Problem set #6

Problem 1 *Virial theorem and Jeans mass*

In the lectures, the Jeans mass was derived using a perturbation ansatz. Here we will derive the Jeans mass using the *virial theorem*. For a self-gravitating spherically-symmetric ideal-gas cloud in (hydrostatic) pressure equilibrium,

$$E_{\mathrm{kin}} = -\frac{1}{2} E_{\mathrm{pot}}.$$

For a cloud to be able to contract, what is the condition that has to be statisfied about the energies involved? What minimum mass must a cloud of given mean density and mean temperature therefore at least have to contract? Compare your result to the one from the lectures and discuss possible differences.

Hint: Assume a monatomic gas. The gravitational potential energy is:

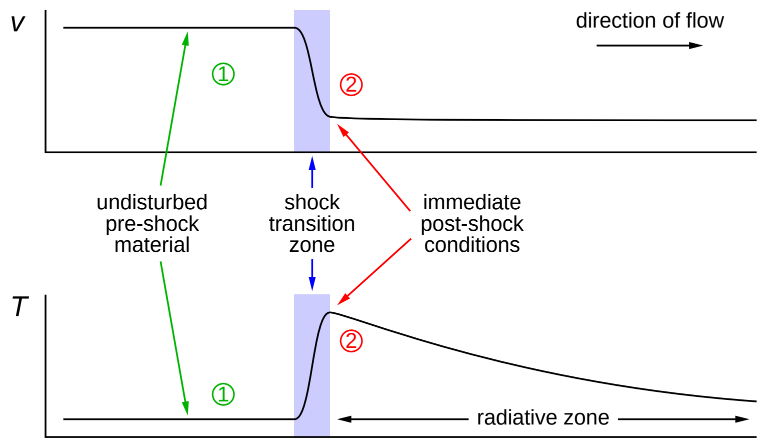
$$E_{\mathrm{pot}} = -\int_0^R G \frac{m(r)\rho 4\pi r^2}{r} \, \mathrm{d}r,$$

where $m(r)$ is the total mass inside of radius r and $\rho 4\pi r^2 \mathrm{d}r$ the mass of the collapsing shell.

Problem 2 *Simple one-dimensional shock model*

Shock waves play an important role in Astrophysics. They occur on all scales in the ISM, for example in supernova blast waves, cloud-cloud collisions, H II regions, interstellar bubbles (very fast winds), accretion and outflow phenomena (jets), spiral shocks in the galactic disk, galaxy mergers, and even at the edges of galaxy clusters.

From a kinetic viewpoint, shocks convert much of the ordered motion of the pre-shock gas into random, thermal motion. If the shock speed is sufficiently high, the hot post-shock gas radiates and heats more distant gas both upstream and downstream from the front itself.



The conservation of mass flow, momentum flow, and energy flow through the shock zone are described by

$$\rho_1 v_1 = \rho_2 v_2 \tag{1}$$

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2 \tag{2}$$

$$v_1 \left(\frac{\rho_1 v_1^2}{2} + \rho_1 U_1 + P_1 \right) = v_2 \left(\frac{\rho_2 v_2^2}{2} + \rho_2 U_2 + P_2 \right) \tag{3}$$

where ρ, v, and P are the mass density, speed, and pressure, and the subscripts "1" and "2" refer to pre- and post-shock gas, respectively. The internal energy per unit mass is

$$U = \frac{P}{(\gamma - 1)\rho},$$

where γ is the adiabatic index of the gas.

The preceding set of equations can be rewritten as the *Rankine-Hugoniot jump conditions*:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \tag{4}$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - \gamma + 1}{\gamma + 1} \tag{5}$$

$$\frac{T_2}{T_1} = \frac{P_2}{P_1} \frac{v_2}{v_1} \tag{6}$$

with $M_1 = v_1/c_1$ the *Mach number* and $c_1 = \sqrt{\gamma P_1/\rho_1}$ the sound speed in the pre-shock region.

(a) For fast shocks ($M_1 \gg 1$), how much denser is the post-shock gas than the pre-shock material? Assume a non-relativistic, monatomic gas ($\gamma = \frac{5}{3}$).

(b) For a shock with $v_1 = 1000$ km/s (a typical value for the speed of the blast wave around a supernova remnant) and an ambient ISM temperature of 1000 K, what is the Mach number (Hint: $c_s = \sqrt{\frac{\gamma k_B T}{\mu m_H}}$)? What is the temperature of the shocked material (Hint: you can simplify the Rankine-Hugoniot conditions assuming $M_1 \gg 1$)?

Problem 3 *Kelvin-Helmholtz timescale*

Pre-main-sequence stars evolve by contracting and radiating away excess gravitational binding energy. Assuming that this process is quasi-static, i.e., the star is always (almost) in hydrostatic equilibrium all the time, and considering that the luminosity is simply the rate of change of the total energy of the star,

$$L = -\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathrm{tot}} = -\frac{\mathrm{d}}{\mathrm{d}t} (E_{\mathrm{kin}}^{\mathrm{therm}} + E_{\mathrm{pot}}^{\mathrm{grav}})$$

(energy conservation), how long would the star have taken to reach its current size if it was shining at a given constant luminosity fueled by gravitational contraction alone? This is known as the Kelvin-Helmholtz time. How long is this time for a star of mass $M = 1M_\odot$, which now has a radius of $R = 1R_\odot$ and which has been shining with a constant luminosity of $L = 1L_\odot$?

Problem 4 *Gas and dust in galaxies*

Consider our Galaxy.

(a) The total mass of gas in the Galaxy is $\sim 5 \cdot 10^9\, M_\odot$. Assume that it is uniformly distributed in a disk of radius 15 kpc and thickness 200 pc, and that it is all HI (atomic hydrogen). What is the average number density of H atoms?

(b) The gas is mixed with dust. What is the extinction between the Sun and the Galactic centre? Assume a distance to the Galactic centre of 10 kpc and the gas to be uniformly distributed. Use the following gas to extinction ratio:

$$N(\mathrm{HI})/A_V = 1.8 \cdot 10^{21} \mathrm{atoms\, cm^{-2}\, mag^{-1}}\,.$$

(Hint: The total extinction A is given by the ratio of the hydrogen column density over $N(\mathrm{HI})/A_V$)

(c) If the dust is in spherical particles of radius $d = 500$ nm and material density 2 g cm^{-3}, what is the gas-to-dust ratio by mass in the Galaxy? Here, assume that the dust absorption cross section is the same at all wavelengths and is the geometric cross section, $\sigma = \pi d^2$. Hint:

$$A = \int dl n_{dust} \sigma$$

(d) Now assume that some of the gas is in molecular clouds of radius 10 pc and mean density n(H_2) = 300 cm^{-3}. What is the extinction through one molecular cloud?

Problem 5 *Key questions*

Summarize the most important facts about the interstellar medium (ISM). These are some questions you should cover:

(a) What are the approximate temperatures of the cold, warm and hot ISM? Where in galaxies do you find these phases?

(b) What is the Jeans criterion?

(c) Explain the importance of dust for the formation of stars.

(d) What are *forbidden* transitions and where / why can we observe them in the ISM?

(e) How did the different variants of the hot ISM become hot?

(f) How can you measure the temperature of an H II region?