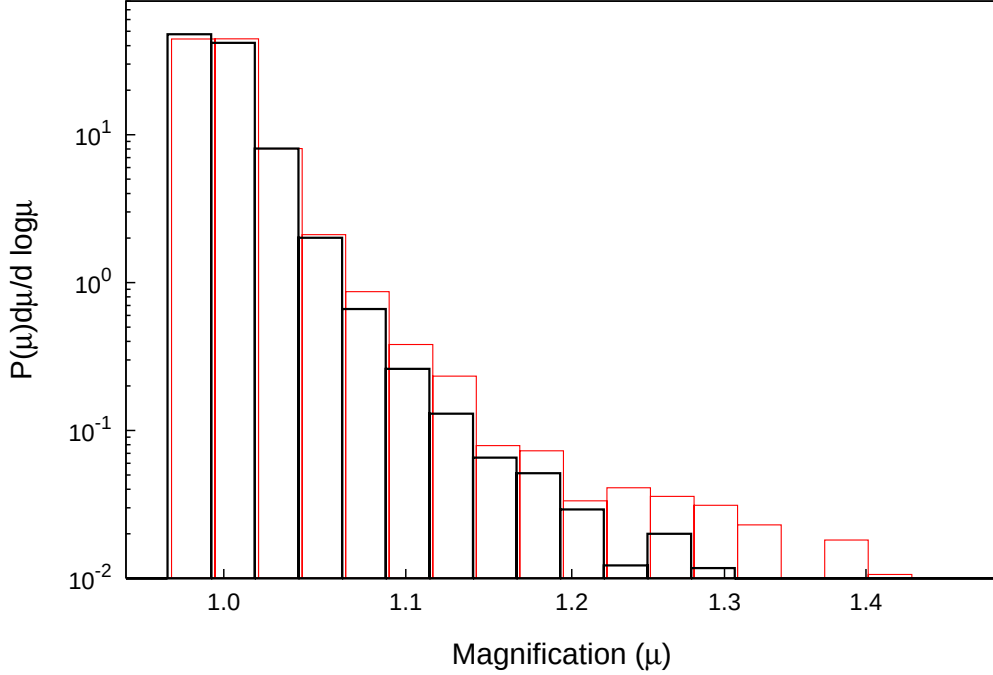


Figure 4.6: The magnification probability distribution for a source at redshift $z = 0.5$ as predicted by the three-dimensional ray-bundle method. The force-softening length in the cosmological simulations is capped at 5 kpc/h (red) and 16 kpc/h (black). The two histograms are shifted slightly for clarity. Both histograms are constructed from sampling 2×10^4 lines of sight.

in Appendix B.1. Since the gravitational potential here is equivalent to the kernel multiplied by the mass of the lens, I directly compute the gravitational field, before perform the ray-tracing. The Einstein radius increases with the mass, so by using an unusually massive lens, it is possible to study the behaviour of ray-bundles that pass near (or inside) the Einstein radius. In cosmological simulations, the RBM explicitly avoids this as the images near the Einstein ring are part of multiple set of images, and the images inside the ring are the demagnified images which contribute only a small portion of the magnification of the associated source. However, for testing purposes, I allow lines of sight that approach the Einstein radius since the analytic solution is known for each individual image. In the fiducial test case, the Schwarzschild lens has a mass of $10^{15} M_{\odot}$. It is placed at a redshift of $z_L = 0.35$, giving it an Einstein radius of $65''.6$ for a source redshift of $z_s = 0.8$, which corresponds to ~ 12 mesh-points on the 256^3 Fourier-grid.

The ray-tracing code developed here is able to reproduce the desired lensing distortion very well. The plots on the left hand side of Figures 4.7 and 4.8 show that even high magnification events, which occur when the image is near the Einstein radius, are recovered by the numerical method. The numerical error, shown on the right hand side, increases for images that are closer to this radius where the strong lensing effects, like flexion, are expected to play a role and an eight-ray bundle has limited application regardless of the choices of other numerical parameters. Note that in each test shown in Figures 4.7 – 4.15, the results break down in the very centre of the lens; the error surpasses 3% when the impact parameter is smaller than

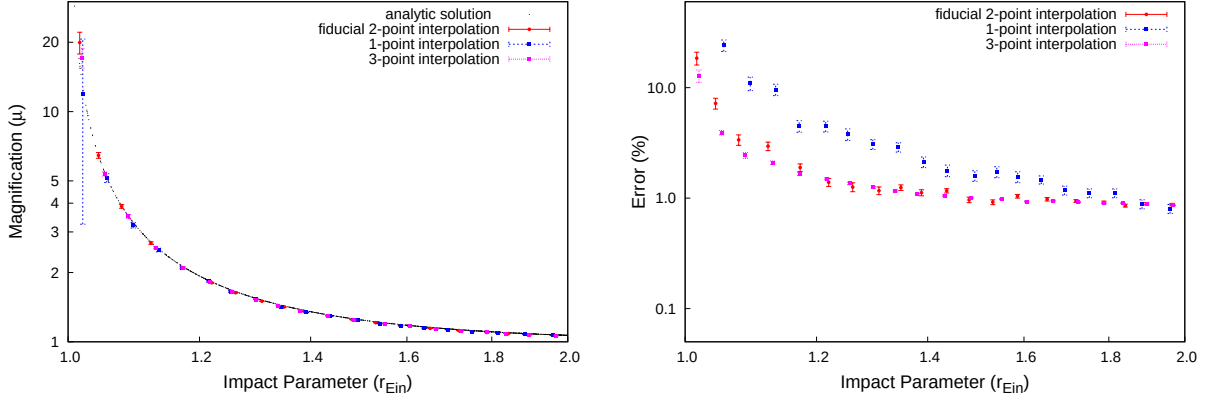


Figure 4.7: The radial dependence of magnification (μ ; left) and the percentage error in the numerical magnification ($\Delta\mu/\mu$; right) when ray-tracing past a Schwarzschild lens. Results are shown for a fiducial choice of the number of points used to interpolate the local potential field and its derivatives: $n_{int} = 2$ (red) as well as $n_{int} = 1$ (blue). In each case, 10^3 lines of sight have been distributed and binned uniformly with $\log(r/r_{vir})$, and the analytic solution at each point is included on the graph on the left. 1σ error-bars are shown.

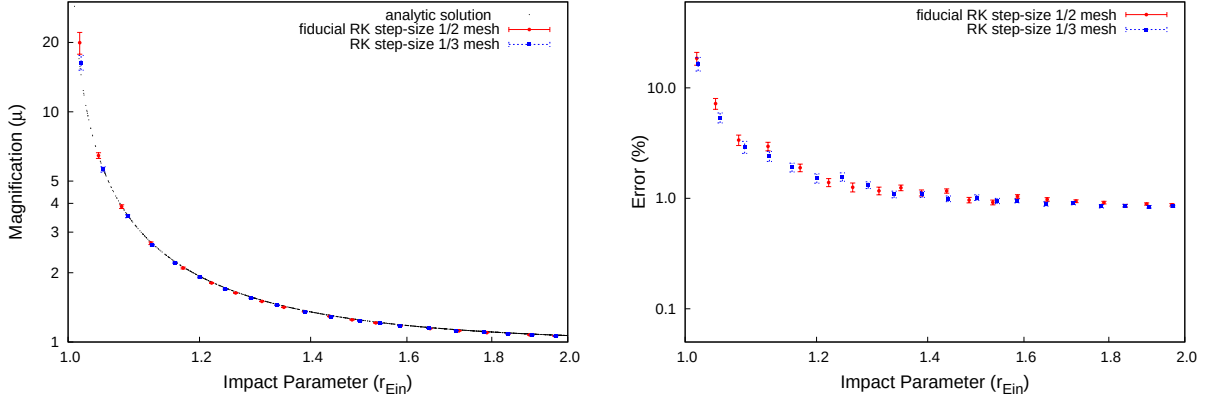


Figure 4.8: The radial dependence of magnification (μ ; left) and the percentage error in the numerical magnification ($\Delta\mu/\mu$; right) when ray-tracing past a Schwarzschild lens. Results are shown for a fiducial choice of Runge-Kutta step-size that is 1/2 the Fourier-grid resolution (red) as well as for 1/3 the grid resolution (blue). In each case, 10^3 lines of sight have been distributed and binned uniformly with $\log(r/r_{vir})$, and the analytic solution at each point is included on the graph on the left. 1σ error-bars are shown.

a limiting radius, hereafter referred to as the ‘minimum reliable radius’, or MRR. The MRR can be used as a measure of numerical accuracy and the effect of numerical parameters. For example, by decreasing the number of interpolation points so that just one Fourier-grid node *on either side* of the current position is used to find the local values of the gravitational field and its derivatives, Figure 4.7 shows that the MRR expands out to a larger impact parameter where only low magnification regions ($\mu \lesssim 1.5$) are recovered with $< 3\%$ accuracy. However, once the number of interpolation points is increased beyond this, this parameter has little influence on the accuracy of the results. Likewise, a Runge-Kutta step-size that is approximately half the size of the Fourier-grid resolution ($f_{\text{RK}} = 0.5$) is sufficient for accuracy at moderate magnifications ($1.5 \lesssim \mu \lesssim 6$). Decreasing this parameter has negligible effect on the results.

4.5.2 Discretised NFW lens

Lenses with the NFW profile (see Section 1.3.3) were modelled by discretising the mass contained in one virial radius into a number of particles of equal masses and constructing a fake simulation output in the **GADGET** format. The numerically determined gravitational potential is therefore *not* equivalent to the analytic solution, but suffers from discretisation effects just as a simulated halo would. The spherically symmetric lens is divided into a fixed number of radial bins of equal width, extending from the centre to the virial radius. Each particle is randomly placed in one of the radial bins with a probability proportional to the mass within that bin; the bin mass is found by appropriately integrating the density profile, given in Eqn 1.9. The lenses modelled for this purpose have a virial mass, and total mass, of $M_{\text{vir}}/M_{\odot} = 10^{14}$ and concentration parameter $c_{\Delta} = 7.2$. At a (lens) redshift of $z_L = 0.35$ and a virial overdensity of $\Delta_c = 200$, their virial radii are 0.55 Mpc/h. The results presented here are for a source redshift of $z_S = 0.8$. Figures 4.9–4.12 show the magnifications determined by ray-tracing over a range of impact parameters. The ray-bundles are allowed to encounter the lens at a distance of up to twice the virial radius, which corresponds to an angular separation of $343''$. Since the lens is only defined out to one virial radius, the metric on the exterior is given by the Schwarzschild metric, by Birkhoff’s theorem. Thus the analytic solution for $\mu(r)$ switches to that appropriate for a Schwarzschild lens outside this impact parameter. The analytic solutions for image magnification and shear by a lens with the NFW profile are presented in Appendix B.2.

By varying the grid size, I notice that the MRR has a very strong dependence on grid size. The interpolation scheme used to measure the local gravitational field would probably smooth over the central cusp, causing the large errors here. I conclude that if small-scale structure is responsible for a caustic (high magnification) then this will be washed out.

Ray-tracing Parameters

The size of the ray-bundle, or equivalently, the radius of the image or beam, has negligible effect on the ray-tracing results. Outside the MRR and within the virial radius, the error remains fairly constant at 1%. The MRR has a strong dependence on the number of points used to interpolate the values of the gravitational potential and its derivatives from the Fourier-grid on which they are calculated. The use of more interpolation points allows for more robust ray-tracing, with a more accurate mean magnification value. However, the computation time rises approximately linearly with the number of interpolation points used, and the interpolation scheme suffers in the vicinity of density peaks. The latter issue should not pose too much of a problem when the lensing mass distribution is large-scale structure, but computational efficiency presents a significant hurdle for ray-tracing procedures such as this. Therefore, the 2-point interpolation scheme is chosen. The parameters that most govern the accuracy of the ray-tracing scheme are the Fourier-grid size and mass smoothing length, which have essentially equivalent effects;

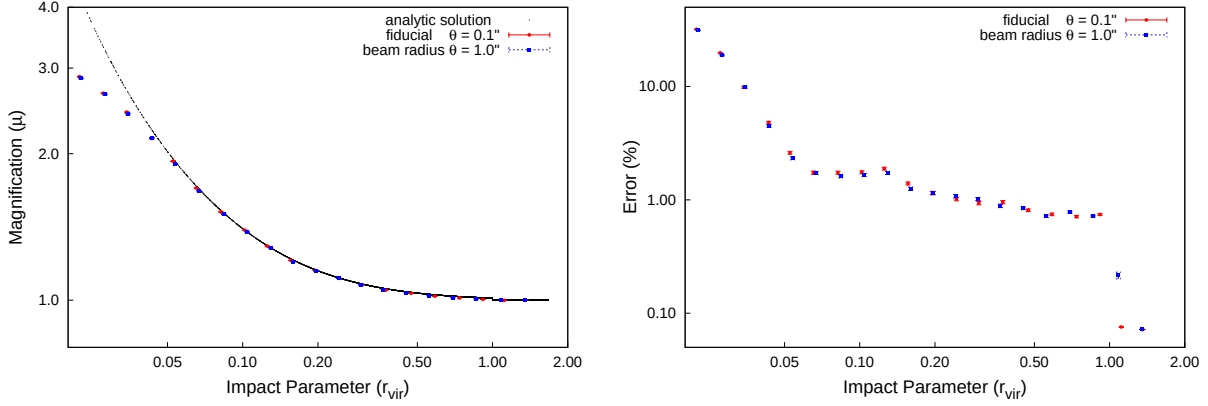


Figure 4.9: The radial dependence of magnification (μ ; left) and the percentage error in the numerical magnification ($\Delta\mu/\mu$; right) when ray-tracing through a discretized NFW lens. Results are shown for a fiducial choice of ray-bundle radius, $\theta = 0.1''$ (red) as well as $\theta = 1''$ (blue). In each case, 10^4 lines of sight have been binned uniformly with $\log(r/r_{\text{vir}})$, and the analytic solution at each point is included on the graph on the left. 1σ error-bars are shown.

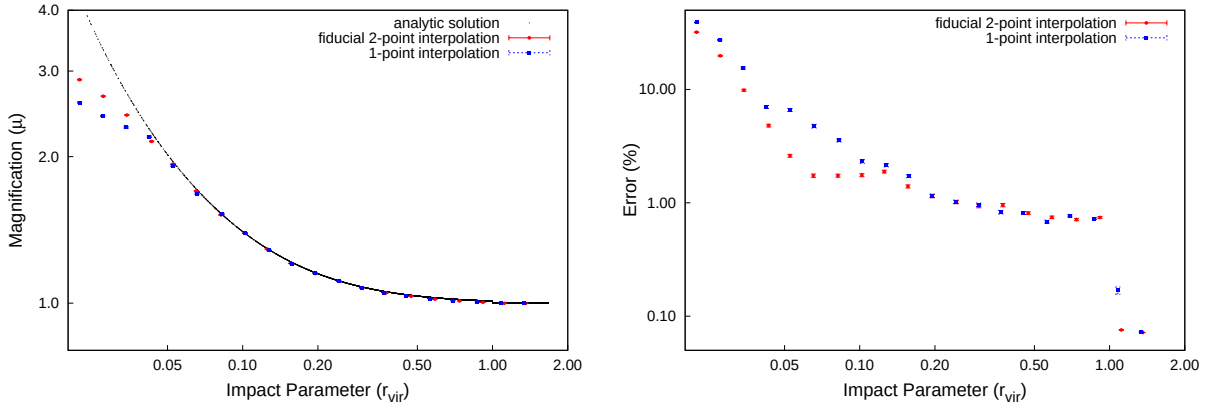


Figure 4.10: The radial dependence of magnification (μ ; left) and the percentage error in the numerical magnification ($\Delta\mu/\mu$; right) when ray-tracing through a discretized NFW lens. Results are shown for a fiducial choice for the number of points used in the interpolation scheme, $n_{\text{int}} = 2$ (red) as well as $n_{\text{int}} = 1$ (blue). In each case, 10^4 lines of sight have been binned uniformly with $\log(r/r_{\text{vir}})$, and the analytic solution at each point is included on the graph on the left. 1σ error-bars are shown.

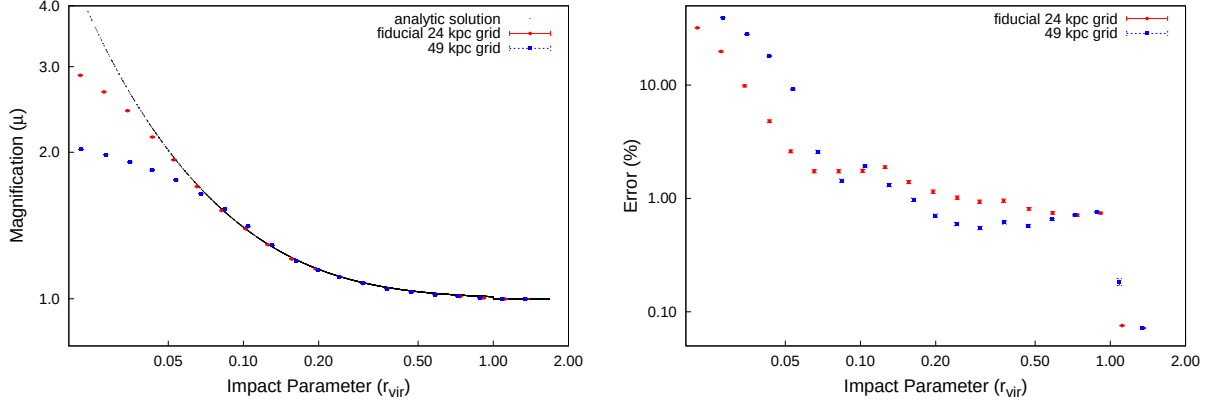


Figure 4.11: The radial dependence of magnification (μ ; left) and the percentage error in the numerical magnification ($\Delta\mu/\mu$; right) when ray-tracing through a discretized NFW lens. Results are shown for a fiducial choice for the Fourier-grid resolution, 24 kpc/h (red), as well as 49 kpc/h (blue). In each case, 10^4 lines of sight have been binned uniformly with $\log(r/r_{\text{vir}})$, and the analytic solution at each point is included on the graph on the left. 1σ error-bars are shown.

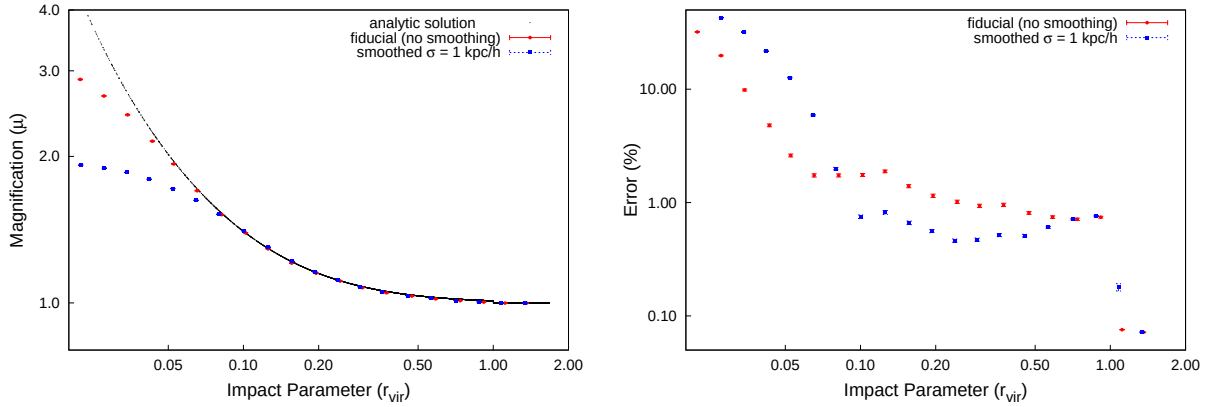


Figure 4.12: The radial dependence of magnification (μ ; left) and the percentage error in the numerical magnification ($\Delta\mu/\mu$; right) when ray-tracing through a discretized NFW lens. Results are shown for a fiducial choice of no smoothing (red) as well as for the case where the lensing mass is smoothed with a filter of width $\sigma = 1$ kpc/h (blue). In each case, 10^4 lines of sight have been binned uniformly with $\log(r/r_{\text{vir}})$, and the analytic solution at each point is included on the graph on the left. 1σ error-bars are shown.

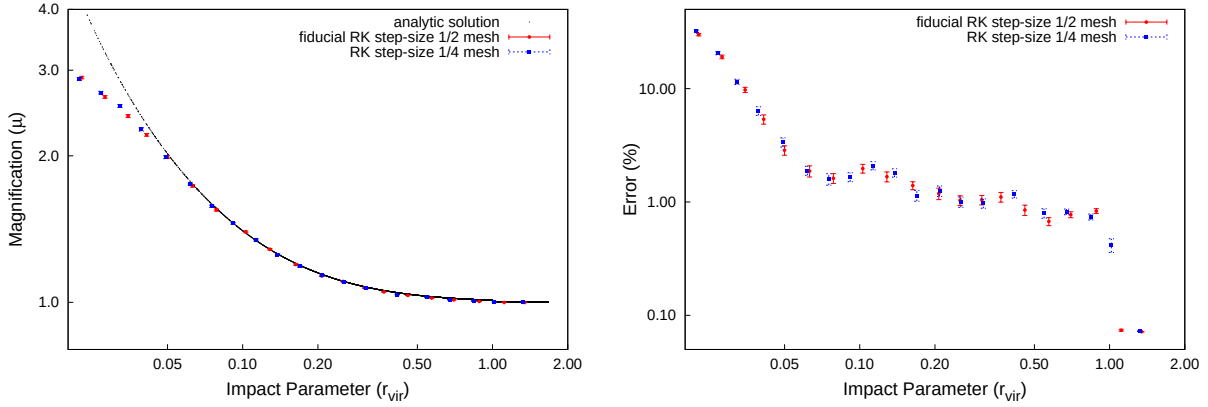


Figure 4.13: The radial dependence of magnification (μ ; left) and the percentage error in the numerical magnification ($\Delta\mu/\mu$; right) when ray-tracing through a discretized NFW lens. Results are shown for a fiducial choice of Runge-Kutta step-size that is 1/2 the Fourier-grid resolution (red) as well as for 1/4 the grid resolution (blue). In each case, 10^3 lines of sight have been binned uniformly with $\log(r/r_{\text{vir}})$, and the analytic solution at each point is included on the graph on the left. 1σ error-bars are shown.

assigning mass to an Fourier-grid acts to spread it out over a fixed number of grid-points. A CIC mass assigning scheme is different to a Gaussian filter smoothing, but in essence, the larger the area over which mass is spread, the more density peaks are suppressed and the larger the error in the tests shown in Figures 4.12 and 4.11.

Lens Model Parameters

The model lens used for this test is subject to a few user-defined variables. A few convergence tests are run to ensure that these were not the cause of the observed features. I repeat the test for particle mass resolutions ranging from 10^9 to $10^{11}M_{\odot}$ for a fixed halo virial mass, $M = 10^{14}M_{\odot}$; the results are shown in Figure 4.14. I find that the resulting scatter increases from less than 1% up to 10% at a fixed impact parameter. One could equivalently say that the MRR is larger for haloes that are not as well resolved. The cosmological mass distributions studied have mass resolutions better than $10^{10}M_{\odot}$, so one can expect a low error in magnification even within dense regions. The number of bins that the lens is divided into is a choice that has negligible effect on the ray-tracing results, as shown in Figure 4.15. One is therefore reassured that the results of the ray-tracing on the discretised NFW profile are relevant to the results of ray-tracing through cosmological mass distributions based on the fiducial parameter choices.

4.6 Summary

In this chapter, I have developed a code that performs ray-tracing with numerical integration of the null geodesic rather than invoking the multiple lens-plane approximation. This allows one to quantify the effects of such approximations when the deflecting mass can include large-scale line-of-sight filaments. This new code implements the efficient 8-ray bundle design developed in previous works yet models high magnification events accurately and is not limited to low

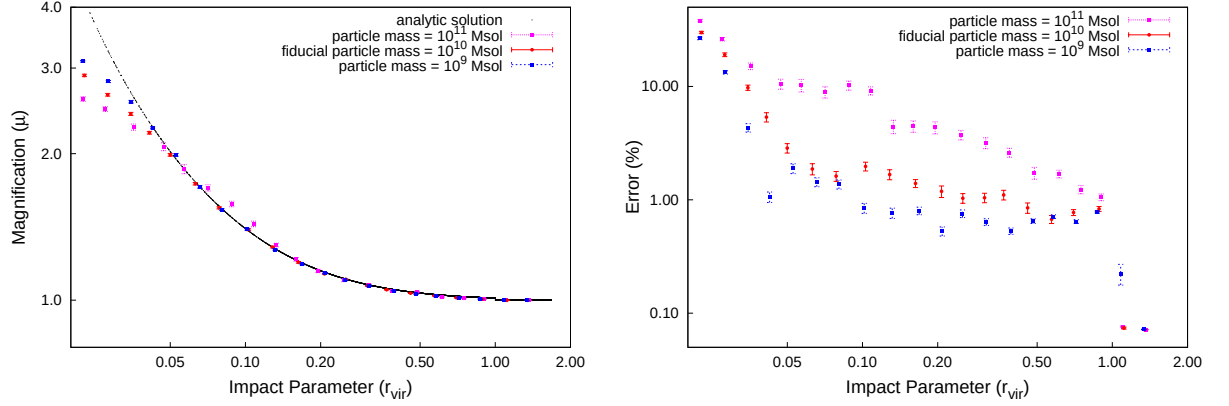


Figure 4.14: The radial dependence of magnification (μ ; left) and the percentage error in the numerical magnification ($\Delta\mu/\mu$; right) when ray-tracing through a discretized NFW lens. Results are shown for a fiducial choice of mass resolution, $m_p/M_\odot = 10^{10}$ (red) as well as for $m_p/M_\odot = 10^{11}$ (pink) and $m_p/M_\odot = 10^9$ (blue). In each case, 10^3 lines of sight have been binned uniformly with $\log(r/r_{\text{vir}})$, and the analytic solution at each point is included on the graph on the left. 1σ error-bars are shown.

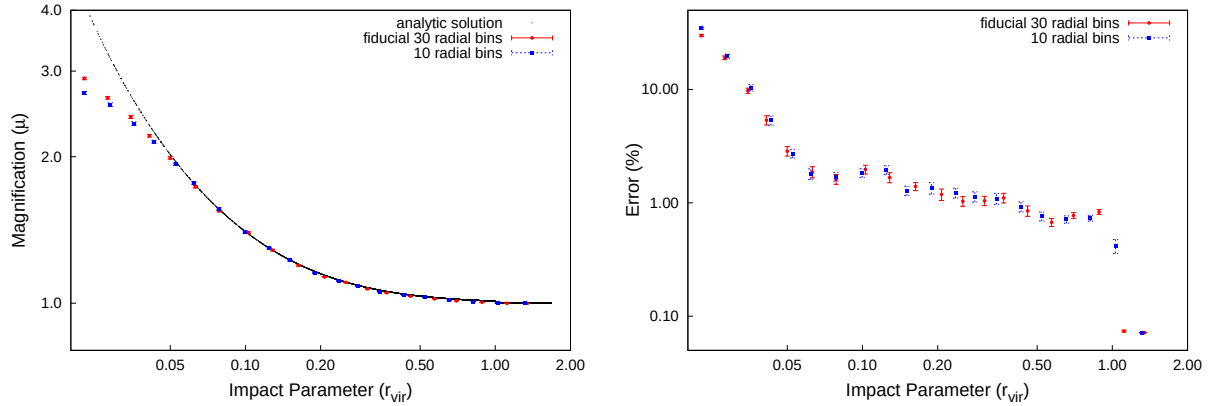


Figure 4.15: The radial dependence of magnification (μ ; left) and the percentage error in the numerical magnification ($\Delta\mu/\mu$; right) when ray-tracing through a discretized NFW lens. Results are shown for a fiducial choice of 30 radial bins used for the lens discretisation (red) as well as 10 bins (blue). In each case, 10^3 lines of sight have been binned uniformly with $\log(r/r_{\text{vir}})$, and the analytic solution at each point is included on the graph on the left. 1σ error-bars are shown.

density regimes as a result of the Fourier methods used in determining the local gravitational potential and its derivatives. The scale-factor is increased along the path of integration while the Newtonian potential (in co-moving coordinates) remains static during the time it takes for a photon to cross a simulation box. The efficiency and accuracy of this computationally challenging approach can be improved by careful choices of numerical parameters; therefore, the results are analysed for the behaviour of the ray-tracing code in the vicinity of Schwarzschild and NFW lenses. The limitations are dominated by the spatial resolution of the Fourier-grid and the mass resolution of the discretized lens. Comparisons to a multiple lens-plane algorithm are drawn in the context of cosmological mass distribution for a source redshift of $z_s = 0.5$. Preliminary results suggest that either the Fourier-grid resolution or shot-noise from a discretized mass distribution are responsible for the observed differences. A number of future studies are proposed: ray-tracing to higher redshifts, comparison to multiple lens-plane methods that incorporate a Fourier method for measuring deflection angles, modelling higher-order statistics as well as the weak lensing correlation function.

Chapter 5

The Redshift Drift Experiment

“Sometimes also, I see objects move when I stare at them long enough.”

– Anonymous (*feedbackfail.com*)

Throughout this thesis we have been concerned with the applications of gravitational lensing to analyse structure on various scales. In some studies, observations of background lensed galaxies provide a new window into the mass distribution of foreground structure; we use gravitational lensing to our benefit. Nevertheless, lensing can introduce biases and noise to the predictions of a given theoretical model, if it is not accounted for. In this chapter, a case in point is explored.

5.1 The Redshift Drift Experiment

Few experiments can truly claim to directly observe the expansion of the universe. Even the presence of redshifted spectral lines in extragalactic objects was not instantly able to convince the community that the universe was expanding; several other explanations, such as the tired-light hypothesis were being invoked as an alternative. Nowadays, discussion is centered around late-time acceleration, accurate mapping of the scale factor as a function of cosmic time and the possibility of evolving dark energy. Redshift drift is a direct probe of the dynamics of expansion, as the Hubble parameter $H(z)$ is measured at particular redshifts. The basic idea is that if the spectroscopic redshift of an extragalactic object is measured and then measured again after a sufficient period of time, the redshift will be seen to have changed.

The spectral resolution required to detect this change is enormous, but while the detection of the cosmic signal was deemed out of reach a few decades ago, the recent suggestion to use the Ly α forest seen in quasar spectra and large cadence (~ 10 yr) observations on next-generation telescopes has allowed us to envision the possibility of conducting this experiment in the near future. The major requirements are ultra-stable spectrographs and thousands of hours of monitoring, so the COsmic Dynamics EXperiment (CODEX) spectrograph proposed for the European-ELT is suitable. As this would be a ground-based study, the redshift range that can be probed is limited by the observed wavelength.

The experiment, also known as the Sandage-Loeb Test, is unique in its ability to probe the “redshift desert” ($2 \lesssim z \lesssim 5$). Many authors have analysed the potential for the test to constrain evolving dark energy models (Corasaniti et al., 2007; Balbi & Quercellini, 2007; Lake, 2007); the results are dependent on the specific redshift dependence of the dark energy equation of state for each of the models; sometimes it is found that large discrepancies between a particular dark energy models and the standard Λ CDM fall outside the redshift range being probed. Thus far, the various studies suggest that Chaplygin gas models, Lemaître-Tolman-Bondi (LTB) void models, interacting dark energy models and certain holographic dark energy models can be well constrained within a decade of observations, or rejected at high confidence if Λ CDM is correct

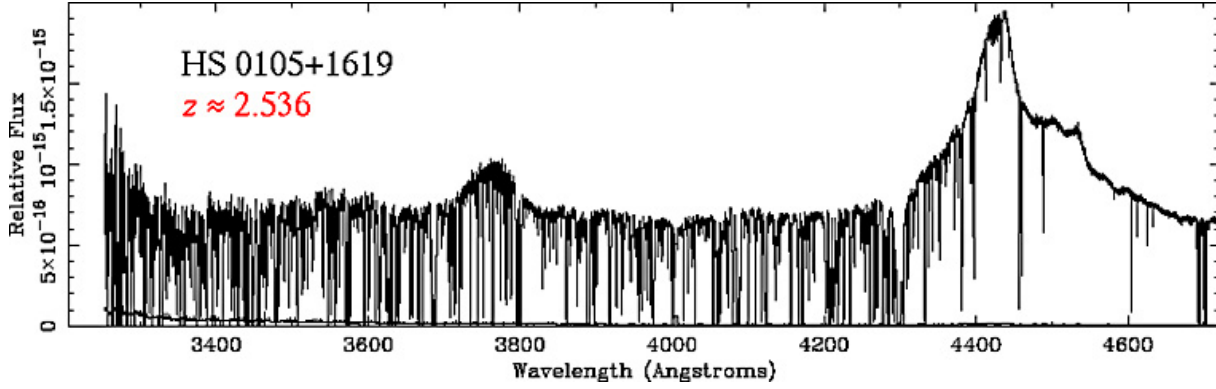


Figure 5.1: spectrum of the quasar HS 0105+1619. Light from distant quasars is absorbed by the neutral hydrogen atoms along the line of sight. This produces characteristic absorption features in the spectra of quasars called the Ly α forest. Taken from O’Meara (2004).

(Balbi & Quercellini, 2007; Quartin & Amendola, 2010). Conversely, the Chevallier-Polarski-Linder (CPL) model, scalar field power law potential model, $f(R)$ gravity and in general dark energy models with many parameters are not as well constrained (Balbi & Quercellini, 2007; Jain & Jhingan, 2010).

Many of these tests rely on accurate measurements of the drift, and detection alone is only possible with next generation large telescopes. All systematic biases and sources of noise must, therefore, be identified and the threat they pose to the detection must be quantified. With this in mind, I examine one such threat: the changes to observed redshifts of objects caused by the transverse motion of Ly α clouds.

5.2 Quasar spectra and the Ly α forest

Any neutral hydrogen present along the line of sight to an observed quasar may manifest itself as absorption features in background quasar spectra; some fraction of the atoms are ionised by rest-frame 912Å photons. Since these so-called Ly α clouds are closer to the observer than the quasar, the absorption lines are blueward of the strong Ly α emission line emitted at the quasar redshift, z_q , which is observed at $912(1+z_q)$ Å. Since neutral hydrogen is distributed throughout the line of sight, one sees absorption features so numerous they cut out almost all emission up to 912Å- this is known as the Ly α forest (see Fig. 5.1). The lines are typically distinguished by their neutral hydrogen column densities, N_{HI} , in units of cm^{-2} . When $\log(N_{\text{HI}}) \gtrsim 17$, the optically thick nature of the absorbers results in a discontinuity at (blueward of) 912Å (rest-frame); these absorbers are classified as Lyman limit systems (LLS). When the column densities surpass a certain threshold, with $\log(N_{\text{HI}}) \gtrsim 20$, the self-shielding mechanism leads to damping wings about the absorption line; the associated cloud is known as a Damped Lyman- α systems (DLA). More recently a new class of absorbers has been identified, sub-DLAs with HI column densities of $10^{19} \lesssim N_{\text{HI}} \lesssim 2 \times 10^{20}$.

5.3 The Nature and Distribution of Absorbers

The existence of the Ly α forest indicates the presence of neutral hydrogen along the line of sight to the quasar being observed. However, the nature of the structures that provide this gas is not well understood, and several studies have aimed to understand exactly what it is that causes the absorption lines. Early suggestions included pancake-like disks of gas formed via dissipative collapse (Zel'Dovich, 1970; Zeldovich, 1978). However, modern simulations provide a very different view of structure formation (See Section 1.3.1). Filamentary structure and galaxy haloes present two distinct possible hosts to neutral hydrogen gas. While the nature of filaments would prevent a large column density (unless an unusual alignment occurred), gas density would clearly drop with distance from the centre of a galaxy halo. Imaging studies of the regions near quasars that exhibit DLAs in their spectra provide the opportunity to identify the nature of the objects that lead to these spectral features. Some of the higher column density absorbers are at small impact parameters from galaxies (e.g. Le Brun et al., 1997). However, more recent studies suggest that the picture is more complicated; subDLAs appear to arise in the vicinity of more massive galaxies than DLAs (Kulkarni et al., 2010).

The redshift distribution of absorbers has been analysed by several studies. One may use these to get an upper bound on the number of galaxy haloes one can expect to disturb the redshift drift experiment. In addition, the line width doppler parameter can be linked to the galaxy mass. In combination with an estimate of transverse peculiar velocities, one may ascertain the lensing effect on the results of the test.

5.4 Gravitational Lensing by Absorbers

When the line of sight to a background source passes a moving lens, the change in the gravitational field has the effect of changing the wavelength of the light (Birkinshaw & Gull, 1983; Birkinshaw, 1989; Pyne & Birkinshaw, 1993). What I have recognised here is that Ly α absorbers, and in particular, galaxy haloes, have the potential to shift the absorption lines in the quasar Ly α forest. Since the signal that will be searched for during the Cosmological Redshift Drift Experiment is weak, one must consider whether the lensing effect will overpower the signal.

The change in wavelength, as a result of the Birkinshaw & Gull effect, occurs in the rest frame of the lens. After the light photon leaves the vicinity of the lens, it continues to be cosmologically redshifted by the expanding scalefactor. Any further interactions with moving lenses will occur after some cosmological redshifting, so the effect is compounded.

Therefore I can determine what the effect is for a single photon that passes the vicinity of a series of lenses, each of which have some non-zero transverse peculiar velocity. I then consider how this effect varies for a photon observed now and then a decade later. Therefore one must also consider how the transverse peculiar velocity of the lens/absorber changes during that time; in other words, one must consider any transverse peculiar *acceleration*.

5.5 Ly- α absorbers in motion: consequences of gravitational lensing for the cosmological redshift drift

The following paper:

Ly- α absorbers in motion: consequences of gravitational lensing for the cosmological redshift drift

Killedar M., Lewis G. F., 2010, *MNRAS*, 402, 650

is concerned with the effects of gravitational lensing on the potential to conduct a future direct cosmological test, that is the redshift drift experiment. The analytical results were developed in collaboration with Geraint Lewis; the text was my own.

Ly α absorbers in motion: consequences of gravitational lensing for the cosmological redshift drift experiment[★]

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ABSTRACT

The evolution of the expansion rate of the Universe results in a drift in the redshift of distant sources over time. A measurement of this drift would provide us with a direct probe of expansion history. The Lyman α (Ly α) forest has been recognized as the best candidate for this experiment, but the signal would be weak and it will take next generation large telescopes coupled with ultrastable high-resolution spectrographs to reach the cm s^{-1} resolution required. One source of noise that has not yet been assessed is the transverse motion of Ly α absorbers, which varies the gravitational potential in the line of sight and subsequently shifts the positions of background absorption lines. We examine the relationship between the pure cosmic signal and the observed redshift drift in the presence of moving Ly α clouds, particularly the collapsed structures associated with Lyman limit systems and damped Ly α systems. Surprisingly, the peculiar velocities and peculiar accelerations *both* enter the expression, although the acceleration term stands alone as an absolute error, whilst the velocity term appears as a fractional noise component. An estimate of the magnitude of the noise reassures us that the motion of the Ly α absorbers will not pose a threat to the detection of the signal.

Key words: gravitational lensing – methods: analytical – intergalactic medium – quasars: absorption lines – cosmology: theory.

1 INTRODUCTION

The large-scale homogeneity and isotropy of the Universe is encompassed in Friedmann–Lemaître–Robertson–Walker geometry (see Hobson, Efstathiou & Lasenby 2006 for further details). One of the key aspects of this cosmology is the evolution of the scale-factor, $R(t)$; modern cosmology aims to characterize this evolution, distinguish between competing cosmological models and predict the fate of the Universe. Observationally, this has been a rather difficult science, with so-called standard candles and standard rulers playing leading roles in indirect measurements. Sandage (1962) recognized that a direct detection of the evolution of the expansion rate was, in theory, possible; just as the cosmological redshifts of extragalactic objects in all directions are a consequence of isotropic expansion, a positive or negative drift in the redshift of a *single object* over time would indicate acceleration or deceleration with respect to the observer. The redshift drift is very small, but as we head towards an era of cm s^{-1} resolution spectroscopy, the prospect of detection becomes very real. In addition to identifying the ap-

propriate instrumentation and spectroscopic techniques required, it is crucial to account for any systematic biases and consider if any source of noise may hinder the detection of the cosmic signal. Previous studies have covered some of this ground but the potential for the transverse motion of Lyman α (Ly α) clouds to hide this signal remains yet unexplored. In the present paper, we aim to address this issue and estimate the magnitude of the bias or noise that is introduced.

In Section 2, we review the current cosmological paradigm along with relevant observational tests. We rederive the expression for the cosmic signal being sought in Section 3. In Section 4, we introduce the transverse moving-lens effect. We derive the expression for the observed redshift drift in the presence of Ly α clouds in Section 5 and discuss the results in Section 6.

2 BACKGROUND

The linear correlation between the recession velocities and luminosity distances of galaxies within a few Mpc was first discovered by Hubble (1929).¹ This result, the first Hubble diagram, set the observational precedent for the notion of an expanding Universe,

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¹ There were large systematic errors in Hubble's distances, and the scales probed were rather small; nevertheless, the result was fundamental to the acceptance of the expanding Universe hypothesis.

at a time when a static Universe was philosophically favoured. More recently, two independent measurements of the luminosities of ‘standard candle’ Type Ia supernovae (SNIa) by Riess et al. (1998) and Perlmutter et al. (1999) have revealed that, assuming a metric theory of gravity, the Universe is currently undergoing a period of acceleration (Shapiro & Turner 2006), and there is evidence for past deceleration beyond a transition redshift of $z \sim 0.5$ from *Hubble Space Telescope* (HST) observations of high-redshift ($0.2 \lesssim z \lesssim 1.6$) SNIa (Riess et al. 2004).

Whilst deceleration is expected in a matter-only universe, recent acceleration requires a ‘dark-energy’ component. To uncover the underlying physical cause of the acceleration, it is necessary to map out the evolution of the scalefactor; the functional form is characterized by the relative densities (in units of the critical density Ω_0) of the various matter–energy components of the cosmological fluid $\Omega_{X,0}$, where X may be one of r (radiation), M (matter), dark energy or k (curvature) and the subscript 0 denotes the current values. The growing evidence points towards a flat universe that is currently dominated by a cosmological constant, Λ (a ‘dark-energy’ component with an equation of state $w = -1$), a subdominant pressureless dark matter component and a small amount of baryonic matter.

The expansion rate is normalized by its value today, i.e. the Hubble constant: $H_0 \equiv 100h$ km s $^{-1}$ Mpc $^{-1}$. The HST Key Project has combined several independent distance methods, establishing the most accurate measurement as $H_0 = 72 \pm 8$ km s $^{-1}$ Mpc $^{-1}$ (Freedman et al. 2001). This is in good agreement with the current best estimates for the values of the cosmological parameters: $\Omega_{M,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$ and $h = 0.71$, based on observations of the cosmic microwave background (CMB) radiation by the *Wilkinson Microwave Anisotropy Probe 5-year data* (WMAP5; Komatsu et al. 2009), SNIa (Kowalski et al. 2008, and references therein) and baryonic acoustic oscillations (BAOs; Percival et al. 2007) in the galaxy distribution. Each of the measurements individually provides constraints on various degenerate combinations of the parameters in question; the Hubble parameter constraints are model dependent.

Whilst precision cosmology converges upon a concordance in the parameters with complementary methods, the expansion remains an underlying assumption upon which observed phenomena, such as cosmological redshifts, are interpreted. Few direct tests are available to support this hypothesis. One observable consequence of expansion is time dilation by a factor of $(1+z)$. Wilson (1939) suggested the use of SNIa light curves as ‘cosmic clocks’, and the results of high-redshift SNIa observations have confirmed the expansion hypothesis and excluded other models (Leibundgut 1996; Goldhaber et al. 2001; Blondin et al. 2008). Another prediction is the decrease in the surface brightness (SB) of galaxies with a $(1+z)^{-4}$ redshift dependence, independent of all other cosmological parameters (Tolman 1930, 1934; Hubble & Tolman 1935). The Tolman SB test has been performed in several studies, each confirming the expanding geometry. Pahre, Djorgovski & de Carvalho (1996) and Lubin & Sandage (2001) rule out the $(1+z)^{-1}$ dependence expected by the alternative non-expanding tired-light model at the 5σ and 10σ confidence level, respectively. Furthermore, the temperature of the CMB is predicted to increase linearly with redshift: $T_{\text{CMB}}(z) = T_0(1+z)$; the temperature at the present epoch has been accurately determined to be $T_0 = 2.725 \pm 0.002$ K (Mather et al. 1999). Atomic fine structure transitions identified in quasar absorption spectra probe the CMB temperature at the absorption redshift (Bahcall & Wolf 1968). This method generally provides an upper limit to the temperature only, but the observations, which cover redshifts up to $z \lesssim 3$ are consistent with the standard cos-

mological model (e.g. Songaila et al. 1994; Srianand, Petitjean & Ledoux 2000; Molaro et al. 2002, and references therein). Alternatively, the CMB temperature can be estimated at the redshift of galaxy clusters by observing the thermal Sunyaev-Zel’dovich effect (Fabbri, Melchiorri & Natale 1978; Rephaeli 1980); measurements using this method have been carried out for clusters at a range of redshifts $0.02 < z < 0.6$, again finding the redshift dependence to be consistent with an expansion hypothesis (Battistelli 2002; De Petris et al. 2002; Luzzi et al. 2009).

Redshift drift is a direct probe of the dynamics of expansion, as the Hubble parameter $H(z)$ is measured at particular redshifts. It allows a unique insight into the validity of our cosmological model, sometimes with complementary observations. For example, by characterizing the redshift dependence of the drift, the experiment can constrain evolving dark-energy models (Corasaniti, Huterer & Melchiorri 2007) and map the equation of state (Lake 2007), although many-parameter dark-energy models are harder to constrain (Balbi & Quercellini 2007). If we live in a Λ CDM universe, the Chaplygin gas model (Kamenshchik, Moschella & Pasquier 2001) and interacting dark-energy model would be rejected at high confidence (Balbi & Quercellini 2007), but the Lemaître–Tolman–Bondi (LTB) void models would be ruled out even earlier (Quartin & Amendola 2009). Furthermore, when combined with luminosity distance data, the redshift drift experiment provides a test of the Copernican principle (Uzan, Clarkson & Ellis 2008b) and an independent measurement of spatial curvature (Nakamura & Chiba 1999).

Many of these tests rely on accurate measurements of the drift, and detection alone is only possible with next generation large telescopes. All systematic biases and sources of noise must, therefore, be identified and the threat they pose to the detection must be quantified. With this in mind, we examine one such threat: the changes to observed redshifts of objects caused by the transverse motion of Ly α clouds.

3 THE COSMIC DECELERATION SIGNAL

The general form for the redshift drift dependence on redshift and the expansion rate was originally derived by McVittie (1962, see equation 4A). We briefly rederive it here.

The Hubble parameter is defined in terms of the scalefactor and expansion rate:

$$H(t) \equiv \frac{\dot{R}(t)}{R(t)}.$$

The cosmological redshift of an emitter is related to the ratio of the scalefactors at emitter and observer:

$$1+z = \frac{R(t_0)}{R(t)},$$

where t_0 is the time of observation and t the time of emission.

As an aside, the cosmological fluid components appear in the redshift dependence of the Hubble parameter as derived from the cosmological field equations. The expression is given by:

$$H(z) = H_0 \sqrt{\Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2},$$

where we neglect radiation and assume dark energy to be in the form of a cosmological constant.

Consider now the redshift after some time interval, when the scalefactors have evolved at both emitter and observer:

$$\begin{aligned} 1 + z + \Delta z &= \frac{R(t_0 + \Delta t_0)}{R(t + \Delta t)} \\ &\approx \frac{R(t_0) + \dot{R}(t_0)\Delta t_0}{R(t) + \dot{R}(t)\Delta t} \\ &= (1 + z) \frac{1 + H(t_0)\Delta t_0}{1 + H(t)\Delta t} \\ &= (1 + z)(1 + H_0\Delta t_0 - H(z)\Delta t). \end{aligned}$$

The scalefactors have been expanded to the first order in Δt . The Hubble parameter at the time of observation is relabelled: $H_0 \equiv H(t_0)$, whilst the Hubble parameter at emission is expressed in terms of the *redshift* of the emitter, rather than the *time* of emission, i.e. $H(z)$ rather than $H(t)$. Subtracting to obtain the change in redshift:

$$\Delta z = (1 + z)[H_0\Delta t_0 - H(z)\Delta t].$$

If both emitter and observer are at rest in comoving coordinates, then time intervals in their respective frames are related by

$$\Delta t_0 = \Delta t(1 + z),$$

and thus we obtain the expression for the redshift drift:

$$\Delta z = [1 + z - E(z)]H_0\Delta t_0, \quad (1)$$

where

$$E(z) \equiv H(z)/H_0.$$

The exact functional form is dependent on the equation of state and density of the cosmological fluid components, but an order of magnitude estimate

$$\Delta z \approx 1 \times 10^{-10} h [1 + z - E(z)] \left(\frac{\Delta t_0}{\text{yr}} \right)$$

using the density parameters determined from *WMAP5*+*BAO*+*SNIa* and evaluated, for example at $z = 3$, gives an estimated cosmic signal of $|\Delta z| \sim 4 \times 10^{-10}$, or $\Delta v \sim 3 \times 10^{-2} \text{ m s}^{-1}$, per decade between epochs of observation.

Sandage (1962) acknowledged that the required spectral resolution ($\sim 1 \text{ cm s}^{-1}$ per decade cadence) was beyond the capabilities of the instrumentation of that generation. The redshift drift and its potential to constrain cosmological parameters have been studied within the contexts of various cosmological models (Ebert & Trümper 1975; Rüdiger 1980; Lake 1981), but without hope of detection.

3.1 The new challenge

Loeb (1998) identified the $\text{Ly}\alpha$ forest as the best candidate for a redshift drift experiment. The $\text{Ly}\alpha$ forest is a phenomenon observed in the spectra of quasars as a series of absorption lines bluewards of the quasar $\text{Ly}\alpha$ emission line, each at a rest wavelength of $\lambda_{\text{Ly}\alpha} \equiv 1216 \text{ \AA}$. The absorption lines are fingerprints of intervening neutral hydrogen clouds. Although absorbers may lie anywhere between the observer and the quasar, they are not detected or distinguishable below a certain redshift cut-off; the reasons are two-fold. Quasar spectra are usually observed by ground-based optical telescopes, but low-redshift absorption lines lie in the ultraviolet, so only absorbers at $z \gtrsim 1.7$ are detected.² Contamination by the $\text{Ly}\beta$ emission line

² Although space-based ultraviolet spectrographs may detect the low-redshift $\text{Ly}\alpha$ forest, they would not have the stability required for such an experiment; the ultrastable high-resolution instrumentation is necessarily ground based.

($\lambda_{\text{Ly}\beta} \equiv 1025 \text{ \AA}$), and the associated $\text{Ly}\beta$ forest, causes confusion below $1 + z = (1 + z_q) \lambda_{\text{Ly}\beta}/\lambda_{\text{Ly}\alpha}$. The redshift range thus probed with this experiment does not include the current acceleration phase. The $\text{Ly}\alpha$ forest supplies a large sample of spectral lines at high redshift, so while the linewidths are relatively large ($\sim 20 \text{ km s}^{-1}$) and the cosmic signal weak, the sheer number density of lines per redshift bin should yield the necessary statistical accuracy. Loeb found that existing spectroscopic techniques, such as those employed in extrasolar planet searches, could produce a marginal detection.

This experiment has become one of the science drivers for next generation 30–60 m extremely large telescopes (ELTs). The instrumental challenges for the Cosmic Dynamics EXperiment (CODEX) spectrograph proposed for the European-ELT have been discussed by Pasquini et al. (2005), Murphy et al. (2007) and Corasaniti et al. (2007); the latter finds CODEX capable of detecting the redshift drift in the $\text{Ly}\alpha$ forest over ~ 10 yr. Liske et al. (2008, hereafter L08) conducted an extensive study into the feasibility of detection with a 42-m ELT, concluding that ~ 4000 h of observing time over a 20 yr interval would reveal the redshift drift at a $\sim 3\sigma$ significance.

But before the search for a weak signal such as this is undertaken, all possible systematic biases and sources of noise must be accounted for. Peculiar accelerations of the $\text{Ly}\alpha$ clouds or their associated galaxies present perhaps the most obvious obstacle. The magnitude of the noise has been studied by Phillipps (1982), Lake (1982), L08 and Uzan, Bernardeau & Mellier (2008a); after some initial controversy, the accelerations were determined to have little impact on the signal.

The noise from galactic feedback and optical depth variations was evaluated by L08 and found to have a negligible effect. L08 also noted that the evolution of gravitational potential wells in the line of sight was a source of error not yet accounted for. We identify $\text{Ly}\alpha$ absorbers to be culprits; the positions of absorption lines may be shifted by foreground $\text{Ly}\alpha$ clouds, which are effectively weak lenses with non-zero transverse velocities.

4 TRANSVERSE MOVING LENSES

As light passes near a massive object, it is gravitationally lensed, but if the lens gravitational potential field varies during the time of passage, the light experiences a net redshift or blue shift. This leads to the well-known Rees–Sciama effect (Rees & Sciama 1968) in the context of a collapsing galaxy cluster lens; a similar phenomenon occurs when a (non-evolving) lens moves across the line of sight. Birkinshaw & Gull (1983) were the first to describe how lenses with transverse relativistic peculiar velocities asymmetrically distort background emission, and the general relativistic treatment was later developed by Pyne & Birkinshaw (1993).

This effect (hereafter the BG effect) has garnered attention largely in the context of galaxy clusters moving across the sky, inducing secondary anisotropies in the CMB brightness temperature (Birkinshaw & Gull 1983; Birkinshaw 1989; Aghanim et al. 1998).³ The influence of cluster lens transverse motion on weakly and strongly lensed background galaxies has also been investigated (Molnar & Birkinshaw 2003).

The induced change in wavelength by the BG effect is proportional to the lens velocity and deflection angle. For a given lens speed, the magnitude of the change in wavelength is maximized if

³ The induced CMB anisotropy derived by Birkinshaw & Gull (1983) was incorrect by a factor of two. This was corrected by Birkinshaw (1989).

the lens is moving in the plane of the sky and towards (or away from) the image of the background source. For the purposes of this study, we need only consider this special case. We thus have

$$\left| \frac{\delta\lambda}{\lambda} \right| = \gamma \frac{v}{c} \hat{\alpha}, \quad (2)$$

where v is the lens speed, γ is the Lorentz factor associated with the total lens velocity and $\hat{\alpha}$ is the deflection angle. This manifests as a change in the observed redshift

$$\begin{aligned} \left| \Delta z \right| &= (1 + z_s) \left| \frac{\delta\lambda}{\lambda} \right| \\ &= (1 + z_s) \gamma \frac{v}{c} \hat{\alpha}, \end{aligned} \quad (3)$$

where z_s is the source (not lens) redshift. It is useful to consider the order of magnitude of the redshift:

$$\left| \Delta z \right| \approx 1.6 \times 10^{-8} (1 + z_s) \gamma \frac{v}{(1000 \text{ km s}^{-1})} \frac{\hat{\alpha}}{1 \text{ arcsec}}.$$

A typical cluster lens ($v \sim 1000 \text{ km s}^{-1}$, $\hat{\alpha} \sim 30 \text{ arcsec}$) can change the redshift of a source at $z \sim 3$ by $\Delta z \sim 2 \times 10^{-6}$ or $\Delta v \sim 150 \text{ m s}^{-1}$, while a galaxy lens may produce an effect an order of magnitude smaller.

In this paper, we interpret Ly α absorbers as a series of numerous moving lenses and consider the impact of all foreground absorbers on the observed wavelengths of absorption lines from more distant Ly α clouds. In the context of the redshift drift, however, we must determine the *differential* BG effect, i.e. how the shift in the observed wavelengths at the second epoch of observation compare to the shift at the first epoch. Thus we will consider the transverse accelerations, a_t , as well.

5 THE EFFECT OF LYMAN α CLOUDS ON THE OBSERVED REDSHIFT DRIFT

We consider here the dynamic nature of Ly α clouds and how, as lenses, they can affect the redshift drift observed over some time interval. The expression we will derive below is parametrized by the cadence between observations, the number of Ly α absorbers, their peculiar velocities and accelerations as well as the deflection angle they induce.

5.1 Ly α clouds

Ly α clouds are typically distinguished by their neutral hydrogen column densities, N_{HI} , in units of cm^{-2} . Beyond $\log(N_{\text{HI}}) \gtrsim 17$, the optically thick nature of the absorbers results in a discontinuity at (bluewards of) $912 \text{ \AA}$ (rest frame); these absorbers are classified as Lyman limit systems (LLSs). Clouds with $\log(N_{\text{HI}}) \gtrsim 20$ exhibit damping wings about the absorption line profile leading to large Ly α equivalent widths; these are labelled damped Ly α systems (DLAs).

The exact nature of these absorbers is somewhat a mystery [see Rauch (1998) for a review]. Low column density absorption occurs in filamentary or sheet-like structures of scalelengths of $0.1\text{--}1 \text{ Mpc}$; LLS absorbers are identified with the outer haloes of spiral and elliptical galaxies, whilst DLAs occur in spiral discs or the haloes of low-mass galaxies. The lensing effect of DLAs (and other types of systems such as those identified by metal lines) is significant enough to introduce a magnification bias and push quasars over magnitude thresholds (Bartelmann & Loeb 1996; Ménard 2005). We must, therefore, consider whether the BG effect associated with these absorbers is significant.

The galaxy haloes and discs responsible for LLSs and DLAs are overdense and therefore result in higher peculiar accelerations, as recognized by Loeb (1998). L08 thus excluded them from their analysis, modelling only the low column density Ly α forest and removing the portions of the spectra blocked out by DLAs from real spectra. *This does not mean that quasar spectra that contain LLSs and DLAs are excluded from the redshift drift experiment.* Indeed, each line of sight may include low-redshift LLSs and DLAs hidden outside the observed bandwidth; in fact, these play an important role in the present study.

5.2 Velocities of lenses

Previous studies have provided insight into the peculiar motions of Ly α clouds. Rauch (2005) measured the velocity shear between pairs of absorbers common to multiple images of gravitationally lensed quasars; their lines of sight are close together. Their results indicate that while the large-scale motions correspond to the Hubble flow, there is tentative evidence for the clouds undergoing gravitational collapse. L08 used hydrodynamic simulations of the intergalactic medium (IGM) to determine the peculiar motions of Ly α absorbers. They find that the distribution of peculiar velocities peak at $v_{\text{pec}} \sim 190\text{--}200 \text{ km s}^{-1}$ for gas at $2 \lesssim z \lesssim 4$ with an approximate dispersion of $\sigma_{v_{\text{pec}}} \sim 80 \text{ km s}^{-1}$.

A single high-velocity cloud that is present in the line of sight, with absorption taking place at an impact parameter of 1 kpc from a $10^{11} \text{ M}_{\odot}$ galactic halo ($v \sim 300 \text{ km s}^{-1}$, $\hat{\alpha} \sim 4 \text{ arcsec}$), could alter positions of background absorption lines at redshift z by as much as $\Delta z \sim 2 \times 10^{-8} (1 + z)$. This is indicative of the BG effect as can be expected from the high column density absorbers, such as LLSs and some DLAs.

The lower column density absorbers are associated with filamentary structures, although overdense knots will move along these, as seen in cosmological N -body simulations. In this context, equation (2), which strictly assumes a compact lens, would require an additional term to compensate for the extended shapes of the overdensities, rendering the BG effect less pronounced. The BG effect here may also arise from galaxy clusters that lie at the ends of the associated filaments. A moderate-sized cluster ($10^{13} \text{ M}_{\odot}$) at a distance of 1 Mpc and with reasonable velocity ($v \sim 300 \text{ km s}^{-1}$) could potentially alter the observed redshifts by $\Delta z \sim 2 \times 10^{-9} (1 + z)$. To ascertain whether this poses a threat to the detection of the cosmic signal, we must include the effect in the derivation of the observed redshift drift.

5.3 Time interval between observation epochs

The duration of the interval between observations of quasar spectra has by far the largest impact on the chance to detect the cosmic signal. The chosen cadence must clearly be large enough to allow for some appreciable evolution in the scalefactor (over all redshifts $2 \lesssim z \lesssim 5$). We will, however, still have $H_0 \Delta t_0 \ll 1$ and $\Delta z \ll z$. By combining the signal from ~ 100 uncorrelated quasars, the signal may be detected over only $\Delta t_0 \sim 10\text{--}20 \text{ yr}$ (Loeb 1998; Corasaniti et al. 2007; Liske et al. 2008). These intervals are proposed in the context of spectrographs mounted on $30\text{--}60 \text{ m}$ ELTs proposed for the near future.

5.4 The observed redshift drift

Here, we derive the observed redshift drift over some time interval and compare this to the drift resulting purely due to cosmic deceleration. Fig. 1 shows how the wavelength of a photon varies as it

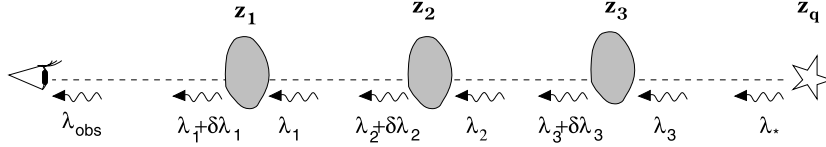


Figure 1. Light from a distant quasar (right-hand side), emitted at rest wavelength λ_* , travels through many Ly α absorbers (three shown here), before reaching the observer (left-hand side) at the observed wavelength λ_{obs} . Each absorber, i , is also a lens with some non-zero transverse velocity (up or down the page) and thus each induces a change in the wavelength, $\delta\lambda_i$, in the frame of the observer. A photon is cosmologically redshifted as it travels between two adjacent clouds (see equation 4).

travels from a distant source to an observer being lensed by moving Ly α clouds along the way. We have $0 < z_i < z_j < z_q$ for $i < j$, although we are not particularly concerned with the redshift of the source quasar nor the wavelength of the photon upon emission, λ_* . We are, instead, interested in the redshift of an absorption line with rest wavelength λ_i , observed at λ_{obs} .

Each photon (or absorption line) is redshifted or blueshifted in the frame of a stationary observer upon passing a Ly α cloud with some non-zero transverse velocity; the photon enters the cloud at λ and exits at $\lambda + \delta\lambda$, where the small shift is given by equation (2). Absorption occurs in the rest frame of the cloud; therefore, in the stationary frame (though at the same redshift), the absorption line is shifted in the same manner. For brevity, we denote

$$\lambda' \equiv \lambda + \delta\lambda.$$

When faced with the situation where the observed wavelength of any emission can be altered by some means other than expansion, we must distinguish between the *true* redshift, z^* , and that which is observed, z^{obs} . The true redshift is given by

$$1 + z^* = \frac{R(t_0)}{R(t)}$$

while the observed redshift is defined as:

$$1 + z^{\text{obs}} = \frac{\lambda_{\text{obs}}}{\lambda}.$$

As the photon travels from cloud j (or the source quasar) to the next cloud $j - 1$ (or observer), it is redshifted as a result of the difference in scalefactors at the two clouds. Thus, the (true) redshifts of the two clouds are related by

$$\frac{1 + z_j^*}{1 + z_{j-1}^*} = \frac{\lambda_{j-1}}{\lambda'_j}. \quad (4)$$

We recognize that the true redshift of the N th cloud is the product of the ratios of scalefactors at consecutive pairs of foreground clouds, and so

$$\begin{aligned} 1 + z_N^* &= \prod_{i=1}^N \frac{\lambda_{i-1}}{\lambda'_i} \\ &= \frac{\lambda_{\text{obs}}}{\lambda_N} \prod_{i=1}^N \frac{\lambda_i}{\lambda'_i} \\ &\approx (1 + z_N^{\text{obs}}) \left[1 - \sum_{i=1}^N \frac{\delta\lambda_i}{\lambda_i} \right], \end{aligned} \quad (5)$$

since $\delta\lambda \ll \lambda$. We have defined $\lambda_0 \equiv \lambda_{\text{obs}}$. After some time interval Δt_0 (in the observer rest frame), the true redshift has evolved to be:

$$\begin{aligned} 1 + z_N^* + \Delta z_N^* &\approx \frac{R_0 + \dot{R}_0 \Delta t_0}{R_N + \dot{R}_N \Delta t_N} \\ &= (1 + z_N^*) \frac{1 + H_0 \Delta t_0}{1 + H(z_N^*) \Delta t_N} \\ &\approx 1 + z_N^* + H_0 \Delta t_0 [1 + z_N^* - E(z_N^*)], \end{aligned} \quad (6)$$

where Δt_N denotes the time passed in the rest frame of the N^{th} cloud:

$$\Delta t_0 = \Delta t_N (1 + z_N^*).$$

We observe a drifted redshift, analogous to equation (5):

$$1 + z_N^* + \Delta z_N^* = (1 + z_N^{\text{obs}} + \Delta z_N^{\text{obs}}) (1 - \mathcal{S}) - (1 + z_N^{\text{obs}}) \mathcal{S}_{\Delta t}, \quad (7)$$

where the summations are denoted thus:

$$\mathcal{S} \equiv \sum_{i=1}^N \frac{\delta\lambda_i}{\lambda_i}$$

$$\mathcal{S}_{\Delta t} \equiv \Delta t_0 \sum_{i=1}^N \frac{a_{t,i}}{c} \hat{\alpha}_i$$

with $a_{t,i}$ the tangential peculiar *acceleration* of the i th Ly α cloud. Any change in the deflection angle is dependent on the mass profile of the cloud and vanishes in the case of a singular isothermal sphere. In any case, the impact parameter changes by only a fraction of a parsec so we neglect this term.

We subtract the observations at the two epochs to find the observed drift rate, i.e. equate equations (6) and (7) and subtract equation (5):

$$H_0 \Delta t [1 + z_N^* - E(z_N^*)] = \Delta z_N^{\text{obs}} (1 - \mathcal{S}) - (1 + z_N^{\text{obs}}) \mathcal{S}_{\Delta t}. \quad (8)$$

As a sanity check, if we neglect the effect of all clouds, we recover the McVittie expression (equation 1).

5.5 The magnitude of the noise

Equation (8) describes how the observed redshift drift after a time interval Δt_0 differs from the drift expected purely due to the evolution of the expansion rate. The peculiar velocities of each absorber lens are contained within the summation $\mathcal{S}$; even in the absence of peculiar acceleration, the velocities will affect the observed magnitude of the redshift drift by entering the expression as a fractional offset. The accelerations are contained within the summation $\mathcal{S}_{\Delta t}$, which stands alone as an absolute error. Below, we estimate the order of magnitude of these effects.

Each term in $\mathcal{S}$ and $\mathcal{S}_{\Delta t}$ may be positive or negative depending on the direction of motion and acceleration of the Ly α cloud. Thus the uncertainties are in the form of noise, rather than a systematic bias; they are the result of a random walk. We would expect the noise to increase with absorption line redshift as there would be more lenses in the foreground.

The denser absorbers, such as LLSs, may produce deflection angles of at most a few arcseconds, and at speeds of $v \sim 300 \text{ km s}^{-1}$ would dominate $\mathcal{S}$ with $\Delta\lambda/\lambda \sim 10^{-8}$, although we are unlikely to find more than one or two in a single quasar spectrum. Even if we coupled the high column density absorbers with the expected large number of low column density absorbers (~ 1000 per line

of sight), and coincidentally aligned the directions of motions, we would still find $|\mathcal{S}| \ll 1$. If $\mathcal{S}$ was non-negligible, the net result would depend on how the lines are binned for the experiment. If the binned absorption lines come from one spectrum only, then $\mathcal{S}$ for each line will be highly correlated, and a strong bias will be present. However, if multiple spectra are observed for the same experiment, which is likely to be the case, the value of $\mathcal{S}$ for each spectrum would be independent, even for a particular redshift bin, and thus the effect would still average out.

L08 also plotted the narrow distribution of peculiar acceleration of the IGM within hydrodynamic simulations, establishing a tight peak at $a_{\text{pec}} \sim 10^{-11} \text{ cm s}^{-2}$. Again, coupling this acceleration with a large deflection angle of a few arcseconds, a large number density (~ 1000) and a time interval of 20 yr between epochs of observation, we find $\mathcal{S}_{\Delta t} \sim 10^{-15}$. Though this appears as an absolute error in the measurement, the redshift drift expected is $\Delta z \sim 10^{-10}$, many orders of magnitude larger. Again, binning within a single spectrum can amplify this offset, but the use of multiple spectra renders the offset small again, leaving the signal unaffected.

6 CONCLUDING REMARKS

The redshift drift experiment has been recognized as a direct test of the cosmic expansion, and carries with it the potential to independently constrain cosmological parameters and distinguish between evolving dark-energy models. The proposed next generation of ELTs and ultrastable high-resolution spectrographs make the detection of the cosmic signal possible in the near future. However, as the signal is weak, it is necessary to account for all possible systematic biases and sources of noise. One source of noise that has, until now, remained unexplored is the variation in the gravitational field due to the transverse motion of Ly α absorbers. In this paper, we have examined this effect and characterized and quantified the noise introduced to the signal. To summarize, we find that the peculiar velocities and accelerations of the Ly α clouds both enter the expression relating the observed redshift to the cosmic signal as noise terms; the former introduces a fractional offset, the latter, absolute. The evolution of the gravitational potential wells in the line of sight due to transverse motion of the Ly α absorbers will not, in fact, introduce significant noise or bias into the detection of the cosmic signal.

We consider now whether this is the final word on evolving gravitational fields. Ly α absorbers were the most obvious source of foreground interference; they, by definition, intersect the line of sight. Galactic haloes at impact parameters of 1–20 kpc are thought to produce the rarer absorption effects, such as LLSs and DLAs; these have been the extreme cases studied in this work. Estimates of the frequency and redshift distribution of these objects are hindered by reddening, which can dim quasars and exclude them from flux-limited samples, although the magnification bias can counteract this effect. Both effects can be expected, particularly if the absorbers are massive galaxy haloes at low impact parameters. The resulting evolution of the gravitational field within the filaments and sheets associated with the low-density Ly α forest is difficult to study analytically. The filaments feed into more massive structures, such as galaxy clusters, with absorption taking place at impact parameters of ~ 1 Mpc. The transverse peculiar motion of such objects will have a similar ‘moving-lens’ effect, the rarity of which has not been assessed. Their presence may not be noticed in the quasar spectra, though perhaps in direct imaging (at least at low redshift). Further studies using dark matter simulations will probe

the expected motions of all mass scales in the vicinity of the line of sight to distant quasars, allowing the determination of the full influence of the BG effect in future redshift drift experiments; this will form the basis of future contributions.

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Chapter 6

Conclusions and Future Work

What are we going to do tomorrow night?

- Pinky (Pinky and the Brain)

Gravitational lensing is a double-edged sword. The statistics of distinctive lensing features such as multiple imaging, distortion and magnification of background sources can form the basis of cosmological predictions. Conversely, the expected fluxes and redshifts of sources can be affected by gravitational lensing, leading to biased results if the effect is not accounted for. Investigations of a selection of these predictions have been the focus of this thesis. In the following chapter, I summarize the achievements of the present work and suggest future directions to extend these results.

Cluster Lensing: The triaxial shape of galaxy clusters generates an orientation dependence on the cross section for multiple-imaging. For the first investigation of this work, I ran a set of **GADGET-2** simulations and identified seven galaxy cluster-sized haloes with masses of $\sim 10^{15} M_{\odot}$. These were analysed and a large number of orientations sampled in order to construct the multiple imaging cross section σ_{MI} and its probability distribution. The mass in the vicinity of the cluster includes filaments that feed galaxy clusters and merging objects, both of which are known to be preferentially aligned with the cluster major axis. Surprisingly, I found that they do not appear to affect the efficiency for multiple imaging. Nor do they enhance the orientation dependence for this cross section. While a single source redshift was used throughout this work, a self-consistent comparison with upcoming surveys may be made with mock observations that mimic the same imaging constraints, such as magnitude cutoff, limited spatial resolution and noise. These could include a realistic source redshift distribution and extended sources (for comparison to galaxy sources, rather than quasars). Mandelbaum et al. (2009) have found minor changes to strong lensing selection biases when they invoke a magnification bias; future studies of σ_{MI} could impose a source luminosity distribution and flux limit, or perhaps simply impose a magnification cut-off for the second brightest image and a minimum separation to ensure the images are resolved. By modelling the emission in various wavebands from these clusters, one may even mimic the selection biases associated with X-ray or SZ selected clusters. Substructure supposedly has negligible effect on the strong lensing efficiency. Yet microlensing studies have long since recognised that substructure can change the flux ratios of multiple images. Although generally studied on galaxy scales, the same applies to clusters. It stands to reason that while the positions and ‘theoretical’ existence of multiple images is unaffected by cluster substructure, the chance that the images are brighter than the observational magnitude limit is. The simulations used throughout this work have modelled the evolution of dark matter only. On cluster scales, there is much uncertainty regarding the effect of gas physics on the structure. Recent studies have demonstrated that baryon cooling can increase the frequency of multiple-imaging by up to

30% depending on the image separation and source redshift (Wambsganss et al., 2008). A comparison between the results presented here and those that use a sample of clusters simulated in the environment where stellar and AGN feedback and cooling mechanisms play a role will make clear how strong lensing cluster surveys should be interpreted in terms of these parameters. In addition, self-interacting dark matter models predict the largest deviation from the standard cold dark matter model in regions of high density, such as cluster cores (Spergel & Steinhardt, 2000). Simulations that include a self-interaction term produce clusters with less concentrated cores and smaller optical depths to multiple imaging, albeit larger magnifications (Wyithe et al., 2001). Thus, strong lensing observations are well-placed to constrain the self-interaction cross section, but the effect of triaxiality has not yet been studied. I therefore propose to conduct further investigations into multiple imaging statistics with simulated clusters that include the footprint of gas physics and self-interacting dark matter.

Raytracing: Modelling the magnification probability distribution of background sources due to weak gravitational lensing relies on the application of ray-tracing methods. In the second study, this work has taken the first steps towards making a direct comparison between the predictions of the multiple lens-plane ray-bundle method (RBM) and one that does not apply the thin-lens approximation; instead, the null geodesic is integrated. The behaviour of the ray-bundles in the vicinity of a Schwarzschild lens demonstrated that the method can reproduce large magnifications given sufficient spatial resolution. A range of tests were able to pin down the numerical parameters that play a critical role in the predicted statistics. This method presents a computational challenge as the three-dimensional FFTs prove memory-expensive, but is justified by the need to quantify the multiple lens-plane approximation. The weak lensing statistics predicted by ray-tracing through simulated cosmological mass distributions found significant differences compared to the results from the original RBM approach. One of the reasons is that structure below 50 kpc/h scales could not be represented on the Fourier-grid used. Small-scale structure can be probed in future studies given either longer computational times or memory. However, the effect of shot-noise in the standard RBM method is identified as another possible reason for the discrepancy. To clarify the dominant factor, a multiple lens-plane ray-bundle method that applies a two-dimensional Fourier-method at each lens-plane is proposed as the next step towards the direct comparison. It has recently been demonstrated that flexion can be modelled using a ray-bundle method (Lasky & Fluke; in prep.). This presents an exciting future application of the method to higher-order statistics, but surely it is even more necessary to justify the use of the multiple lens-plane formalism.

Redshift Drift Experiment: Finally, I have drawn attention to a source of potential biases that gravitational lensing may impose on a future experiment that has strong implications for direct testing of cosmological expansion. The redshift-drift experiment will be an ambitious project which will use state-of-the-art high-resolution spectroscopy from CODEX proposed for the E-ELT. A significant detection will require a few thousand hours of observing time over a few decades. That said, the potential to directly detect expansion in a redshift range rarely probed, namely $2 < z < 5$, makes it a unique opportunity to constrain cosmological models. Specifically several dark energy models will be constrained well and the Copernican principle will be tested. It has its limitations, however. For example, $f(R)$ gravity models cannot be constrained well given the redshift range probed. Given the weak nature of the detection, it's vital to be aware of all possible biases. In the present work, I have looked at gravitational lensing by the foreground absorbers (particularly those that arise in the vicinity of massive galaxies) and the danger they pose. Massive galaxies that move across the line of sight, with significant peculiar velocity would change the wavelength of light from background spectral lines (Birkinshaw & Gull, 1983), effectively changing the observed redshift. And any peculiar acceleration will lead to shifts that could change over the decade of observation. The effect is small, but so is the cosmic signal. By

constructing an analytic expression that includes the effect of gravitational lensing, I found that the combined effect of foreground absorbers is not linear; however, a rough estimate for the error is orders of magnitude below the expected cosmological signal. These results have informed the more recent works of Jain & Jhingan (2010), which judge how well cosmological parameters will be constrained by the redshift drift experiment. Even though gravitational lensing will not overcome the cosmic signal, it is not clear if it will affect the ability to distinguish between dark energy models, since the spectral resolution required would be even greater. In future studies, one may also make more self-consistent predictions of the gravitational lensing effect using hydrodynamical simulations by coupling mock quasar spectra with the foreground mass that affects each spectra individually, whether that mass is associated with an absorption line or not. On a more practical note, the recent release of the US Decadal Survey has put the large telescopes (GMT and TMT) behind other projects, which means the E-ELT bears a greater weight of responsibility for opening the door to a range of studies that require the advanced capabilities of large telescopes, such as the redshift drift experiment.

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Appendix A

Cosmography

The distance to observed sources is an essential parameter in almost every astronomical dataset. Yet the definition of distance is laden with common misinterpretations and often, observable distance measures are not simply related to fundamental distance measures. Here, I describe the relationship between these measures, distinguished by their method of measurement. Throughout this appendix chapter, c and H_0 denote the speed of light and the Hubble constant, respectively. I stress that the expressions presented here are only correct for the case of a flat universe, i.e. $\Omega_k = 0$; for further details, the reader is directed to Hogg (1999).

Redshift is not a distance measure, *per se*, but it is a fundamental observable that is widely used to determine the distance to an astrophysical object; the distance measures are thus presented in terms of their dependence with redshift. If a single redshift is the only dependent variable, it is understood that the term represents the distance from the observer (at $z = 0$) to that redshift. If two redshifts, z_1 and z_2 , are presented as the dependent variables, the distance is being measured from z_1 to z_2 . A fundamental cosmological distance scale is the Hubble distance, D_H , given by

$$D_H = \frac{c}{H_0}. \quad (\text{A.1})$$

The proper distance, D_P , is the distance measured by a ruler at the time of observation:

$$D_P(z) = \frac{c}{1+z} \int_0^z \frac{dz'}{H(z')}. \quad (\text{A.2})$$

The line-of-sight co-moving distance, D_C , is the proper distance divided by the ratio of scale factors at the object and at the observer, so

$$\begin{aligned} D_C(z) &= (1+z)D_P(z) \\ &= c \int_0^z \frac{dz'}{H(z')}. \end{aligned} \quad (\text{A.3})$$

The co-moving distance is a fundamental distance measure seeing as it is additive, i.e.

$$D_C(z_1, z_2) + D_C(z_2, z_3) = D_C(z_1, z_3). \quad (\text{A.4})$$

The angular diameter distance $D_A(z)$ to an object at redshift z is the ratio of the object's transverse length, l , to its angular size θ , for small angular sizes:

$$D_A(z) = \frac{l}{\theta}. \quad (\text{A.5})$$

Since gravitational lensing can alter the angular size subtended by a source, angular diameter distances are commonly used in describing lensing scale lengths. The angular diameter distance

between the lens and source plays an important role, and so the relationship for an arbitrary ‘observer’ redshift is required. The angular diameter distance between two objects at redshifts z_1 and z_2 in a flat universe is given by

$$D_A(z_1, z_2) = \frac{D_H}{1 + z_2} \int_0^z \frac{dz'}{[\Omega_{M,0}(1 + z')^3 + \Omega_{\Lambda,0}]^{1/2}} \quad (\text{A.6})$$

$$= \frac{1}{1 + z_2} [D_C(z_2) - D_C(z_1)]. \quad (\text{A.7})$$

It is clear from Eqn. A.6 that unlike co-moving distances, the angular diameter distances are not additive, i.e.

$$D_A(z_1, z_2) + D_A(z_2, z_3) \neq D_A(z_1, z_3). \quad (\text{A.8})$$

For completeness, I also include the luminosity distance for comparison. This distance measure relates the intrinsic flux, F , of a luminous object to its apparent luminosity, L , and is inferred by:

$$D_L(z) = \sqrt{\frac{L}{4\pi F}}. \quad (\text{A.9})$$

The luminosity distance is particularly relevant to studies of ‘standard candles’, such as SNIa, for which intrinsic fluxes are known or can be determined from the shapes of their light curves. Given redshift measurements, the fundamental cosmological distances are then derived from the Etherington principle (Etherington, 1933):

$$D_L(z) = (1 + z)D_C(z) = (1 + z)^2 D_A(z). \quad (\text{A.10})$$

Appendix B

Analytic Solutions for Lensing

B.1 Schwarzschild Lens

A single peak of infinite density is one of the simplest lens models, and is often used to model a spherically symmetric lens for idealised analytical studies. The gravitational potential field around a so-called Schwarzschild lens of mass M is given by

$$\phi(\mathbf{r}) = -\frac{GM}{c^2 r}. \quad (\text{B.1})$$

The relevant (angular) scale length for such a lens is the Einstein radius, given by

$$\theta_{\text{Ein}} = \sqrt{\frac{D_{ds}}{D_d D_s}} \frac{\sqrt{4GM}}{c}. \quad (\text{B.2})$$

At the lens plane, this corresponds to a physical scale length given by:

$$r_{\text{Ein}} = \theta_{\text{Ein}} D_d. \quad (\text{B.3})$$

The gravitational lensing quantities of interest are found by combining the derivatives of this potential, as described in Sec. 1.4. They are defined in terms of a dimensionless radial distance, $x = r/r_{\text{Ein}}$. All lines of sight probe regions outside the lens, so the convergence is zero. The solution for the shear for images lensed by a Schwarzschild lens, γ_{Schw} , is simply

$$\gamma_{\text{Schw}}(x) = x. \quad (\text{B.4})$$

The analytic solution for the magnification for an image at any location is well known and is given by

$$\mu_{\text{Schw}}(x) = \left(1 - \frac{1}{x^4}\right)^{-1}. \quad (\text{B.5})$$

B.2 NFW lens

The NFW profile, described in Sec. 1.3.3, can be considered a thin lens for cases where the observer, lens and source are separated by large angular diameter distances. The following analytic solutions, B.6-B.13, are therefore derived from projected surface densities. A dimensionless radial distance, $x = r/r_s$, has been adopted. Here, the quantities ρ_c , r_s , δ_c and Σ_{crit} are those defined in Eqns. 1.6, 1.9, 1.10 and 1.20. Following Wright & Brainerd (2000) and Coe (2010), the analytic solution for the radial dependence of the convergence is given by

$$\kappa_{\text{NFW}}(x) = \frac{r_s \delta_c \rho_c}{\Sigma_{\text{crit}}} K(x), \quad (\text{B.6})$$

where $K(x)$ is given by

$$K(x) = \begin{cases} \frac{2}{(x^2-1)} \left[1 - \frac{2}{\sqrt{1-x^2}} \tanh^{-1} \sqrt{\frac{1-x}{1+x}} \right] & \text{if } x < 1 \\ \frac{2}{3} & \text{if } x = 1 \\ \frac{2}{(x^2-1)} \left[1 - \frac{2}{\sqrt{x^2-1}} \tan^{-1} \sqrt{\frac{x-1}{1+x}} \right] & \text{if } x > 1 \end{cases} \quad (\text{B.7})$$

By integrating Eqn. B.6 over the area within r , one finds the mass within a *cylinder* of radius r :

$$M_{\text{NFW,cyl}}(x) = 4\pi r_s^3 \delta_c \rho_c C(x), \quad (\text{B.8})$$

where $C(x)$ is given by

$$C(x) = \ln \frac{x}{2} + \begin{cases} \frac{1}{\sqrt{1-x^2}} \cosh^{-1} \frac{1}{x} & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ \frac{1}{\sqrt{x^2-1}} \cos^{-1} \frac{1}{x} & \text{if } x > 1 \end{cases} \quad (\text{B.9})$$

The radial dependence of shear is thus given by

$$\gamma_{\text{NFW}}(x) = \frac{r_s \delta_c \rho_c}{\Sigma_{\text{crit}}} G(x), \quad (\text{B.10})$$

where $G(x)$ is given by

$$G(x) = \begin{cases} g_{<}(x) & \text{if } x < 1 \\ \left[\frac{10}{3} + 4 \ln \left(\frac{1}{2} \right) \right] & \text{if } x = 1 \\ g_{>}(x) & \text{if } x > 1 \end{cases} \quad (\text{B.11})$$

The functions $g_{<}(x)$ and $g_{>}(x)$ are given by

$$g_{<}(x) = \frac{8 \tanh^{-1} \sqrt{(1-x)/(1+x)}}{x^2 \sqrt{1-x^2}} + \frac{4}{x^2} \ln \left(\frac{x}{2} \right) - \frac{2}{(x^2-1)} + \frac{4 \tanh^{-1} \sqrt{(1-x)/(1+x)}}{(x^2-1) \sqrt{1-x^2}} \quad (\text{B.12})$$

and

$$g_{>}(x) = \frac{8 \tan^{-1} \sqrt{(x-1)/(1+x)}}{x^2 \sqrt{x^2-1}} + \frac{4}{x^2} \ln \left(\frac{x}{2} \right) - \frac{2}{(x^2-1)} + \frac{4 \tan^{-1} \sqrt{(x-1)/(1+x)}}{(x^2-1)^{3/2}}. \quad (\text{B.13})$$

Combining these with Eqns. B.6 and B.10 with Eqns. 1.19 and 1.24 allows one to determine the analytic value of the magnification, μ , of an image at a given projected distance from the centre of the NFW lens.