Universitäts-Sternwarte
München

Radiative Transfer, Stellar Atmospheres and Winds

Five lectures (four hours each) within the IMPRS advanced course

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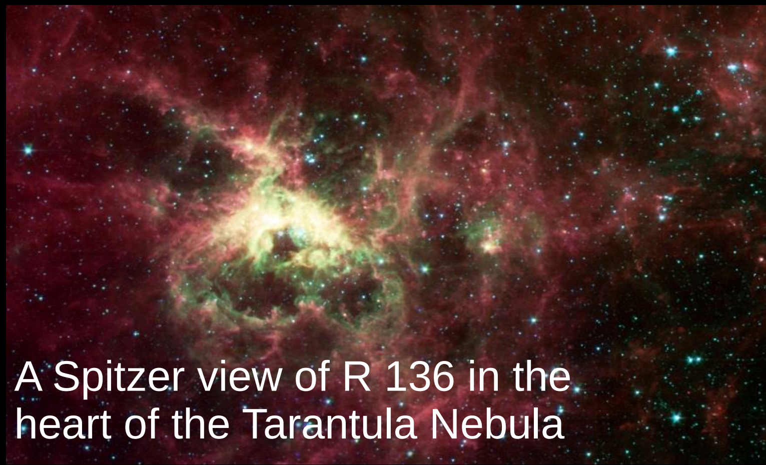
NLTE Group

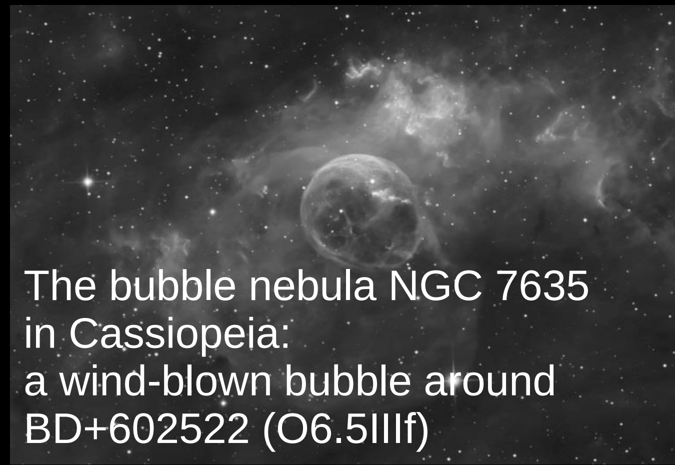
Radiative transfer, stellar atmospheres and winds

The Tarantula Nebula
(30 Dor) in the LMC

The wind-blown bubble
N44F in the LMC



A Spitzer view of R 136 in the
heart of the Tarantula Nebula



The bubble nebula NGC 7635
in Cassiopeia:
a wind-blown bubble around
BD+602522 (O6.5IIIf)

Content

1. *Prelude:* What are stars good for? A brief tour through present hot topics (not complete, personally biased)
2. *Quantitative spectroscopy:* the astrophysical tool to measure stellar and interstellar properties
3. *The radiation field:* specific and mean intensity, radiative flux and pressure, Planck function
4. *Coupling with matter:* opacity, emissivity and the equation of radiative transfer (incl. angular moments)
5. *Radiative transfer:* simple solutions, spectral lines and limb darkening
6. *Stellar atmospheres:* basic assumptions, hydrostatic, radiative and local thermodynamic equilibrium, temperature stratification and convection
7. *Microscopic theory*
   1. *Line transitions:* Einstein-coefficients, line-broadening and curve of growth, continuous processes and scattering
   2. *Ionization and excitation in LTE:* Saha- and Boltzmann-equation
   3. *Non-LTE:* motivation and introduction

Intermezzo: Stellar Atmospheres in practice -- A tour de modeling and analysis of stellar atmospheres throughout the HRD

A first application: The D4000 break in early-type galaxies

8. *Stellar winds* – overview, pressure and radiation driven winds
9. *Quantitative spectroscopy:* stellar/atmospheric parameters and how to determine them, for the exemplary case of hot stars

Literature

- Carroll, B.W., Ostlie, D.A., “An Introduction to Modern Astrophysics”, 2nd edition, Pearson International Edition, San Francisco, 2007, Chap. 3,5,8,9
- Mihalas, D., "Stellar atmospheres", 2nd edition, Freeman & Co., San Francisco, 1978
- Hubeny, I., & Mihalas, D., "Stellar atmospheres", 3rd edition, in press
- Unsöld, A., "Physik der Sternatmosphären", 2nd edition, Springer Verlag, Heidelberg, 1968
- Shu, F.H., "The physics of astrophysics, Volume I: radiation", University science books, Mill Valley, 1991
- Rybicki, G.B., Lightman, A., "Radiative Processes in Astrophysics", New York, Wiley, 1979
- Osterbrock, D.E., "Astrophysics of Gaseous Nebulae and Active Galactic Nuclei", University science books, Mill Valley, 1989
- Mihalas, D., Weibel Mihalas, B., “Foundations of Radiation Hydrodynamics”, Oxford University Press, New York, 1984
- Cercignani, C., "The Boltzmann Equation and Its Applications", Appl. Math. Sciences 67, Springer, 1987
- Kudritzki, R.-P., Hummer, D.G., "Quantitative spectroscopy of hot stars", Annual Review of Astronomy and Astrophysics, Vol. 28, p. 303, 1990
- Sobolev, V.V., “Moving envelopes of stars”, Cambridge: Harvard University Press, 1960
- Kudritzki, R.-P., Puls, J., "Winds from hot stars", Annual Review of Astronomy and Astrophysics, Vol. 38, p. 613, 2000
- Puls, J., Vink, J.S., Najarro, F., "Mass loss from hot massive stars", Astronomy & Astrophysics Review Vol. 16, ISSUE 3, p. 209, Springer, 2008

- cosmology, galaxies, dark energy, dark matter, …

What are stars good for?

- … and who cares for radiative transfer and stellar atmospheres?

- Remember
  - galaxies consist of stars (and gas, dust)
  - most of the (visible) light originates from stars
  - astronomical experiments are (mostly) observations of light: have to understand how it is created and transported

The cosmic circuit of matter

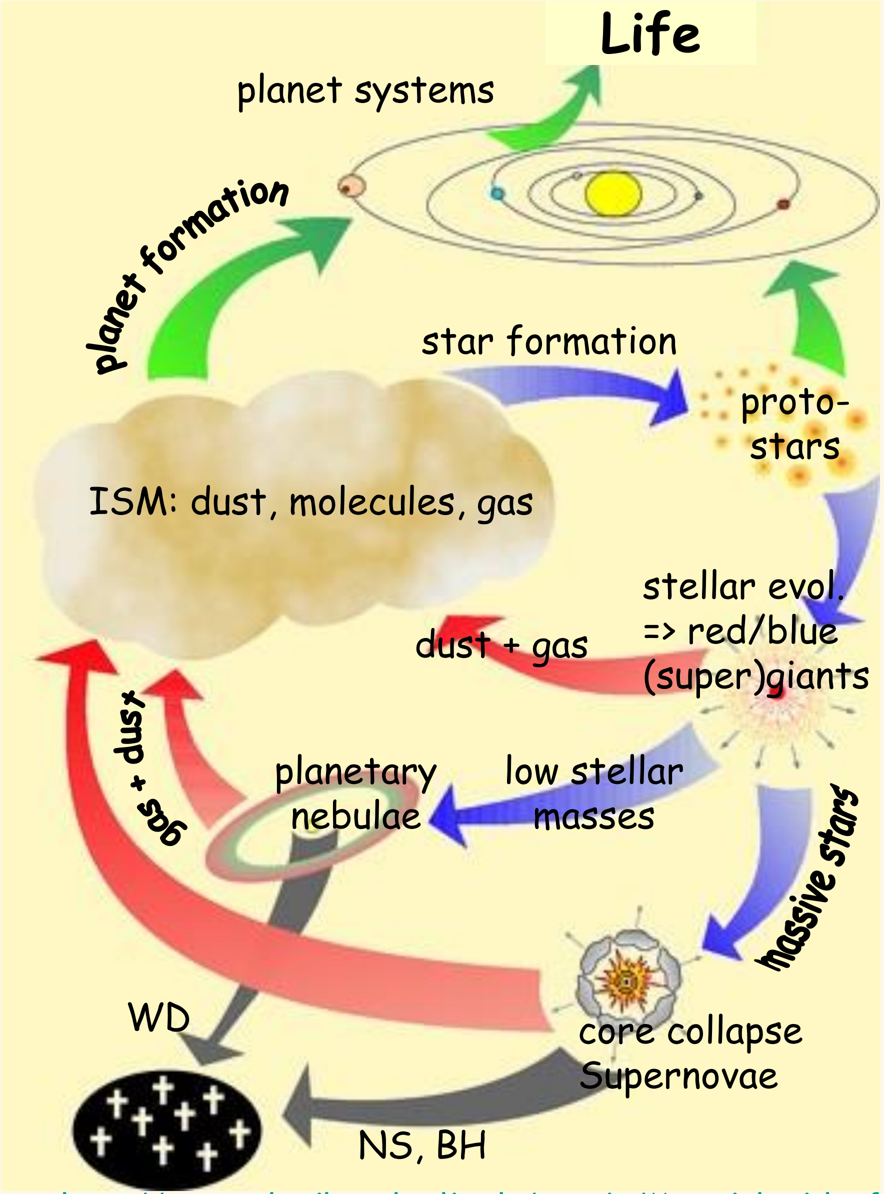
What are stars good for?

- Us!
- (whether this is really good, is another question...)

Joni Mitchell - Woodstock (1970!)

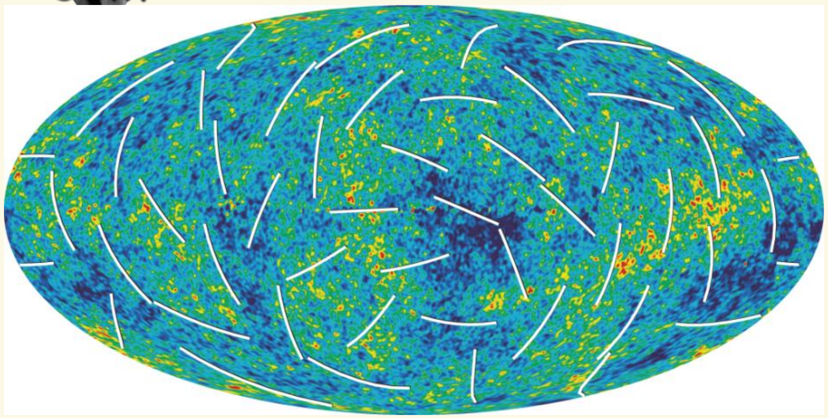
"... We are stardust

Billion year old carbon..."



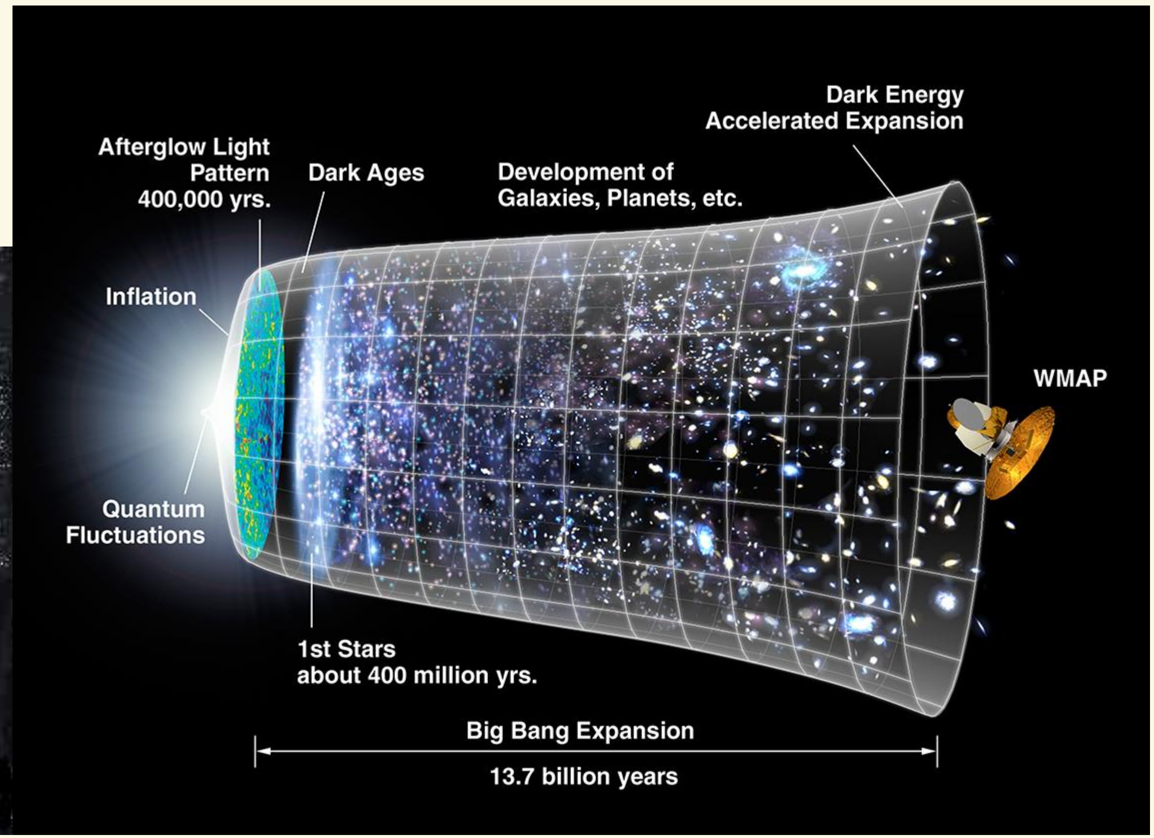
adapted from http://astro.physik.tu-berlin.de/~sonja/Materiekreislauf/index.html

First stars and reionization



credit: NASA/WMAP Science Team

WMAP = Wilkinson Microwave Anisotropy Probe
color coding: ΔT range $\pm$ 200 μK, $\Delta T/T \sim$ few 10^{-5}
=> "anisotropy" of last scattering surface (before recomb.)
white bars: polarization vector
=> CMB photons scattered at electrons (reionzed gas)

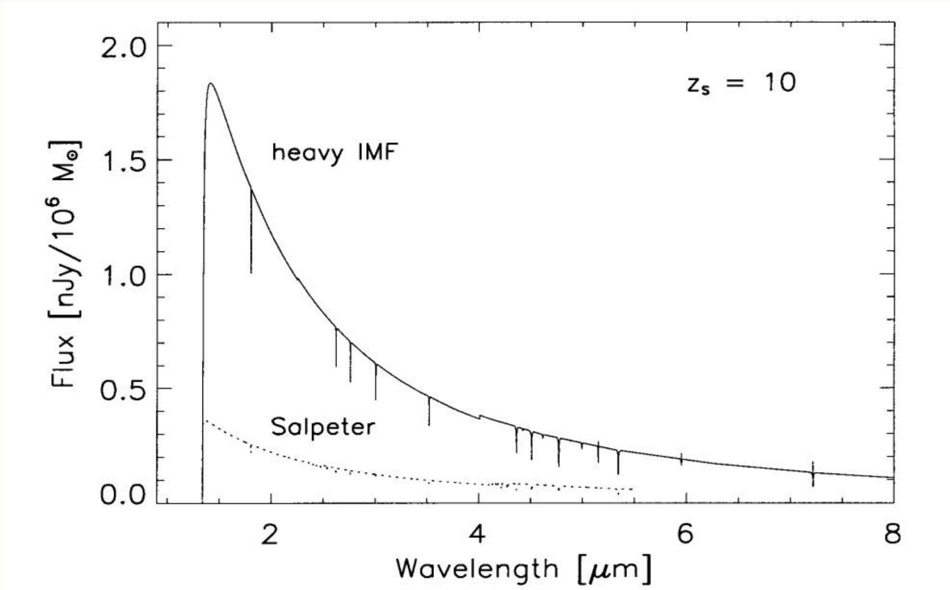


The first stars …

- begin of reionization: $z \approx 11$ (from WMAP, polarization, assuming instantaneous reionization) to $z \approx 15 \dots 30$ (modeling)
- complete (for hydrogen) at $z \sim 6.0$
- quasars alone not capable to reionize universe at that high redshift ($z > 6$) since rapid decline in space density for $z > 3$ (Madau et al.1999, ApJ 514, Fan et al. 2006, ARA&A 44)

Bromm et al. (2001, ApJ 552)

- (almost) metal free: Pop III
- very massive stars (VMS) with
  $1000\ M_\odot > M > 100\ M_\odot$
- hotter ($\approx 10^5$ K), more compact
- $L \propto M$, spectrum almost BB,
- large H/He ionizing fluxes:
  10^{48} (10^{47}) H (He) ionizing photons
  per second *and solar mass*
- assume that primordial IMF
  favours formation of VMS



IF heavy IMF,
then capable to reionize universe
(at least in a first step, cf. Cen 2003, ApJ 591)

see also

Abel et al. 2000, ApJ 540; Bromm et al. 2002, ApJ 564; Furnaletto & Loeb 2005, ApJ 634; Wise & Abel 2008, ApJ 684; Johnson et al. 2008, Proc IAU Symp 250 (review); Maio et al. 2009, A&A 503; Maio et al. 2010, MNRAS 407; Weber et al. 2013, A&A 555

… and many more publications

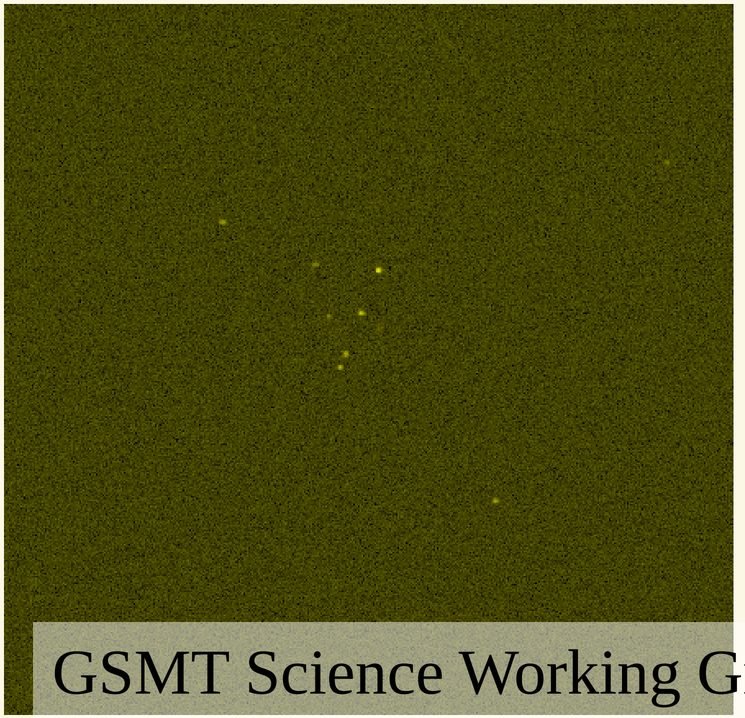
… might be observable in the NIR

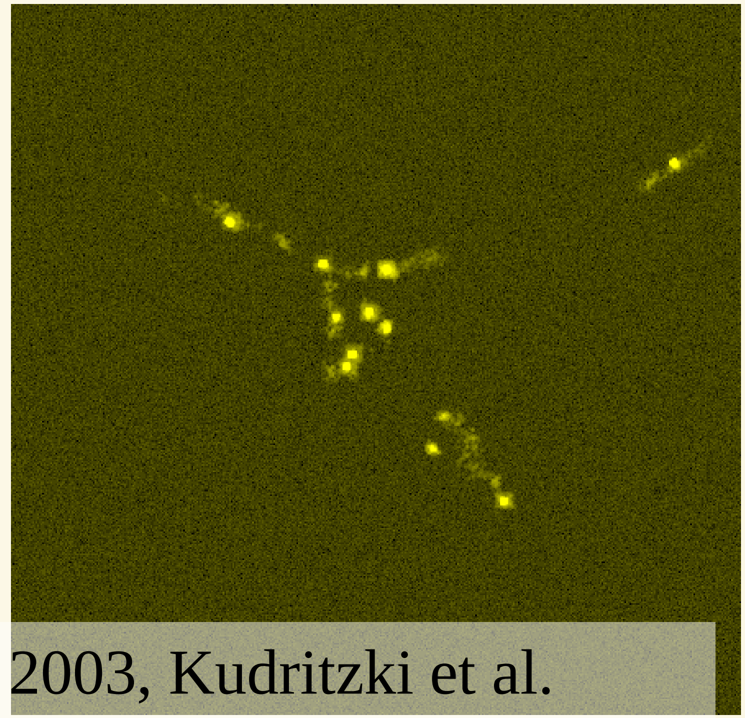
with a ≥ 30m telescope, e.g. via HeII λ1640 Å (strong ISM recomb. line)

Standard IMF

1 Mpc (comoving)

Heavy IMF, zero metallicity





GSMT Science Working Group Report, 2003, Kudritzki et al.

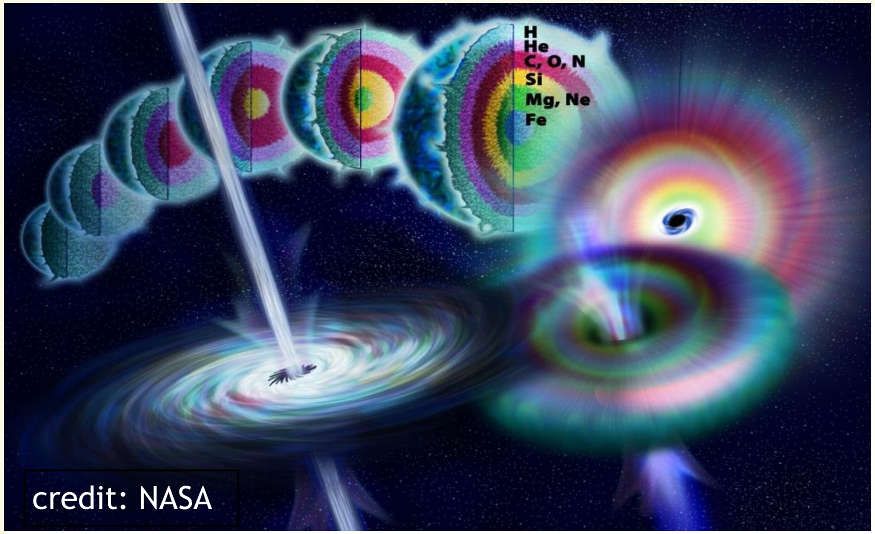
http://www.aura-nio.noao.edu/gsmt_swg/SWG_Report/SWG_Report_7.2.03.pdf

(Hydro-simulations by Davé, Katz, & Weinberg)

As observed through 30-meter telescope R=3000, 10^5 seconds (favourable conditions, see also Barton et al., 2004, ApJ 604, L1)

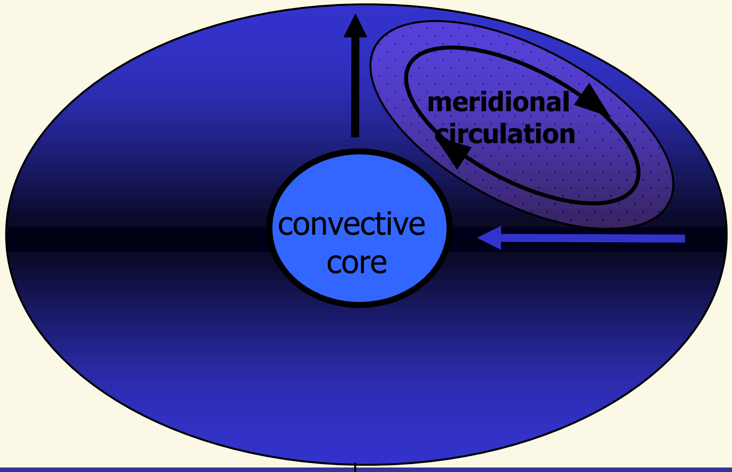
Long Gamma Ray Bursts

- long: >2s
- Collapsar: death of a massive star



Collapsar Scenario for Long GRB (Woosley 1993)

- massive core (enough to produce a BH)
- removal of hydrogen envelope
- rapidly rotating core (enough to produce an accretion disk)



requires **chemically homogeneous evolution** of **rapidly rotating** massive star

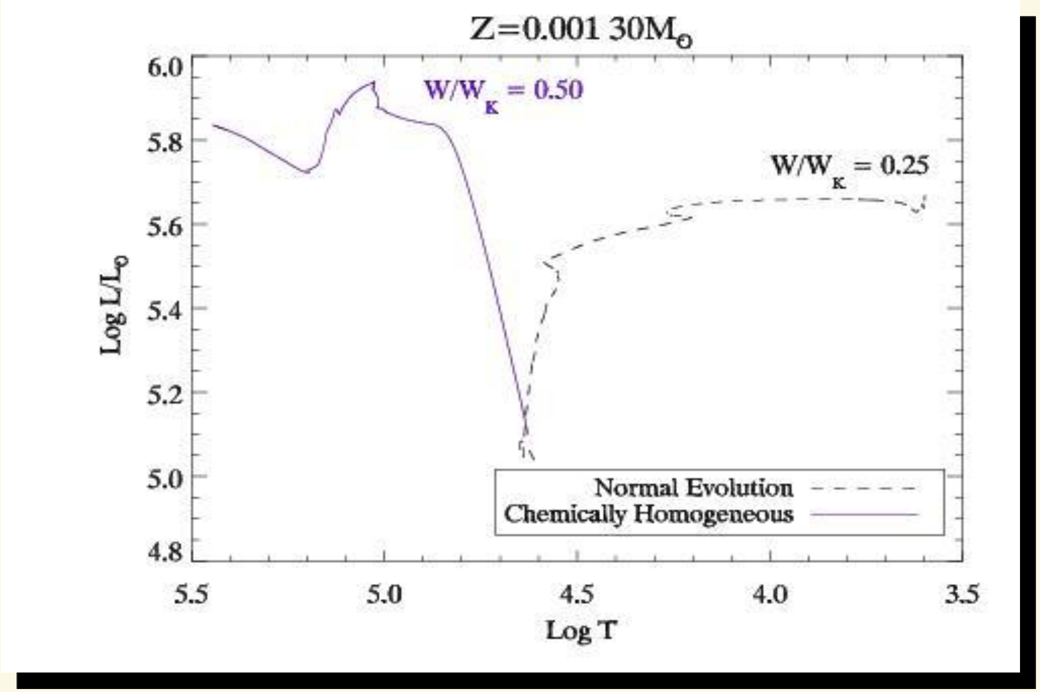
- pole hotter than equator (von Zeipel)
- rotational mixing due to meridional circulation (Eddington-Sweet)

- …if rotational mixing during main sequence *faster than* built-up of chemical gradients due to nuclear fusion (*Maeder 1987*)

- bluewards evolution directly towards Wolf-Rayet phase (no RSG phase).
  Due to meridional circulation, envelope and core are mixed ->
  no hydrogen envelope

- since no RSG phase, higher angular momentum in the core
  (*Yoon & Langer 2005*)

W/W_k: rotational frequency in units of critical one



massive stars as progenitors of high redshift GRBs:

✓ early work: Bromm & Loeb 2002, Ciardi & Loeb 2001, Kulkarni et al. 2000, Djorgovski et al. 2001, Lamb & Reichart 2000

✓ At low metallicity stars are expected to be rotating faster because of weaker stellar winds

Feedback

- massive stars determine energy (kinetic and radiation) and momentum budget of surrounding ISM
- massive stars have winds with different strengths, in dependence of evolution. status
- massive stars enrich environment with metals, via winds and SNe, determine chemo-dynamical evolution of Galaxies (exclusively before onset of SNe Ia)

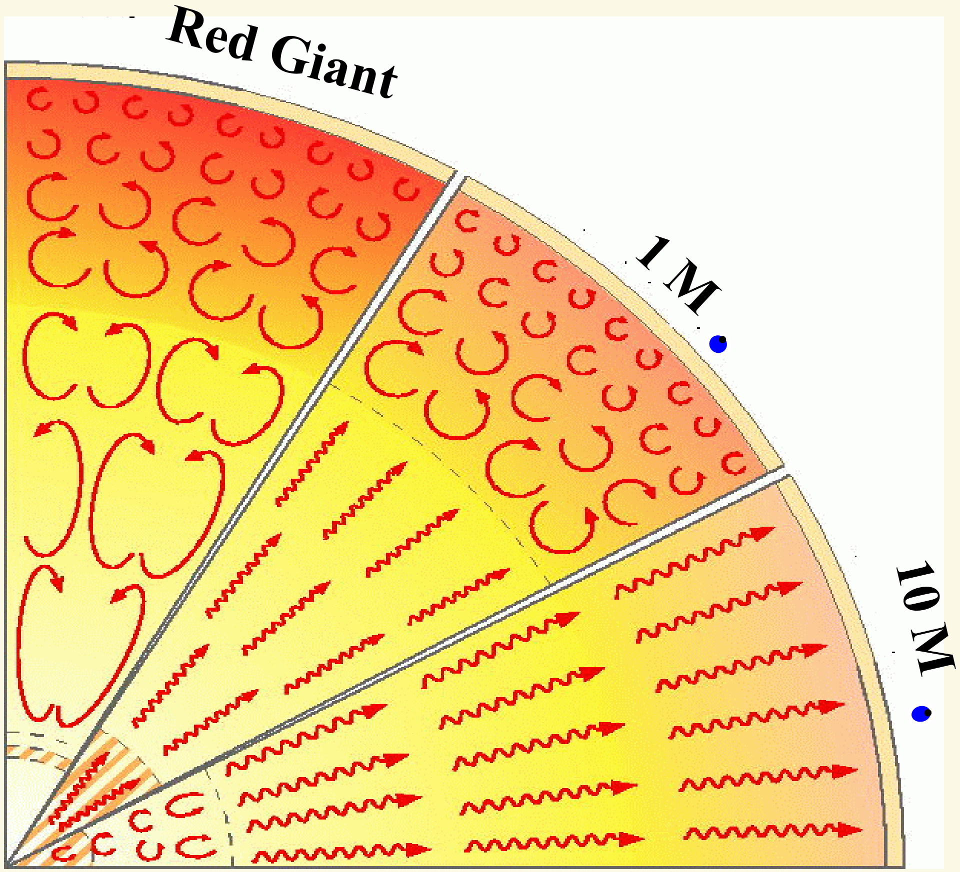
→"FEEDBACK"

Bubble Nebula (NGC 7635) in Cassiopeia

wind-blown bubble around BD+602522 (O6.5IIIf)

Asteroseismology – Revealing the internal structure

following slides adapted from a talk given by C. Aerts (Univ. Leuven)



Large variation of internal structure

We need to do much better than this !

What appliance can pierce through the outer layers of a star and test the conditions within?

Some open questions in stellar astropyhsics

- Effects of (differential) internal rotation?
- Effects of convective overshooting ? how does mixing occur inside the stars ?
- Preamble of supernova explosion ?
- Evolution of close binary systems ?

Asteroseismology will imply (and implies already now) major steps forward in answering these questions

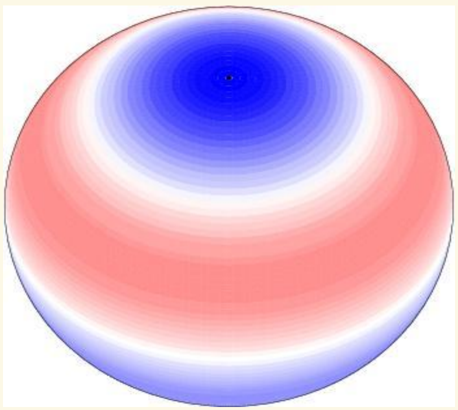
Stellar pulsations - non-radial modes

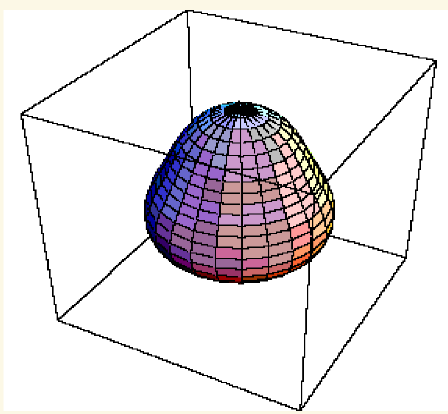
Blue: Moving towards Observer

Red: Moving away from Observer

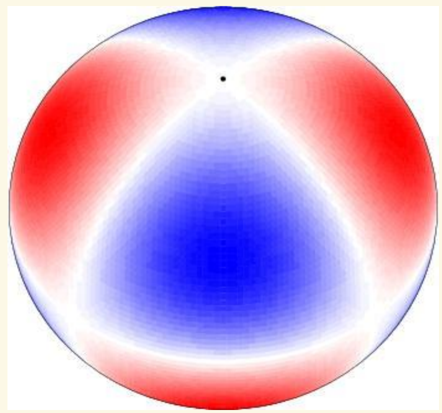
l: nonradial degree, **m**: azimuthal order

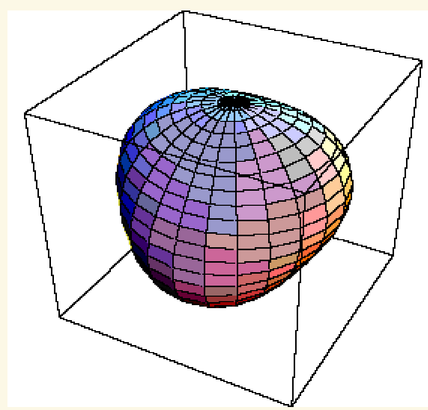
(l,m)=(3,0)
axisymmetric



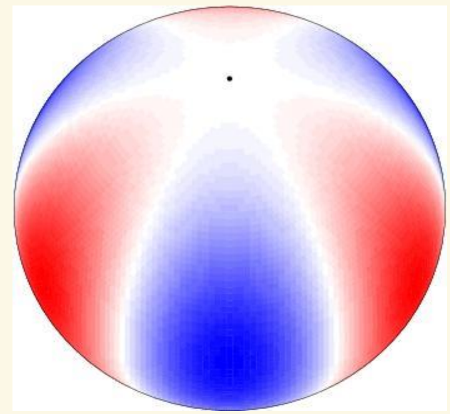


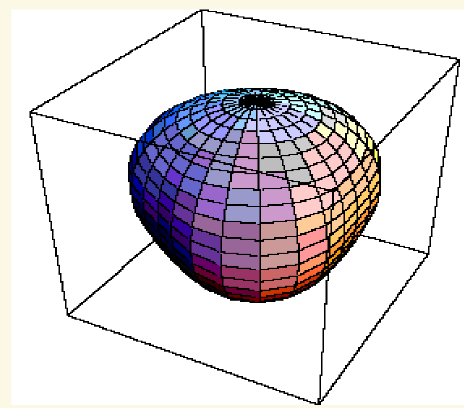
(l,m) = (3,2)
tesseral



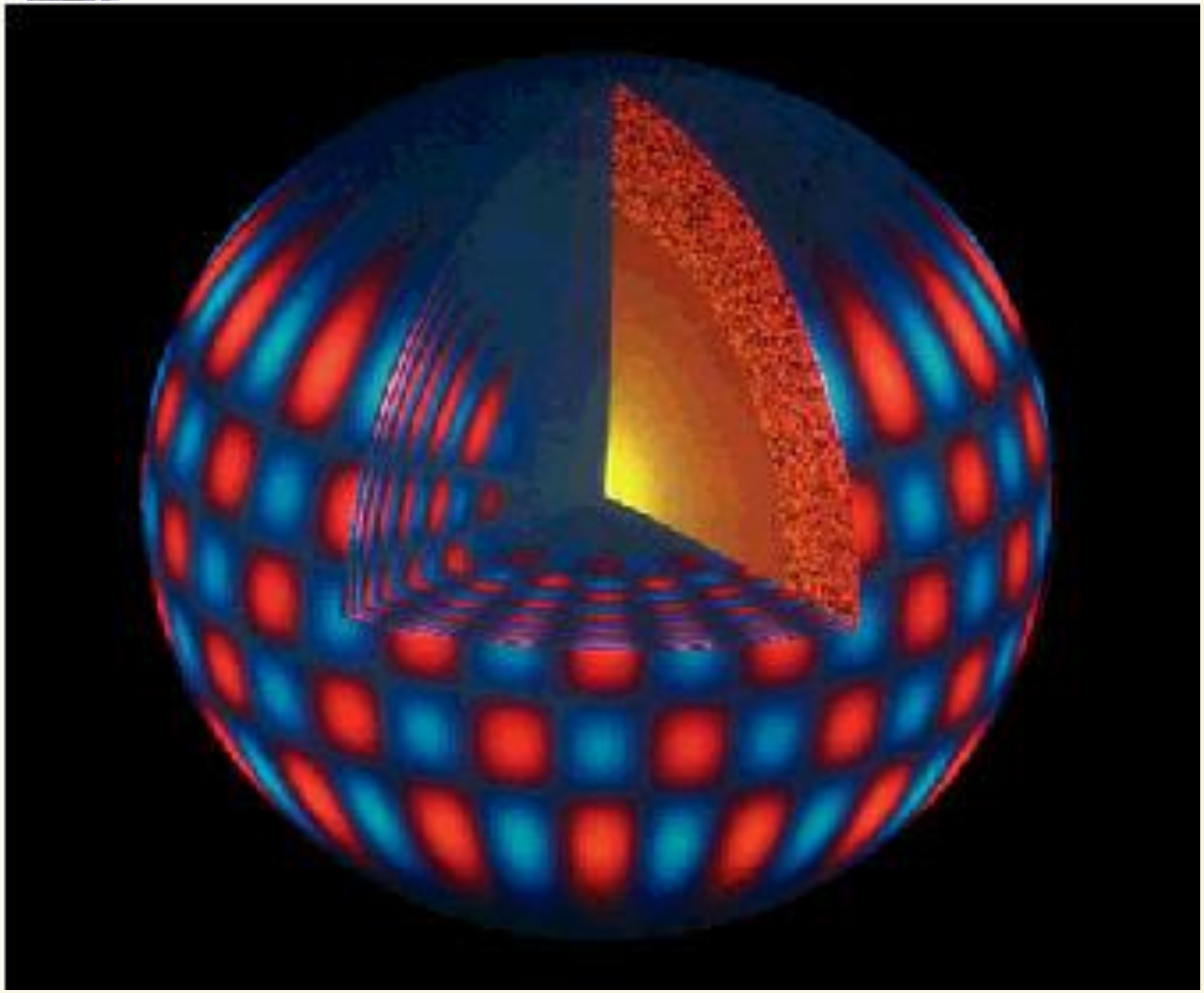


(l,m)=(3,3)
sectoral





Internal behaviour of the oscillations



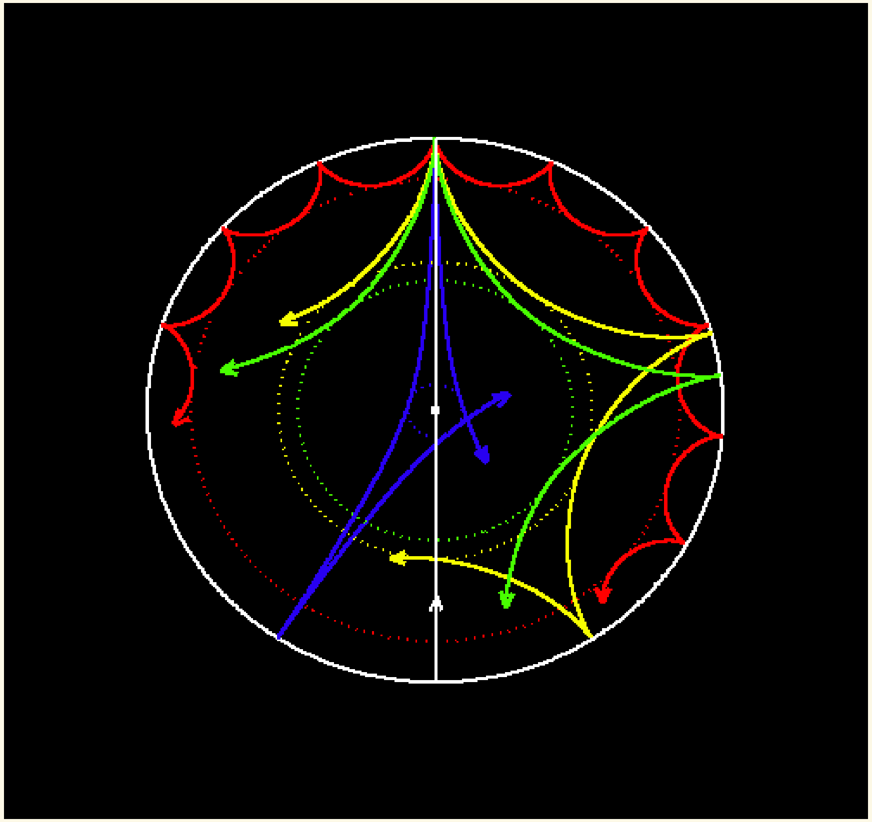
The oscillation pattern at the surface propagates in a continuous way towards the stellar centre.

Study of the surface patterns hence allows to characterize the oscillation throughout the star.

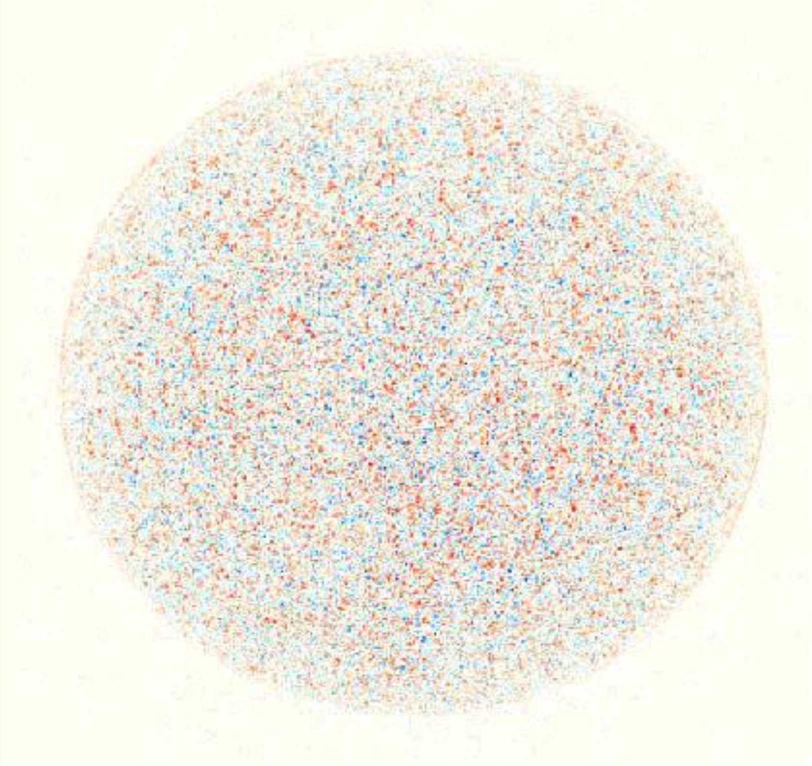
Inversion of the frequencies

The oscillations are standing sound waves that are reflected within a cavity

Different oscillations penetrate to different depths and hence probe different layers



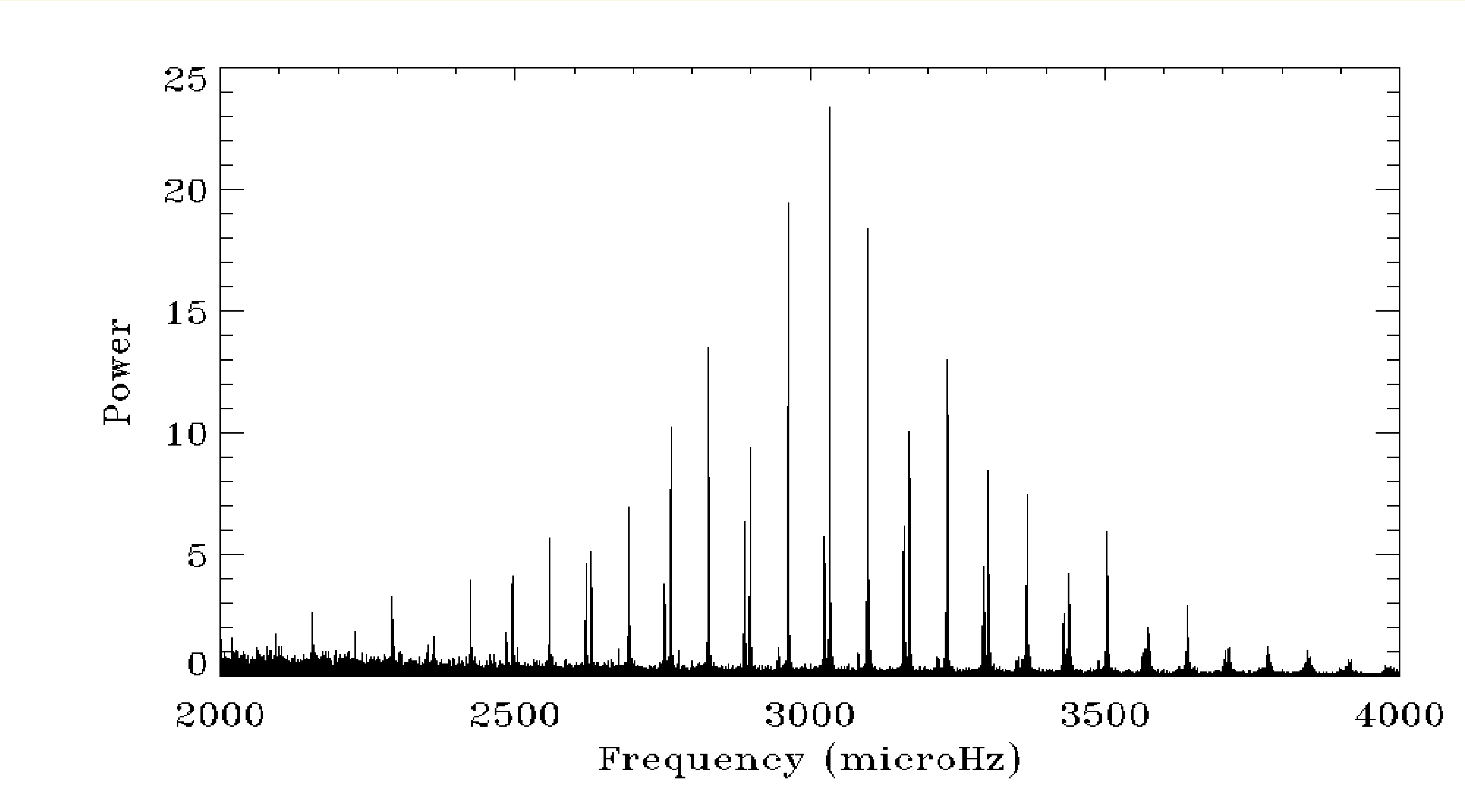
Doppler map of the Sun



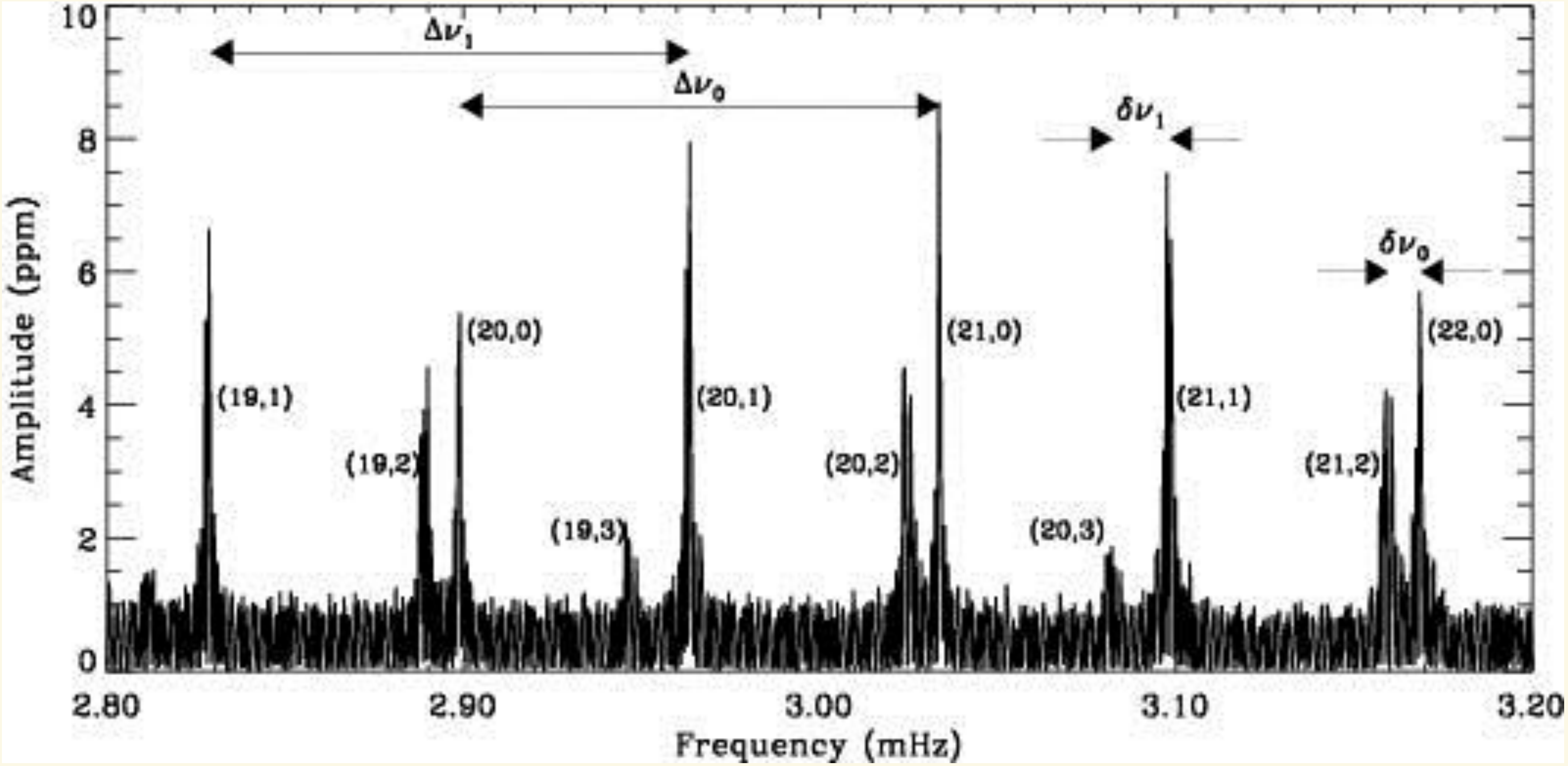
The Sun oscillates in thousands of non-radial modes with periods of ~5 minutes

The Dopplermap shows velocities of the order of some cm/s

Solar frequency spectrum from ESA/NASA satellite SoHO: systematics !

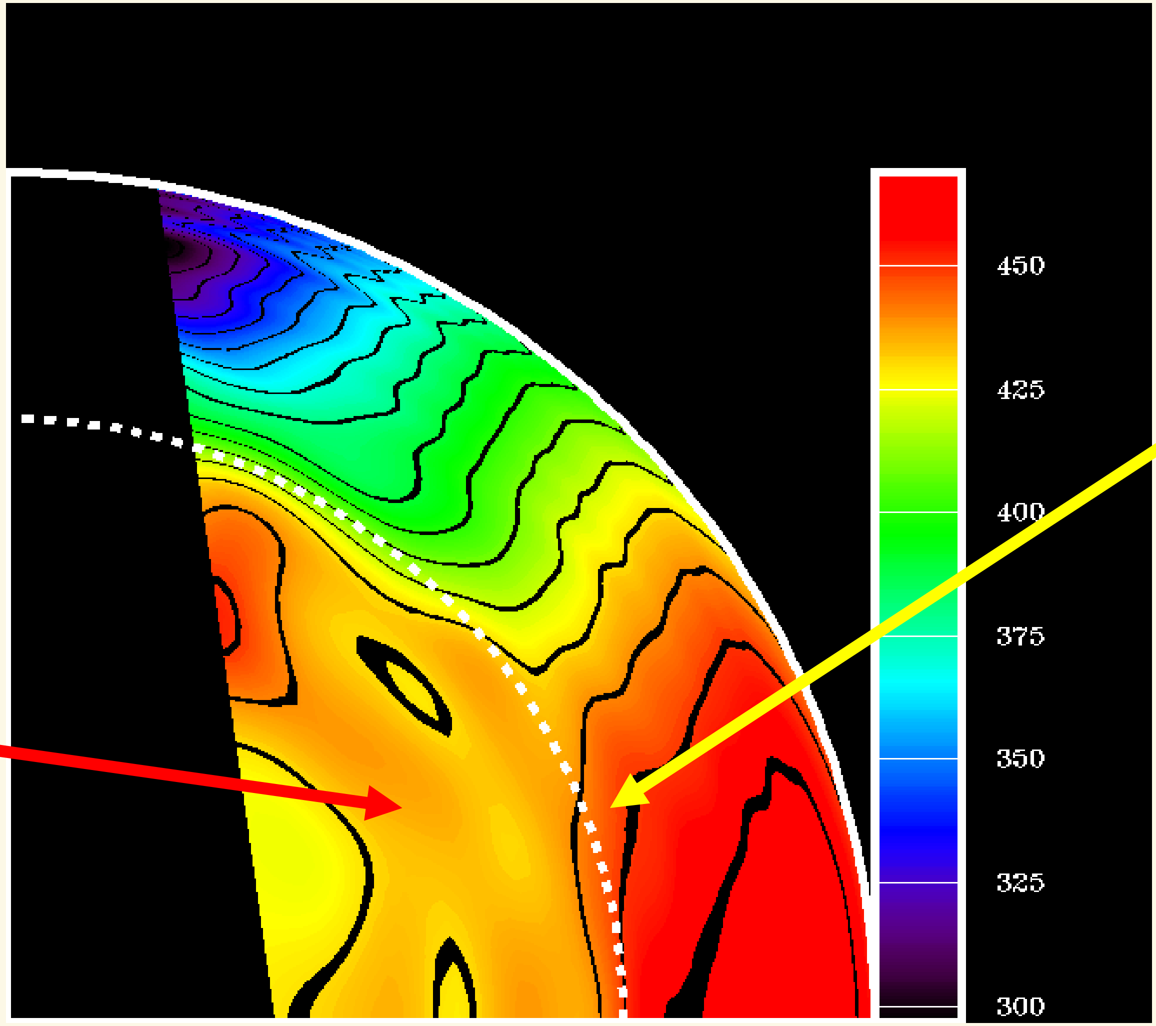


Frequency separations in the Sun



Result: internal sound speed and internal rotation could be determined very accurately by means of helioseismic data from SoHO

Internal rotation of the Sun



Solar interior has rigid rotation

Beginning of outer convection zone

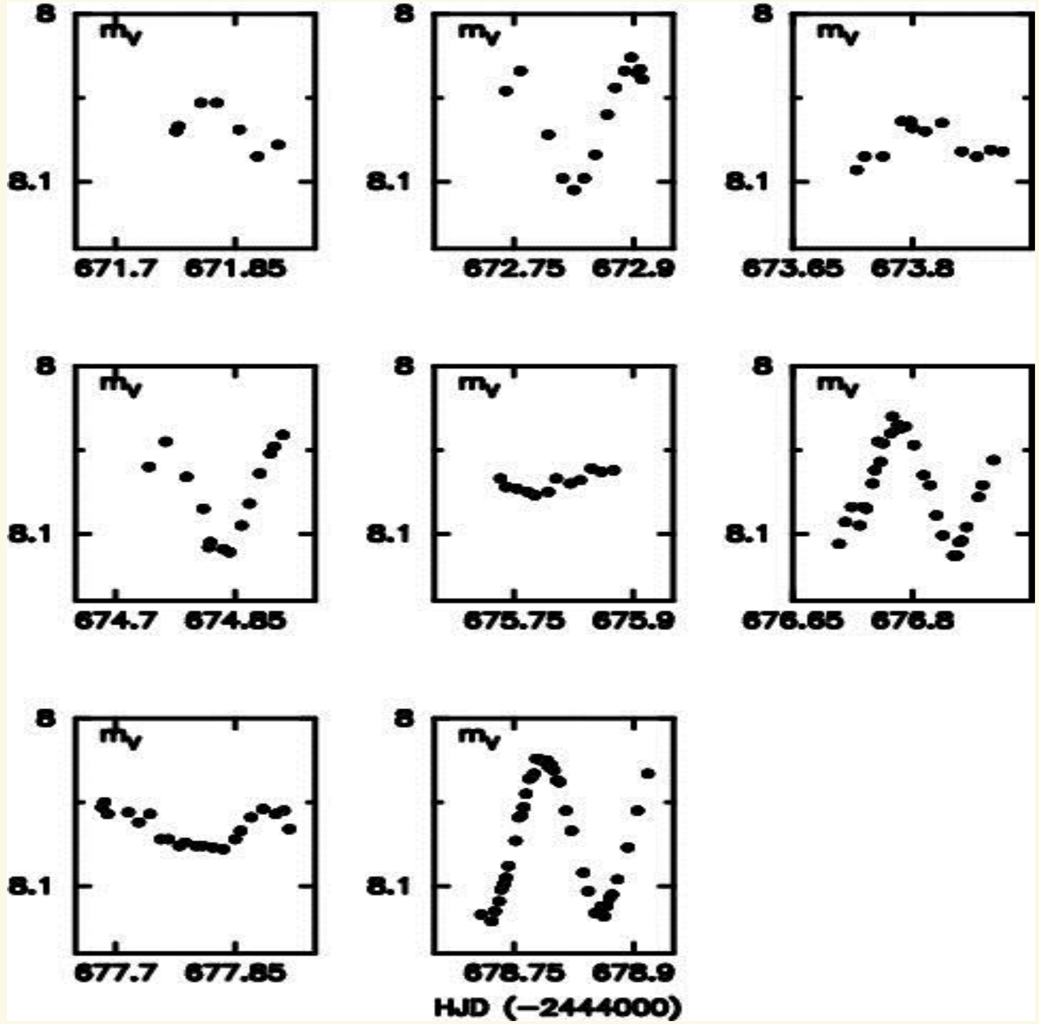
Strategy: forward modeling

- Derive set of frequencies & amplitudes from data
- Compute stellar models through error box in HR diagram, with parameters from stellar atmospheres
  + predict their unstable oscillation modes
- Identify observed modes
- Confrontation: does the input physics of the models explain the seismic data?
  - if yes: models OK
  - if no: great! Input physics is insufficient and must be upgraded to include additional effects or better descriptions until frequencies can be matched...

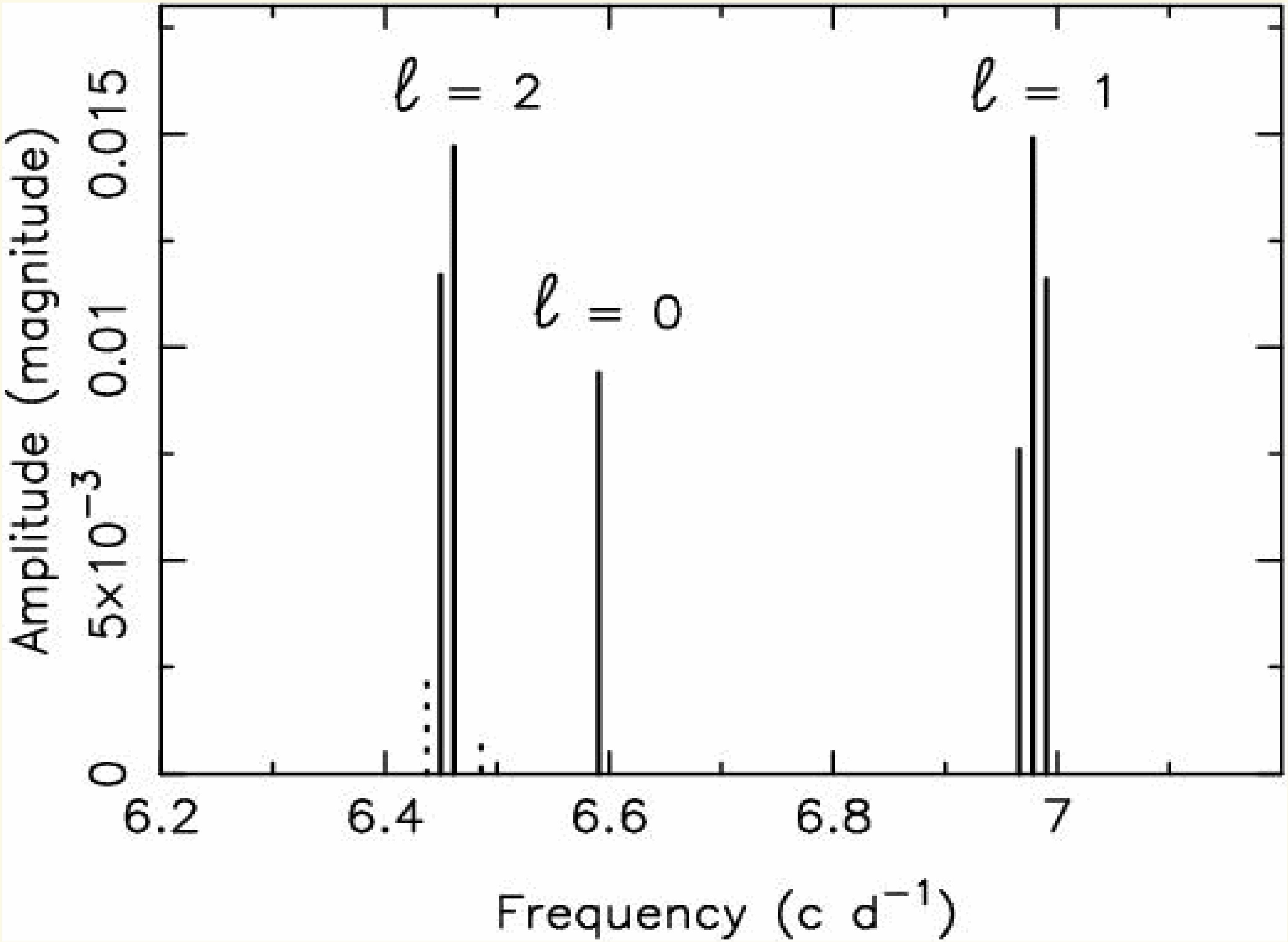
Massive star seismology: HD 129929

HD 129929: B3V
ß Ceph variable

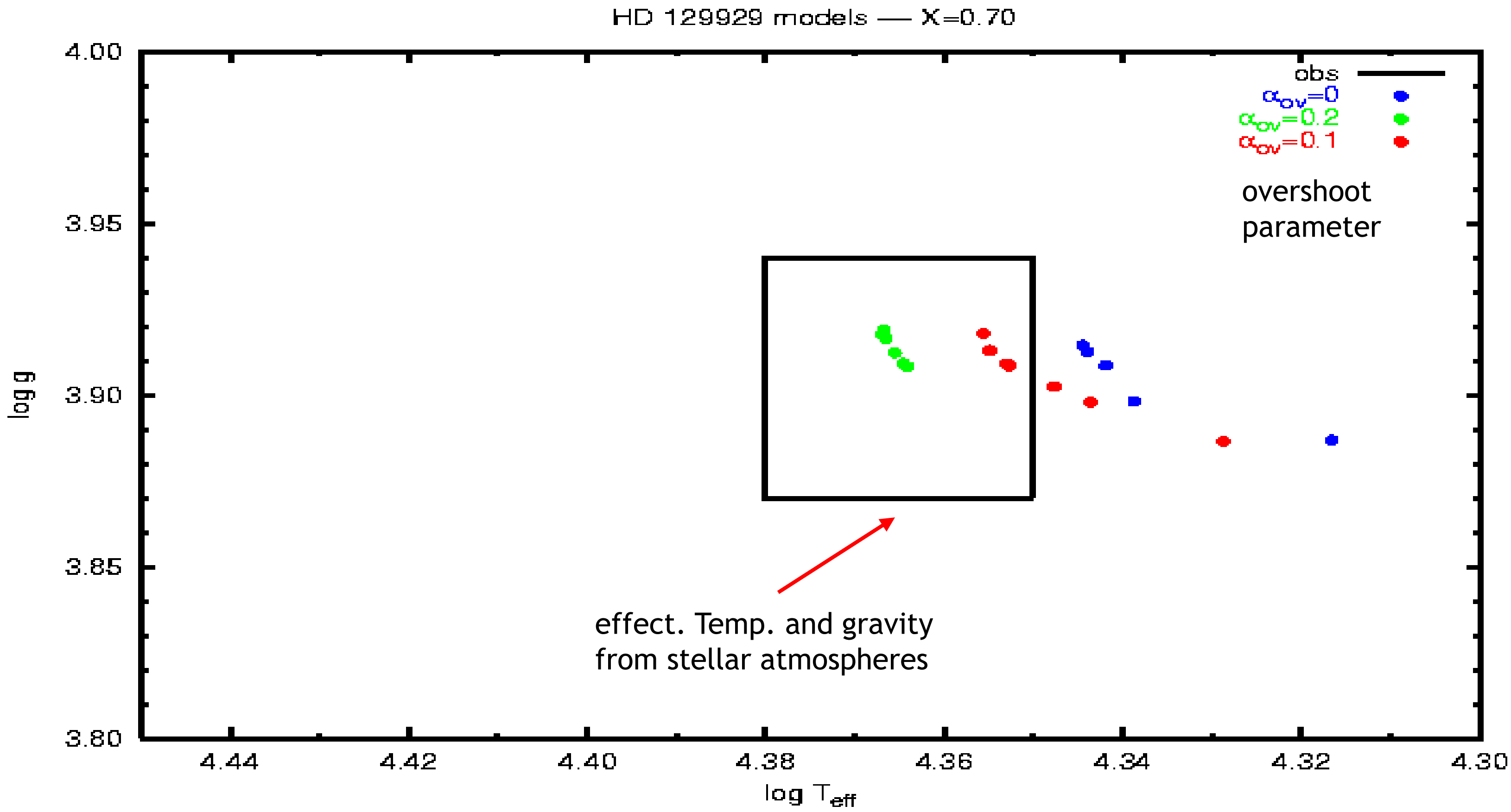
this star has been observed during 21 years (Waelkens & Aerts)



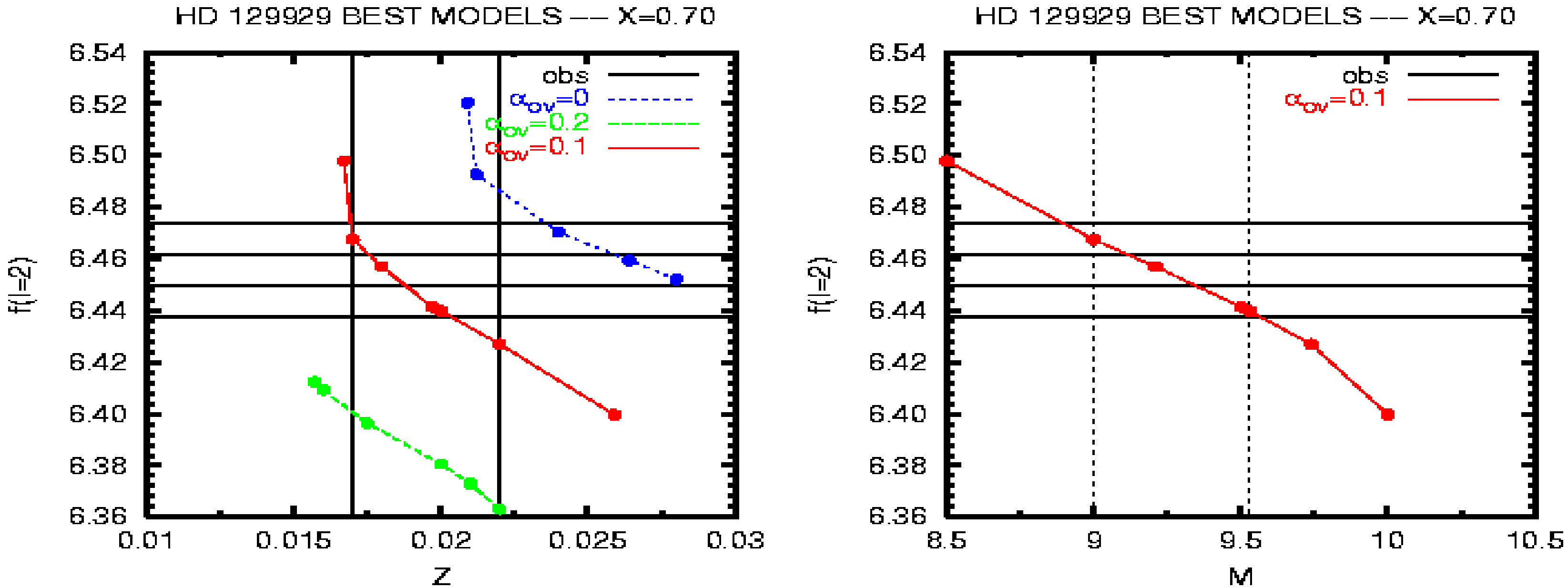
HD 129929: frequency spectrum



HD129929: Position in ‘HR’ diagram



Acceptable range in M,Z,overshoot



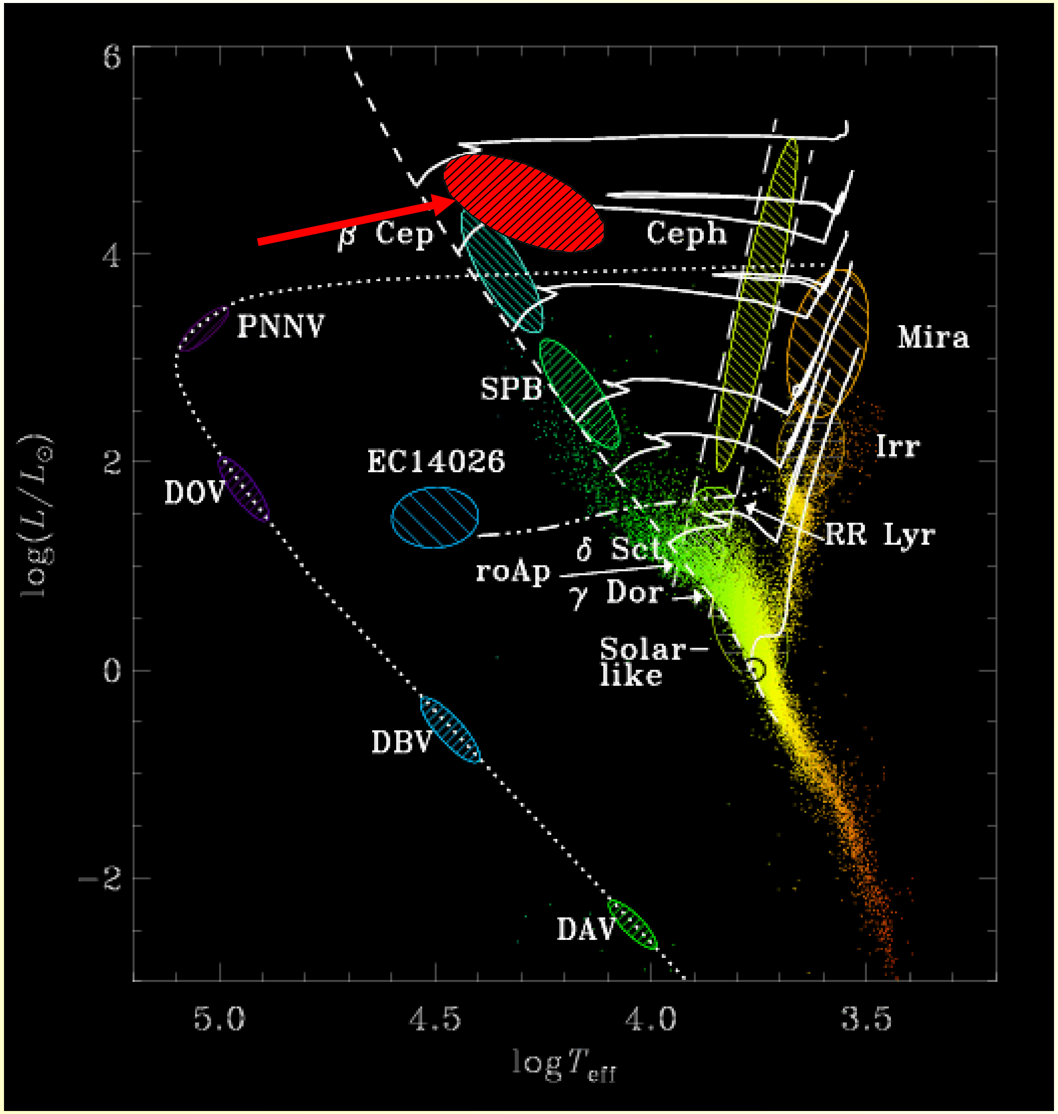
RESULT: very small core-overshooting needed to explain the frequencies of the star using standard opacities and standard solar mixture

FIRST star besides the Sun in which non-rigid rotation is proven:
core 4 times faster than surface

Table 1. Summary of the seismic results obtained so far. Reference numbers refer to (1): Aerts et al. (2006), (2): Pamyatnykh et al. (2004) and Ausseloos et al. (2004), (3): Mazumdar et al. 06), (4): Aerts et al. (2003, 2004) and Dupret et al. (2004), (5): Briquet et al. (2007).

Ref.	Star	Mass ($M_\odot$)	SpT	α_{ov} (H_p)	ΩR (km s^{-1})	$\Omega_{core}/\Omega_{env}$
(1)	HD 16582	10.2±0.2	B2IV	0.20±0.10	28(14?)	
(2)	HD 29248	9.2±0.6	B2III	0.10±0.05	6±2	~ 5
(3)	HD 44743	13.5±0.5	B1III	0.20±0.05	31±5	
(4)	HD 129929	9.4±0.1	B3V	0.10±0.05	2±1	3.6
(5)	HD 157056	8.2±0.3	B2IV	0.44±0.07	29±7	~ 1

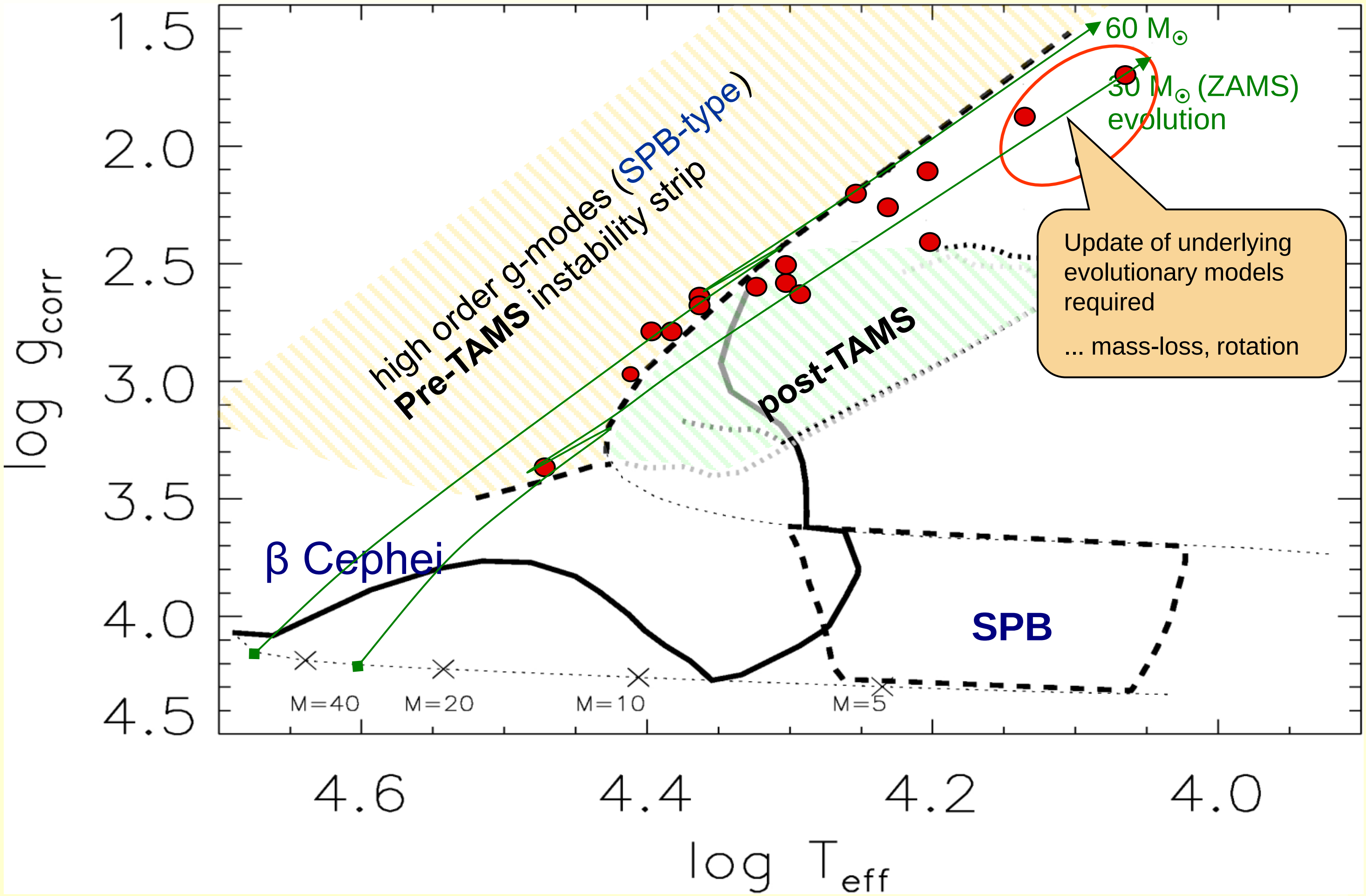
… towards massive star seismology



- β Cep:
  low order p- and g-modes
- SPB
  slowly pulsating B-stars
  high order g-modes
- Hipparcos:
  29 periodically variable
  B-supergiants
  (Waelkens et al. 1998)
- no instability region
  predicted at that time
- nowdays: additional
  region for high order
  g-mode instability

➔ asteroseismology of
evolved massive stars
becomes possible

... towards massive star seismology



dashed - Pre-TAMS instability strips: Pamyatnykh (1999)

dotted - Post-TAMS predictions of Saio et al (2006) l = 1, l = 2

- quant. spectroscopy by Lefever, Puls & Aerts (2007, A&A 463): 17 from 29 objects with sufficient spectral info
- most of these objects

  very close to the high gravity limit of the predicted pre-TAMS instability strip and

  within the predicted post-TAMS instability strip of g-modes in evolved stars

\+ multiperiodic behaviour

⇒ suggests to be opacity-driven non-radial pulsators

⇒ asteroseismology of **evolved** massive stars becomes possible

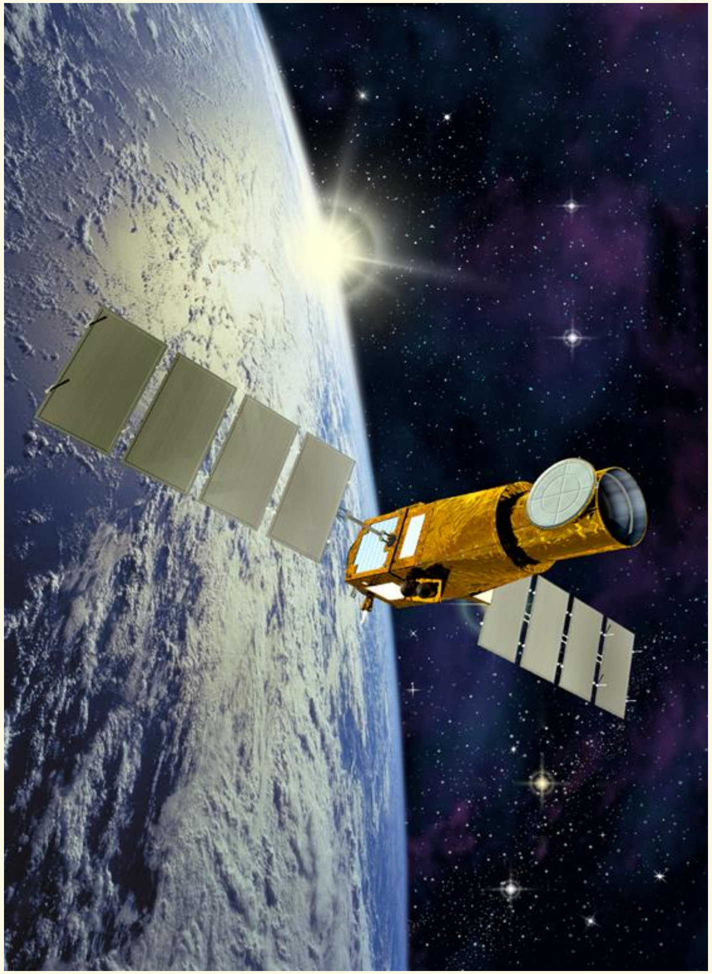
Space Asteroseismology

MOST: Canadian mission (15 cm) launched in June 2003

COROT: COnvection ROtation and planetary Transits

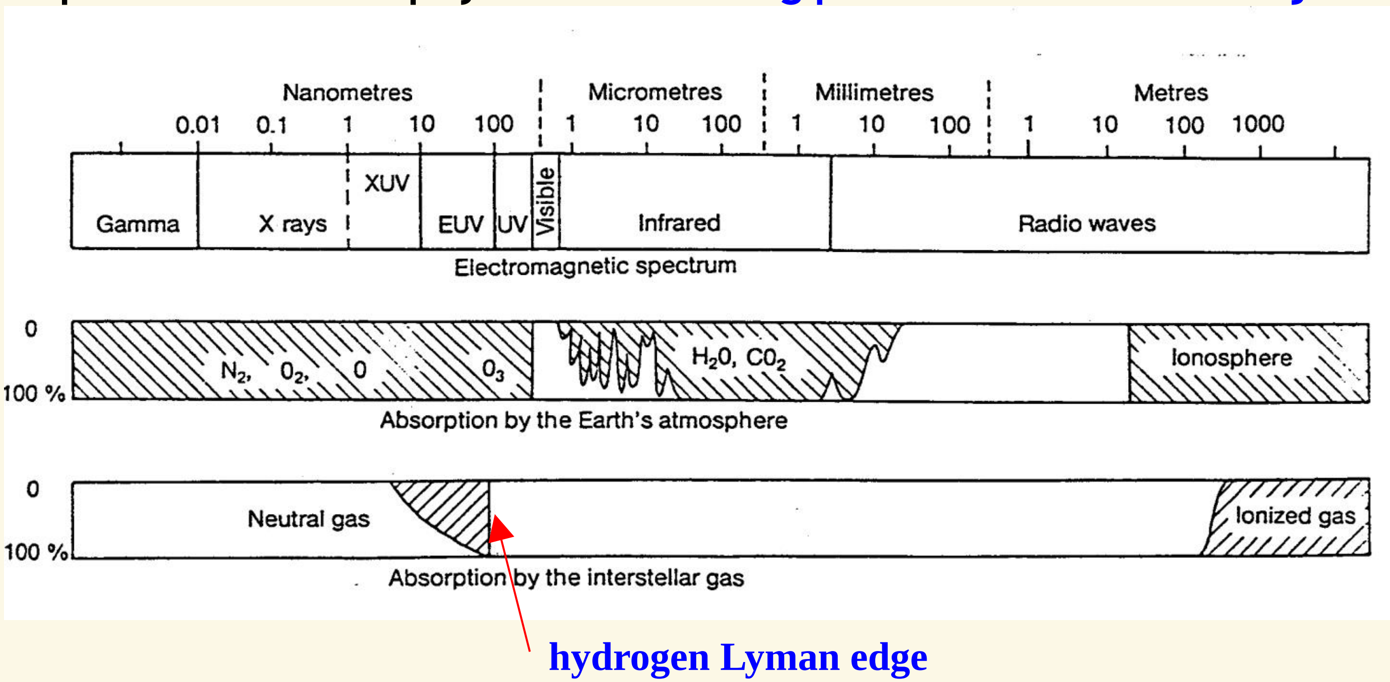
French-European mission (27cm), launched December 2006

Kepler : NASA mission (1.2m), launched March 2009



Astrophysics → theory
→ experiment

Experiment in astrophysics = Collecting photons from cosmic objects



1 Å = 10^{-8} cm = 10^{-4} µm (micron);   1 nm = 10 Å

Collecting: earthbound and via satellites!

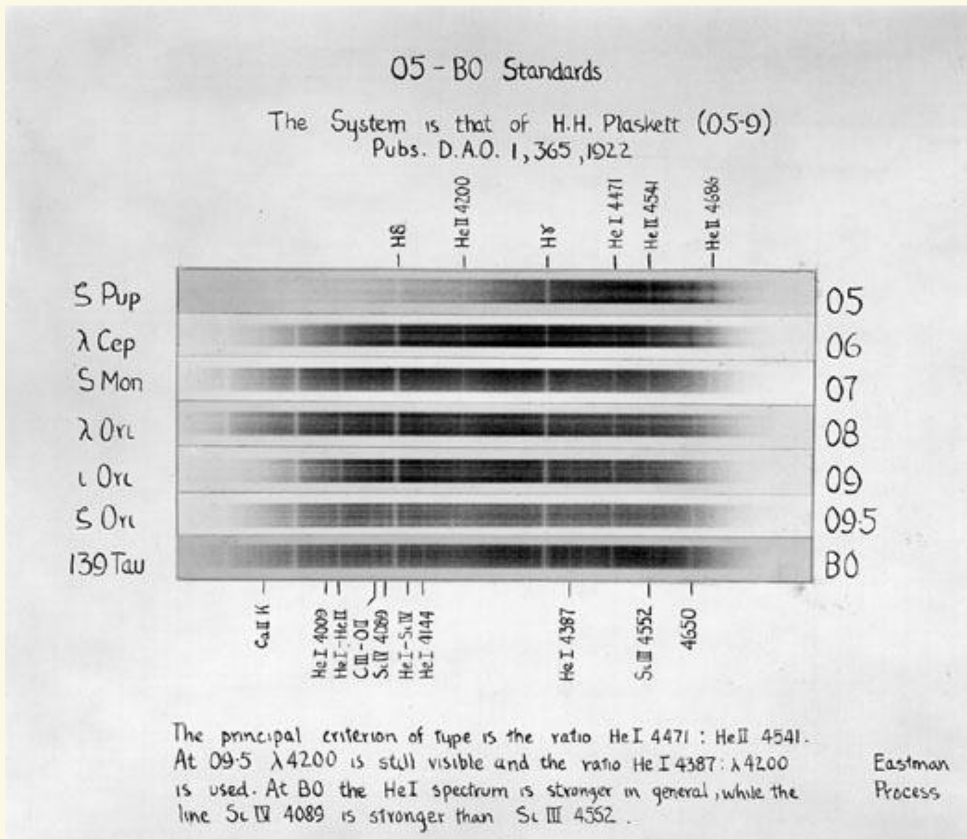
Note: Most of these photons originate from the atmospheres of stellar(-like) objects.
Even galaxies consist of stars!

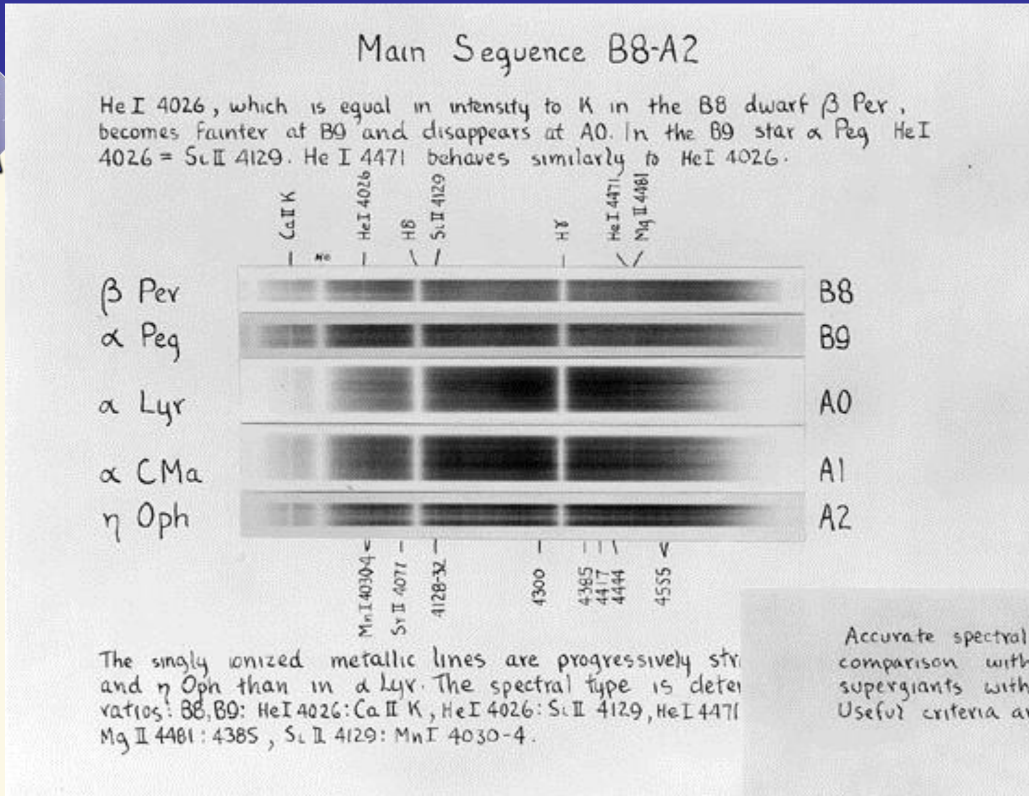
Astrophys. monographs, Univ. Chicago Press (1943)

AN ATLAS OF STELLAR SPECTRA

WITH AN OUTLINE OF SPECTRAL CLASSIFICATION

Morgan, Keenan, Kellman

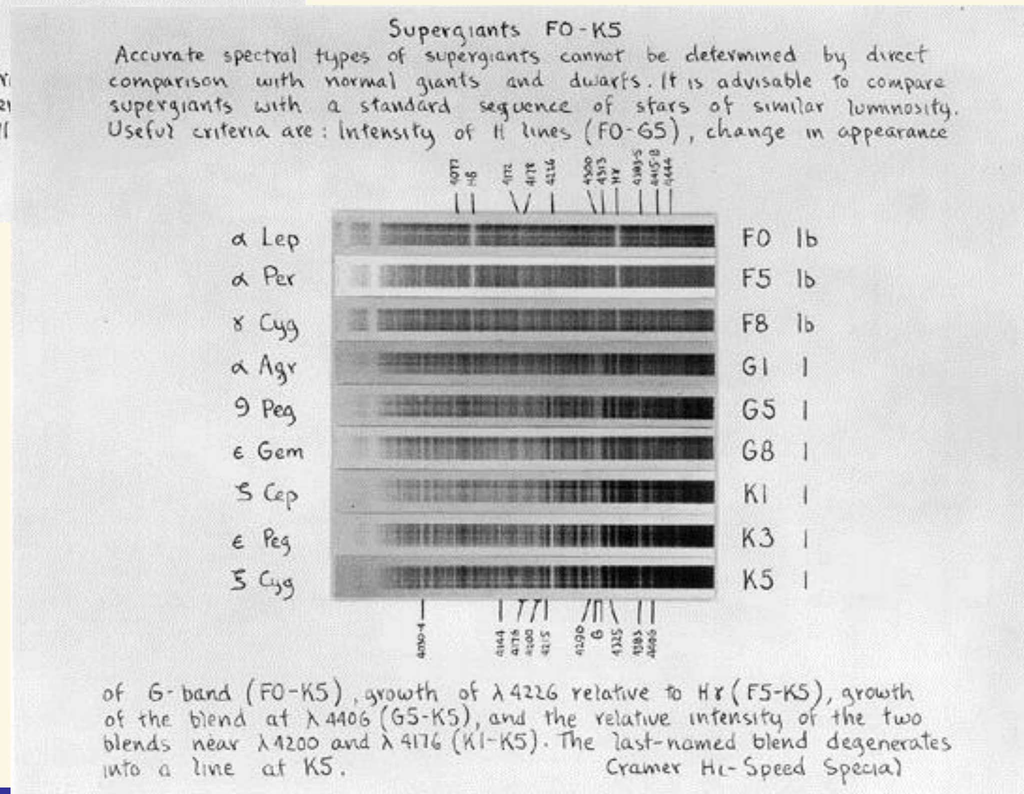




Empirical system

=>

Physical system



Digitized spectra

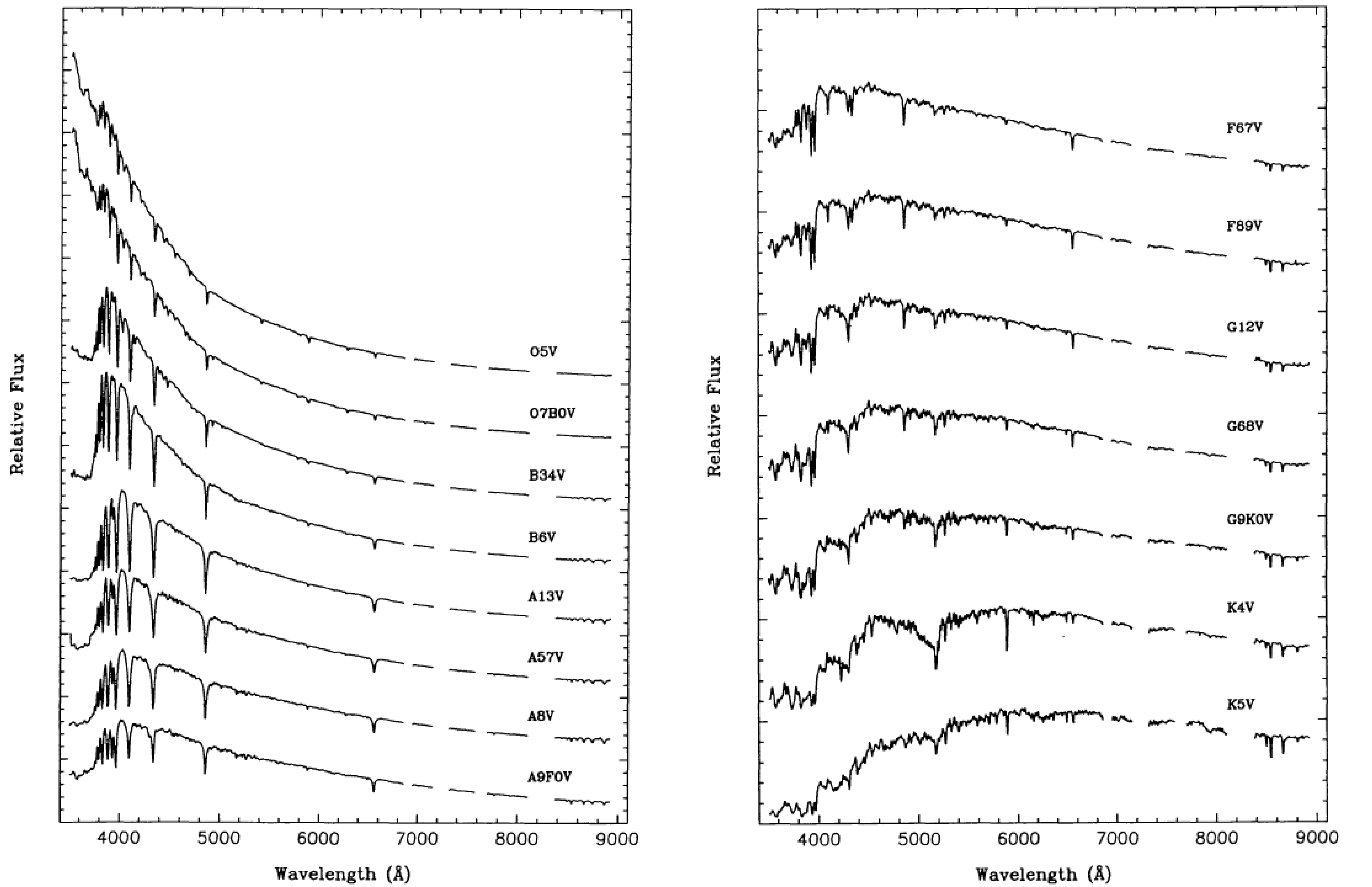


FIG. 1.—Dwarf-type library stars. Near-IR gaps are excised telluric absorption bands. All spectra have been normalized to 100 at 5450 Å. Major tick marks on "Relative Flux" axis are separated by 100 relative units. The M dwarf library stars are displayed with the M giants in Fig. 3.

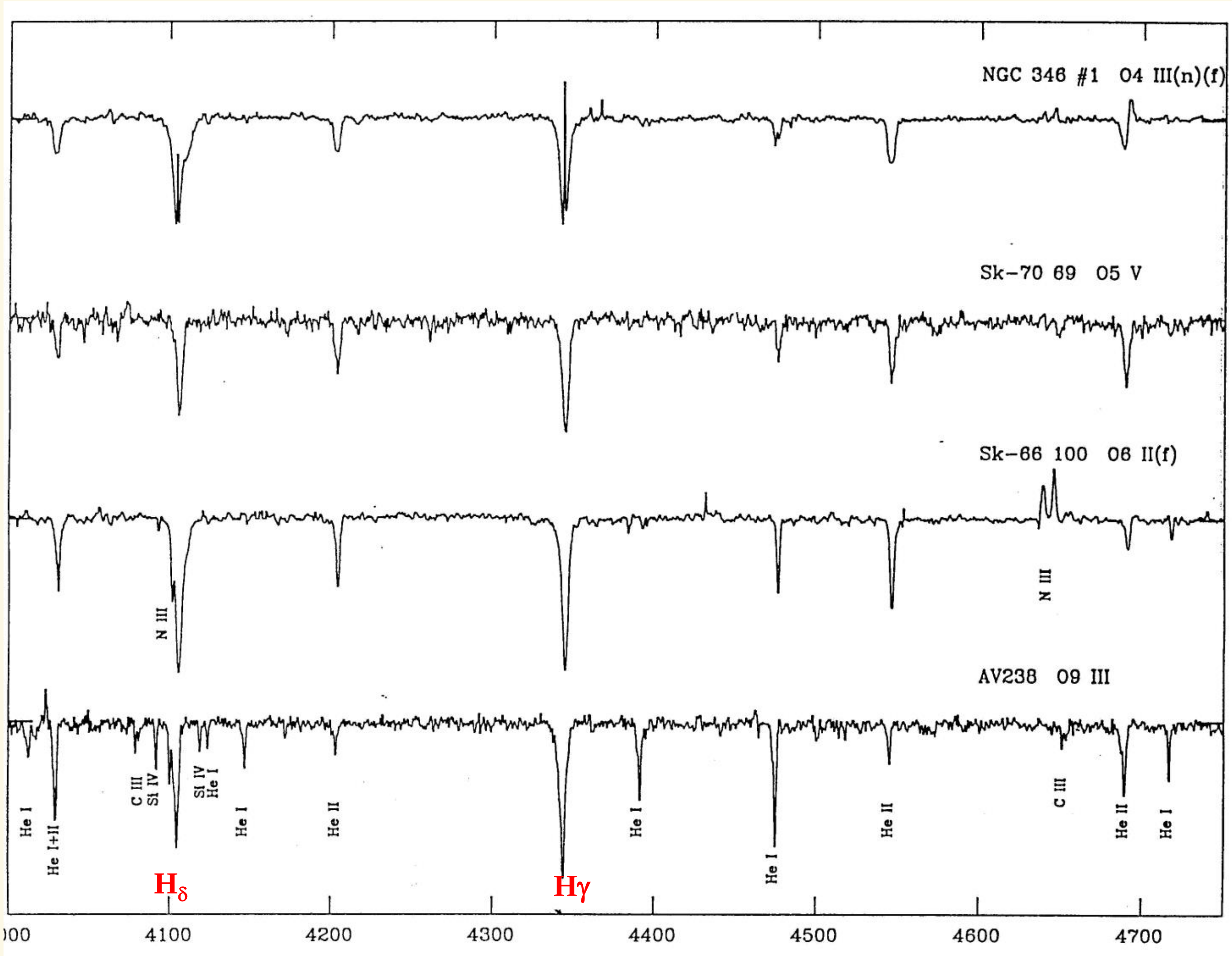
from Silva & Cornell, 1992

Spectral lines formed in (quasi-)hydrostatic atmospheres

ESO 3.6m
CASPEC

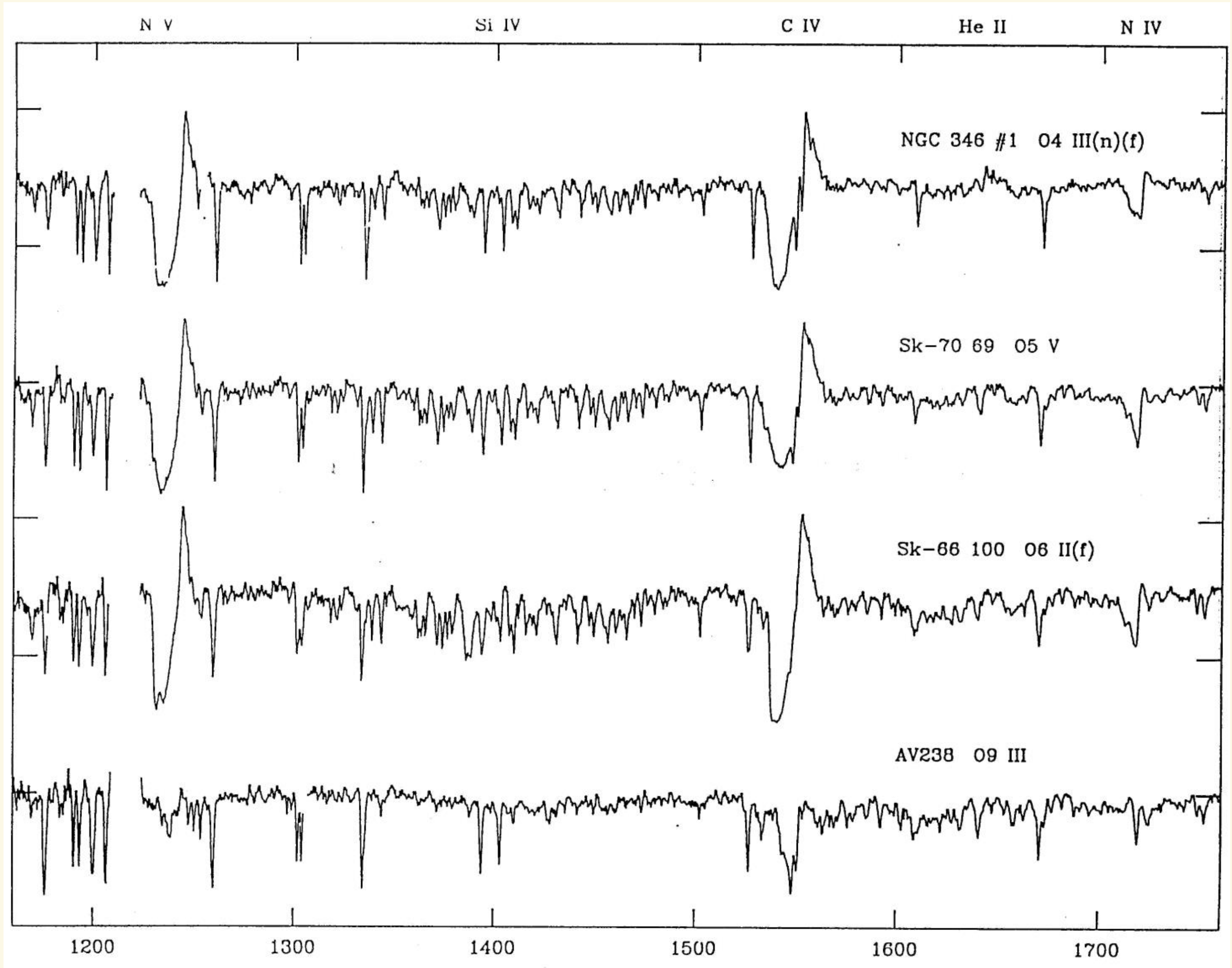
$\Delta\lambda \approx 0.5$Å
S/N 30...70

(Walborn et al.,1995



P-Cygni lines formed in hydrodynamic atmospheres

HST-FOS



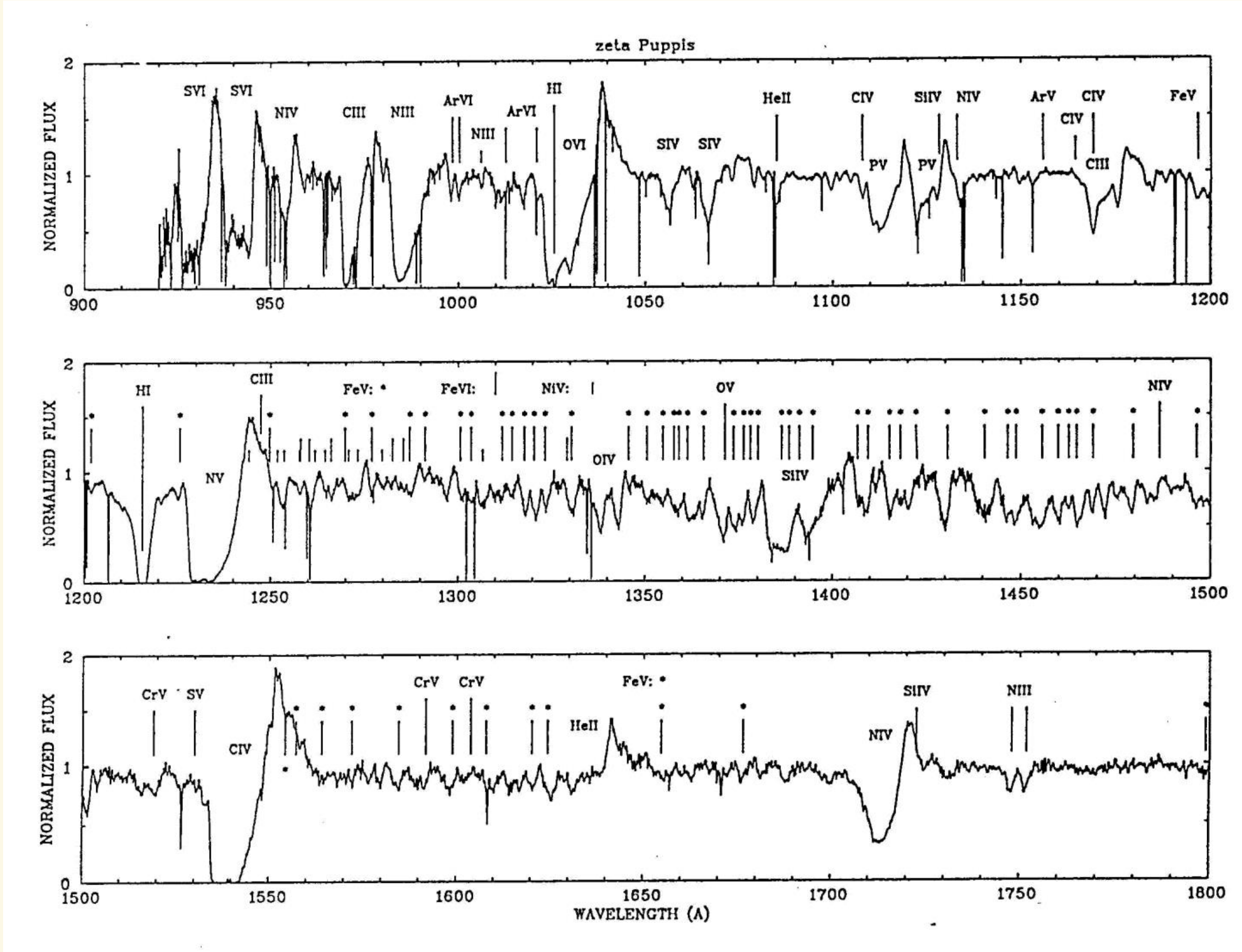
SMC

LMC

LMC

SMC

UV spectrum of the O4I(f) supergiant ζ Pup



montage of Copernicus (λ < 1500 Å, high res. mode, Δλ ≈ 0.05 Å, Morton & Underhill 1977) and IUE (Δλ ≈ 0.1 Å) observations

photospheric phase

transition to nebular phase

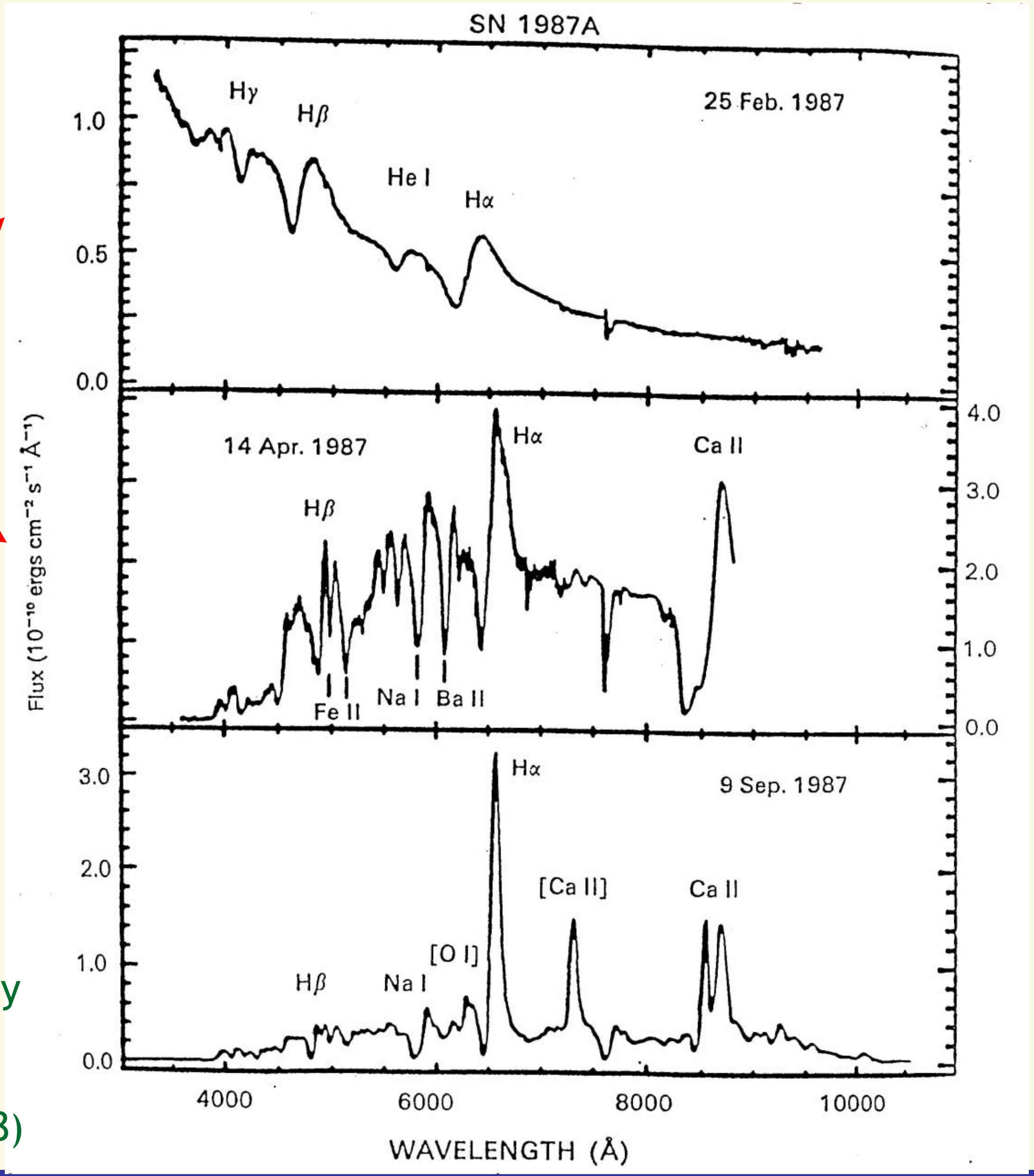
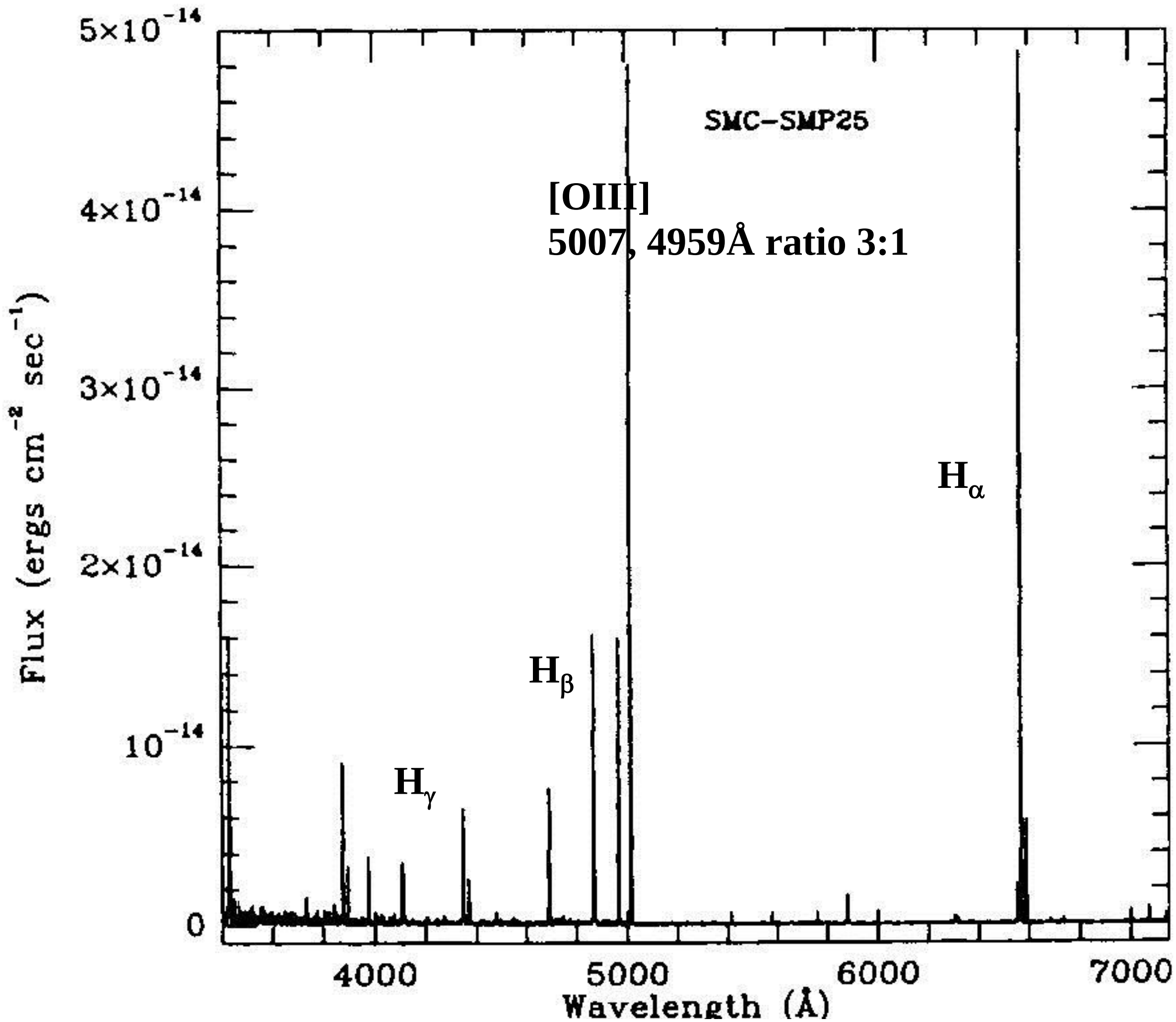


figure prepared by Mark M. Phillips, reproduced from McCray & Li (1988)

Spectrum of Planetary Nebula

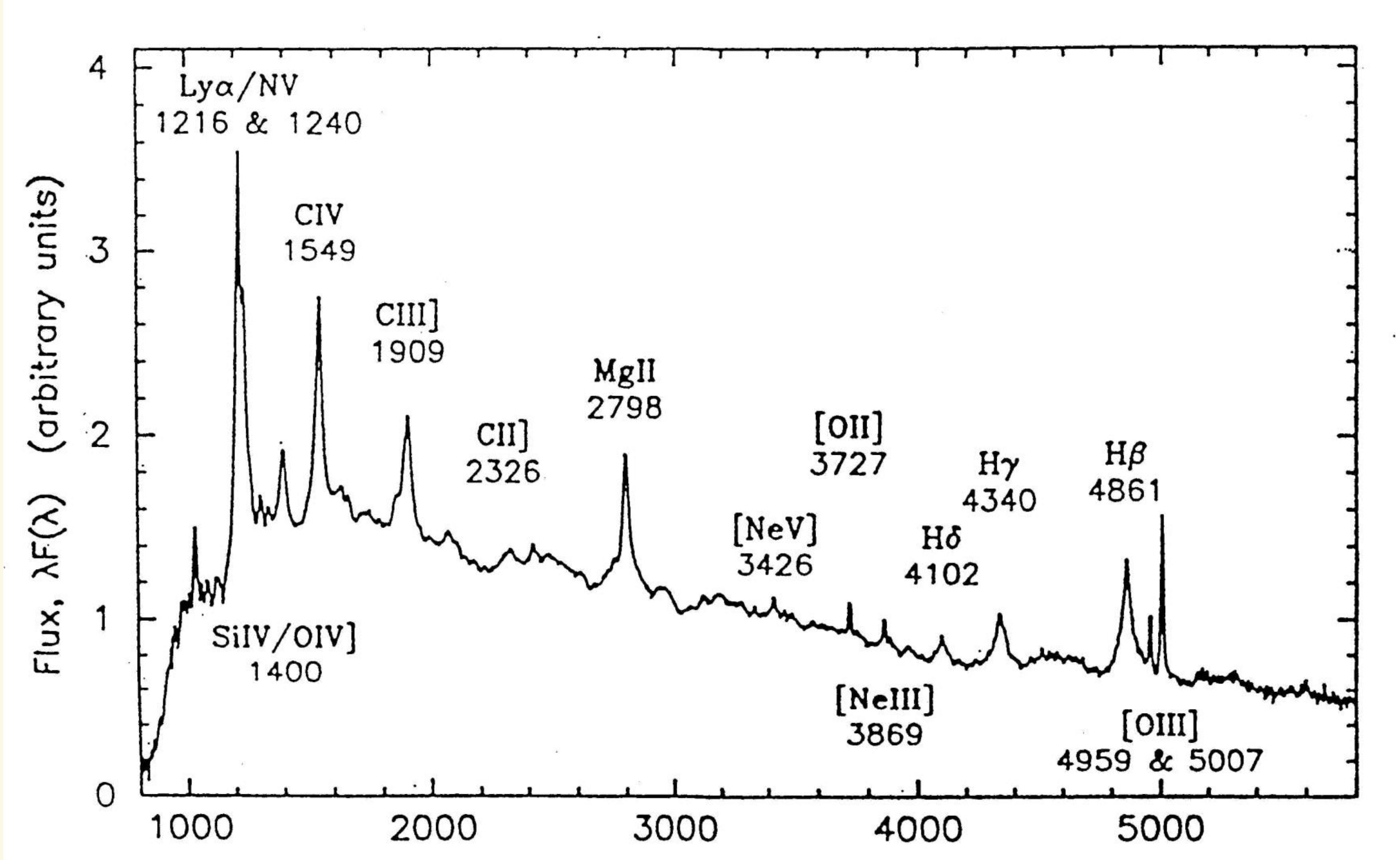
pure emission
line spectrum
with forbidden
lines of O III



From Meatheringham & Dopita, 1991, ApJS 75

FIG. 1*a*

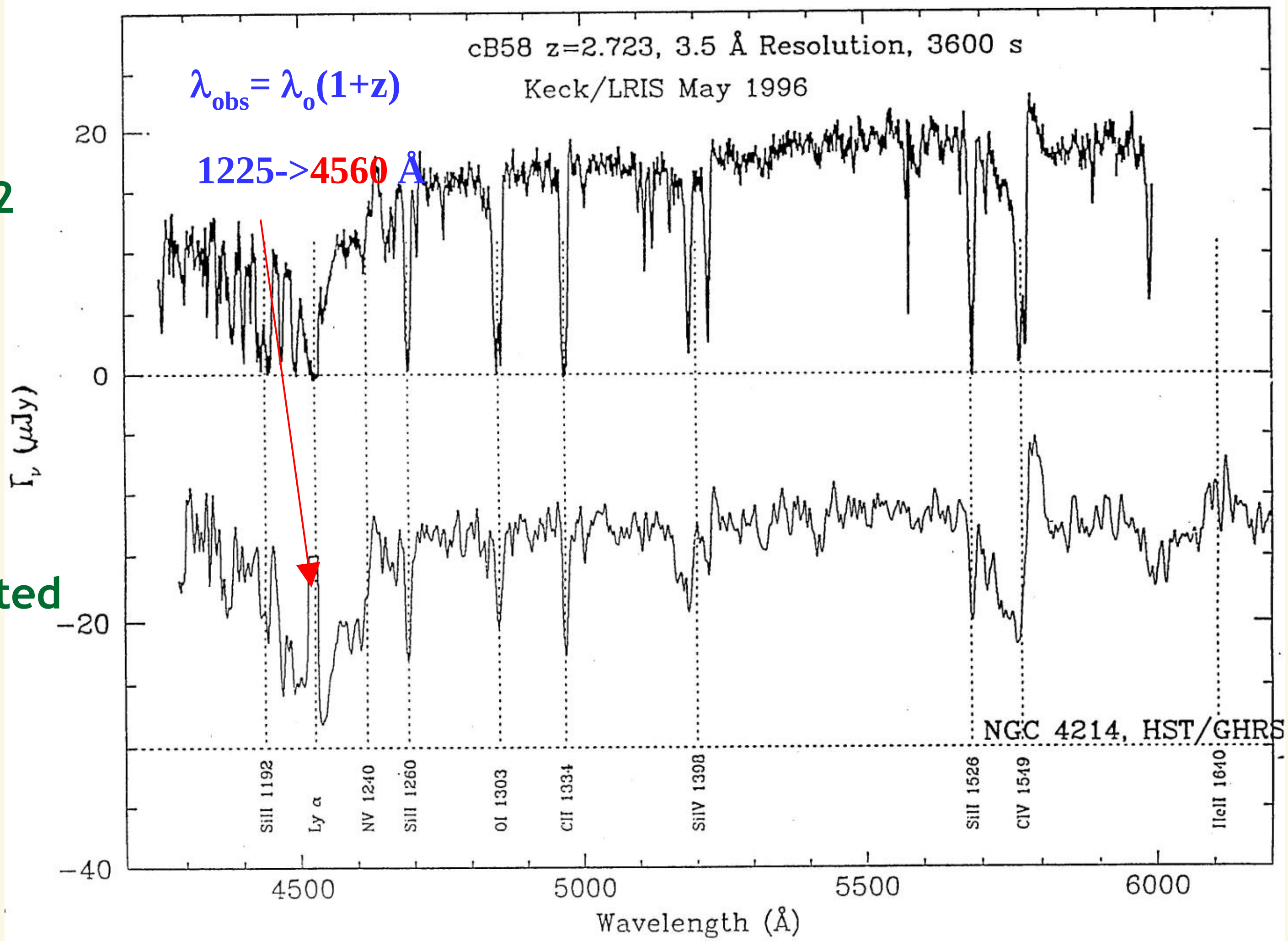
Quasar spectrum in rest frame of quasar



“UV”-spectra of starburst galaxies

galaxy at z = 2.72

local
starburst galaxy,
wavelengths shifted



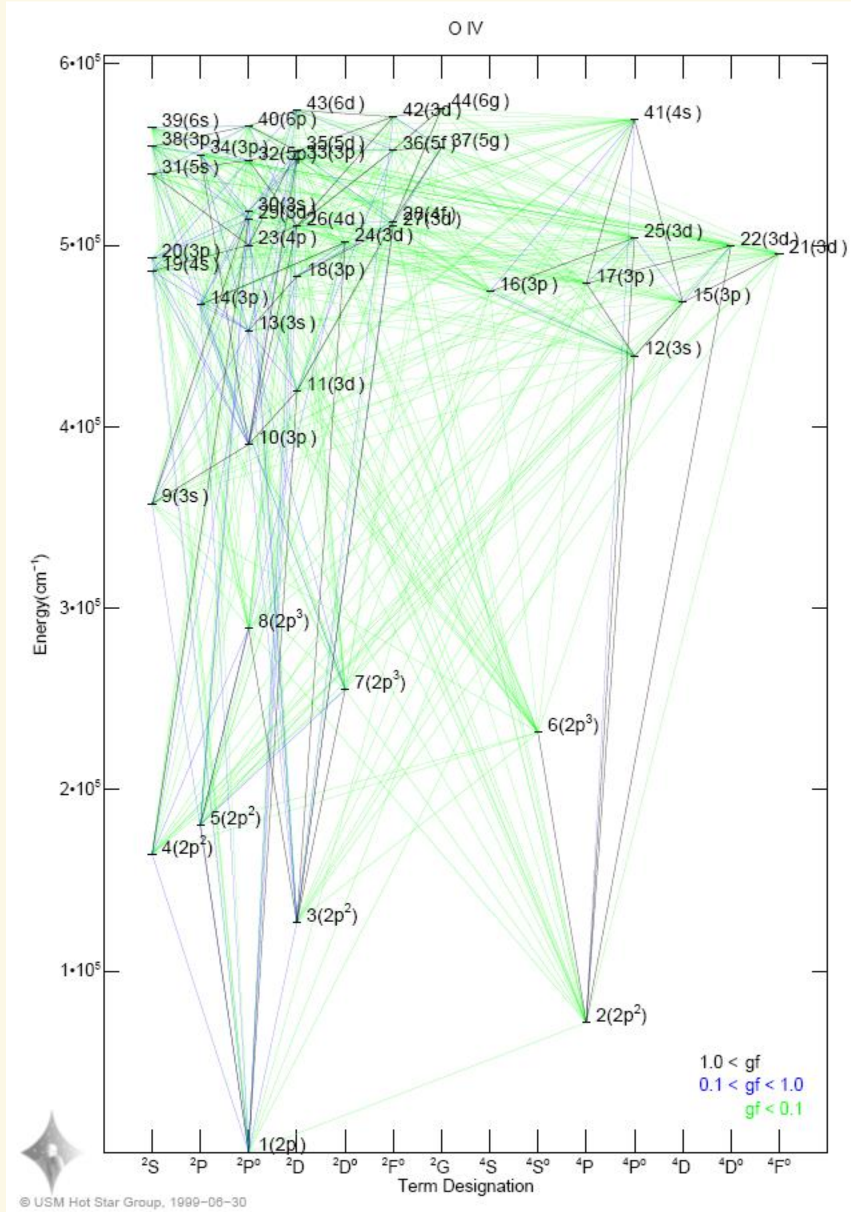
From Steidel et al. (1997)

Quantitative spectroscopy…

…gives insight into and understanding of our cosmos

- requires
  - plasma physics, plasma is "normal" state of atmospheres and interstellar matter (plasma diagnostics, line broadening, influence of magnetic fields,...)
  - atomic physics/quantum mechanics, interaction light/matter (micro quantities)
  - radiative transfer, interaction light/matter (macroscopic description)
  - thermodynamics, thermodynamic equilibria: TE, LTE (local), NLTE (non-local)
  - hydrodynamics, atmospheric structure, velocity fields, shockwaves,...
- provides
  - stellar properties, mass, radius, luminosity, energy production, chemical composition,properties of outflows
  - properties of (inter) stellar plasmas, temperature, density, excitation, chemical comp., magnetic fields
- INPUT for stellar, galactic and cosmologic evolution and for stellar and galactic structure

atomic levels and allowed transitions ("Grotrian-diagram") in OIV

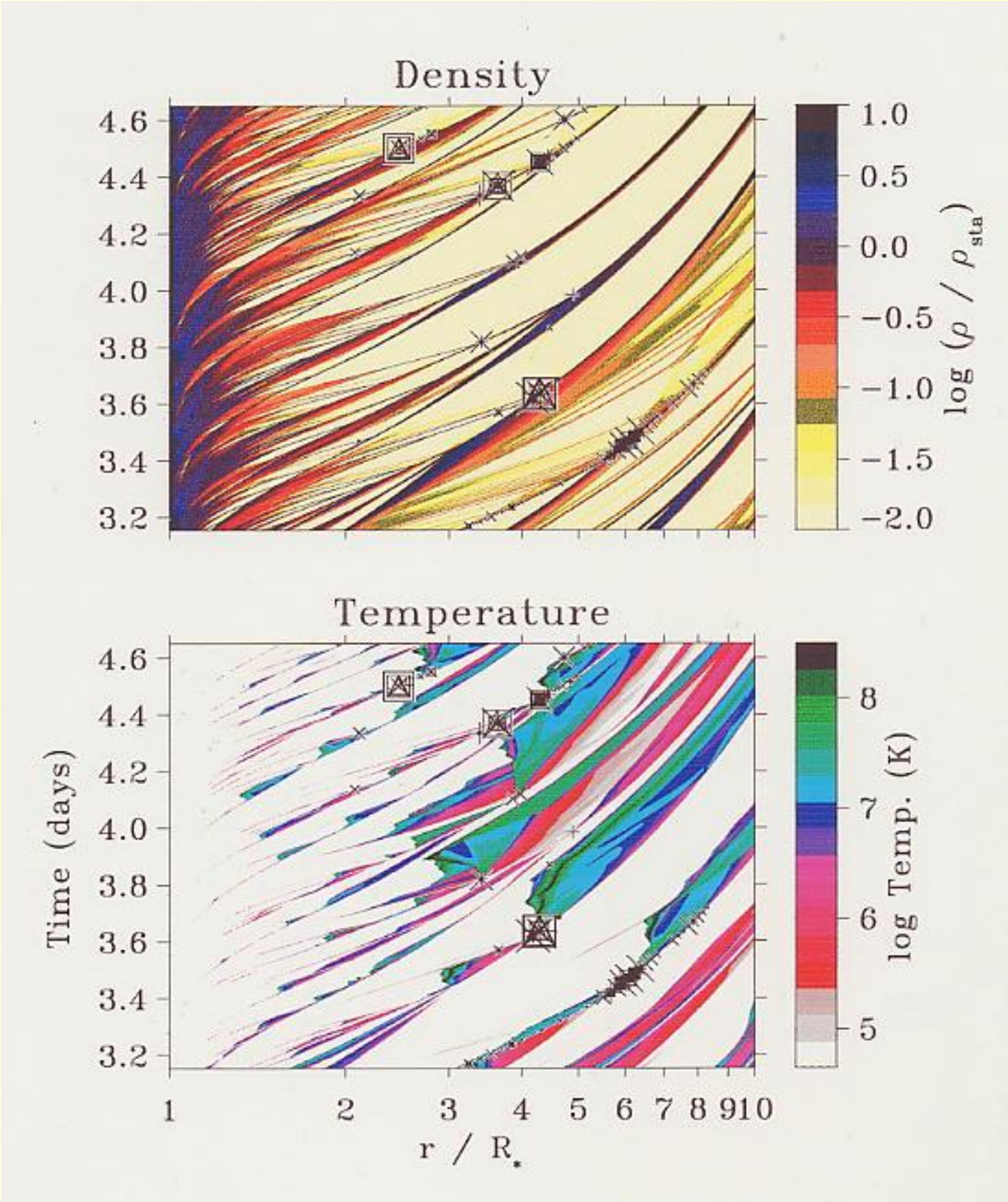


gf oscillator strength, measures "strength" of transition (cf. Chap 7)

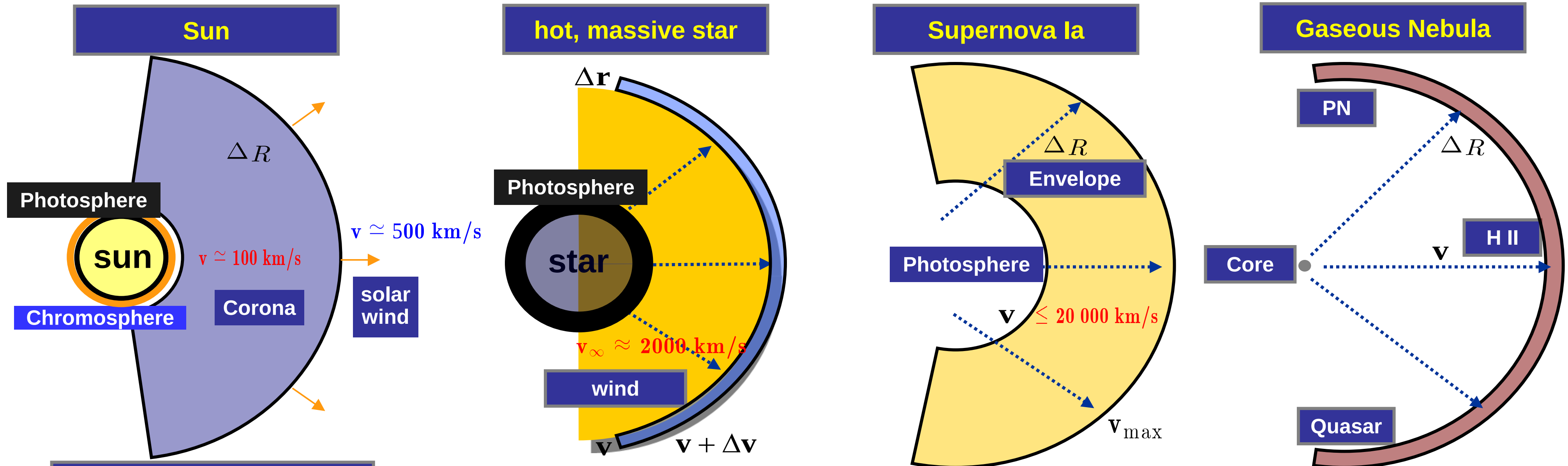
sites of X-ray emission in
hot stars:

shell collisions

hydrodynamical simulations
of instable hot star winds,
from A. Feldmeier,
by permission



Stellar atmospheres - an overview



Core:

$M_\odot = 2\ 10^{33}$ g

$R_\odot = 7\ 10^{10}$ cm

$L_\odot = 4\ 10^{33}$ erg s^{-1}

Photosphere:

$\Delta R \sim 200$ km

$T \sim 6000$ K

$n \sim 10^{15}$ cm^{-3}

Chromosphere:

$\Delta R \sim 1000$ km

$T \sim 20000$ K

$n \sim 10^{12}$ cm^{-3}

Corona:

$\Delta R \sim R_\odot$

$T \sim 10^6$ K

$n \sim 2 \cdot 10^6$ cm^{-3}

Core:

$M \sim 50\ M_\odot$

$R \sim 20\ R_\odot$

$L \sim 10^5 \mapsto 10^6\ L_\odot$

Photosphere:

$\Delta R / R \sim \Delta R_\odot / R_\odot$

$T \sim 20000 \mapsto 50000$ K

$n \sim 10^{14} \mapsto 10^{12}$ cm^{-3}

Wind:

$\Delta R \sim 100 R_*$

$T \sim 0.9 \ldots 0.3 \cdot T_{eff}$

$n \sim 10^{12} \ldots 10^8$ cm^{-3}

Core:

$M \sim 1.44\ M_\odot$

exploded

$L_{max} \sim 10^{11}\ L_\odot$

Envelope:

$\Delta R \sim 10^5 R_\odot$

$T \sim 12000$ K

$n \sim 10^6$ cm^{-3}

Core:

$M_{PN} \sim 1\ M_\odot$

$M_{HII} \sim 50 \mapsto 10^5\ M_\odot$

$M_Q \sim 10^9 \mapsto 10^{11}\ M_\odot$

$L_{PN} \sim 10^4\ L_\odot$

$L_{HII} \sim 10^6\ L_\odot$

$L_Q \sim 10^{12}\ L_\odot$

Envelope:

$\Delta R \sim 0.1$pc (PN)

$\Delta R \sim 10$ pc (H II)

$\Delta R \sim 10^3$ pc (Quasar)

Concept of spectral analysis

observed spectrum ⟷ **comparison** ⟷ **synthetic spectrum**

detector

spectrograph

telescope

theory of atmospheres

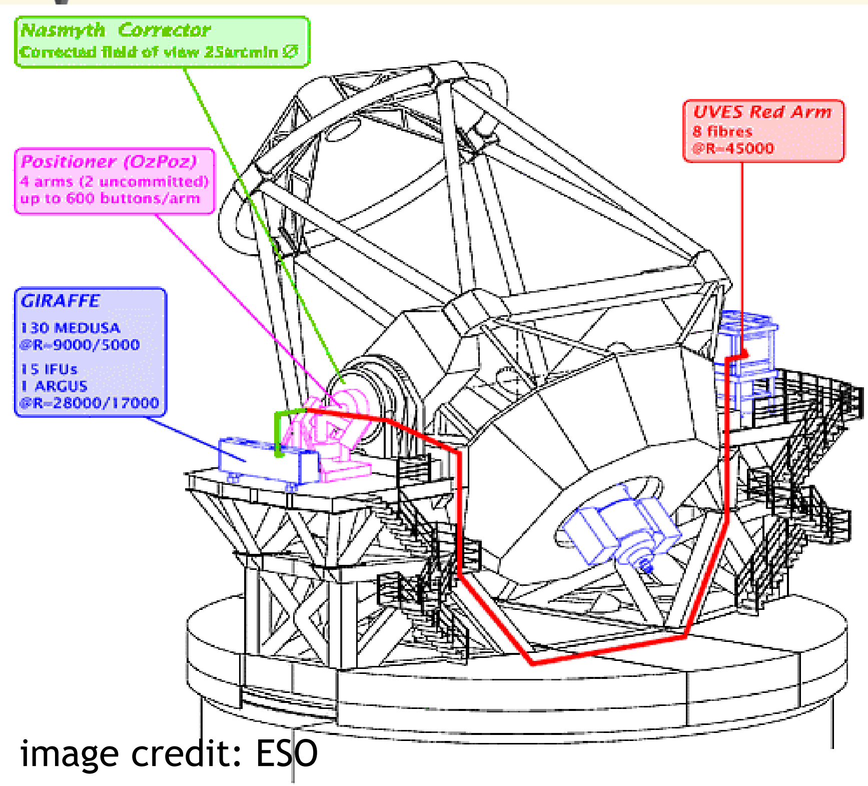
physical concept
- hydro-/thermodynamics
- atomic physics
- radiative transfer

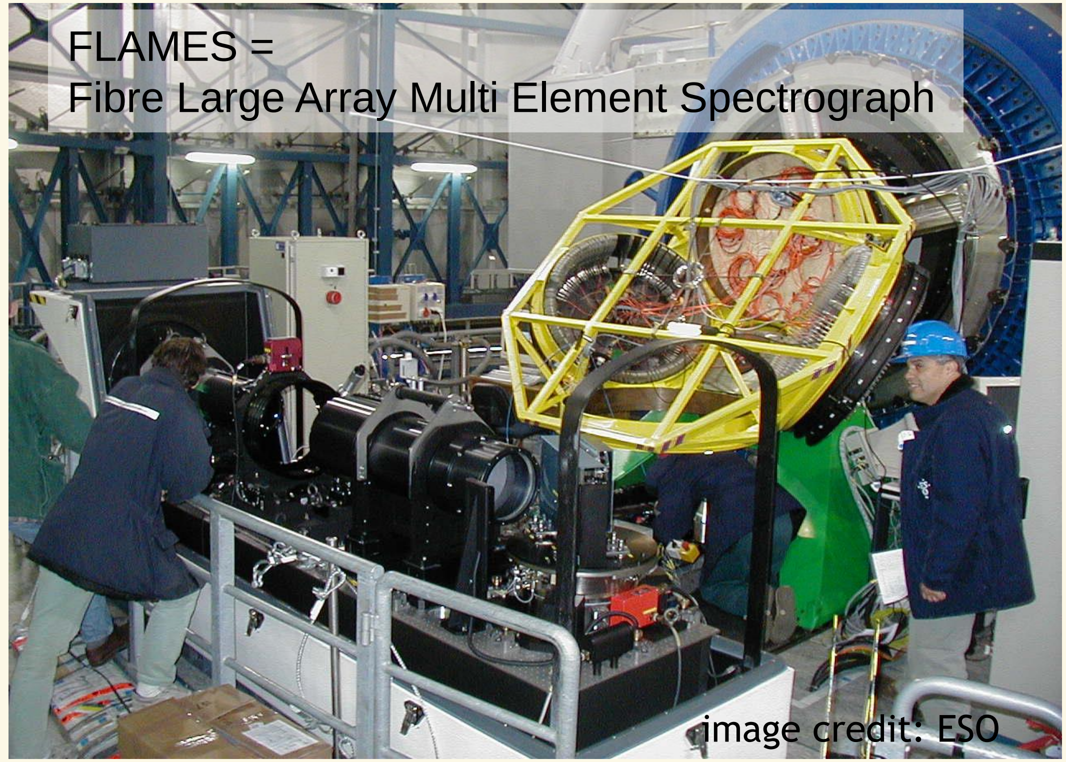
approximations for specific objects
- geometry, symmetry, homogeneity, stationarity

model calculations
(simulation)

numerical solution of theoretical equations

INPUT of L, M, R, chemical composition





- **FLAMES I:** high resolution spectroscopy of massive stars in 3 Galactic, 2 LMC and 2 SMC clusters (young and old)
  - total of 86 O- and 615 B-stars
- **FLAMES II**: high resolution spectroscopy of more than 1000 massive stars in Tarantula Nebula (incl. 300 O-type stars)

Major objectives

- rotation and abundances (test rotational mixing)
- stellar mass-loss as a function of metallicity
- binarity/multiplicity (fraction, impact)
- detailed investigation of the closest ‘proto-starburst’

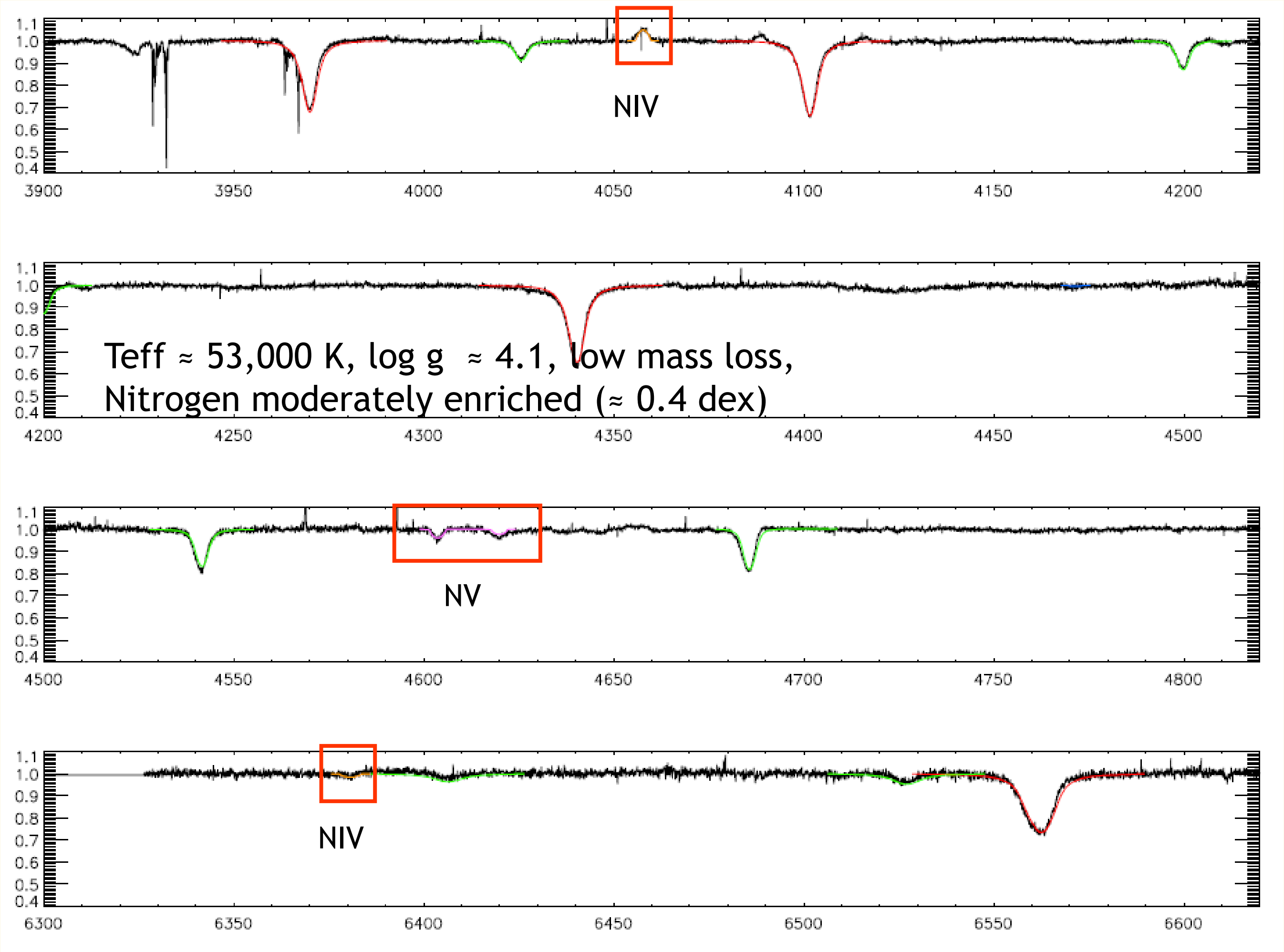
summary of FLAMES I results: Evans et al. (2008)

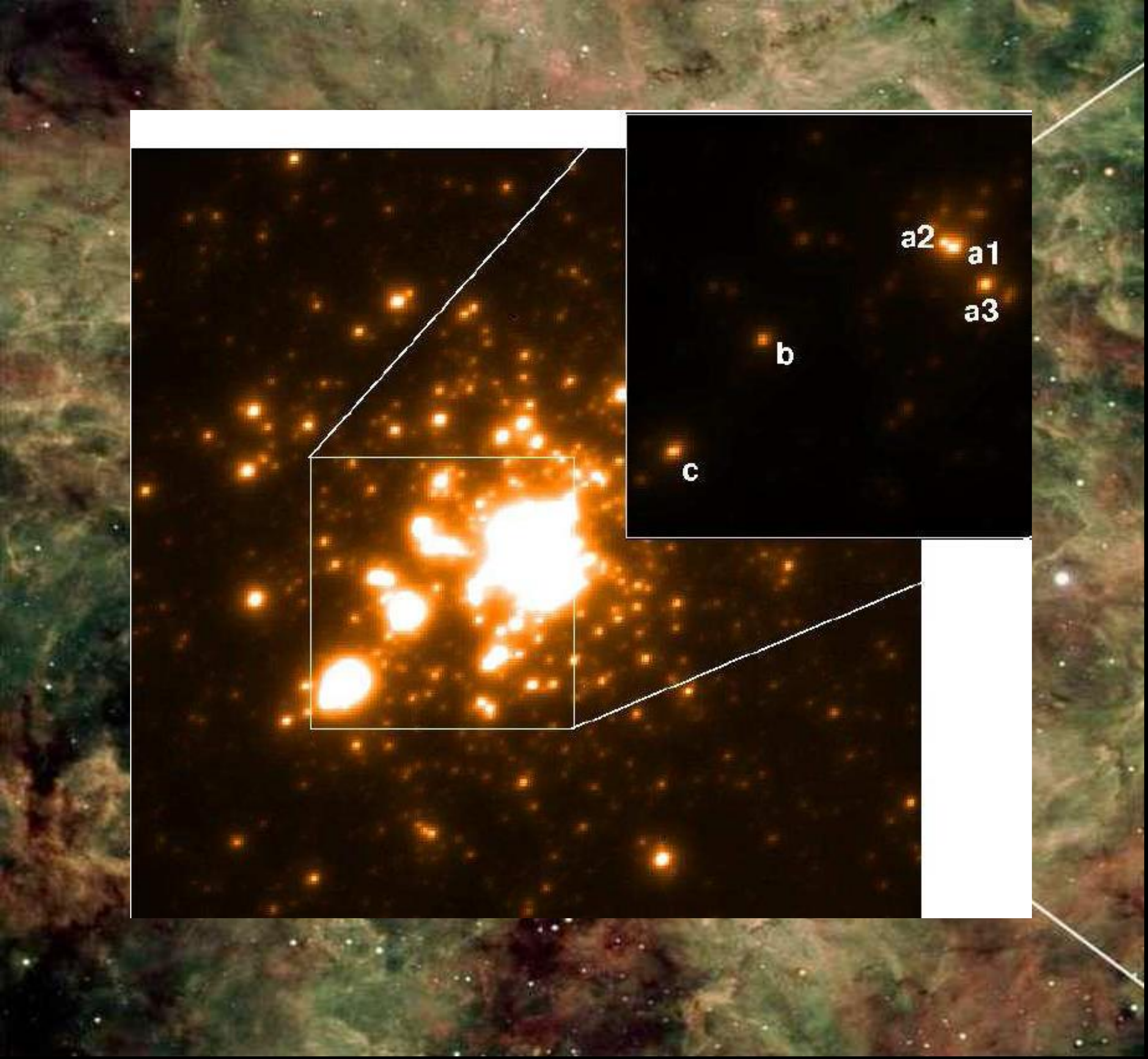
Optical spectrum of a very hot O-star

BI237 O2V (f*) (LMC) – vsini = 140 km/s

- Synthetic spectra from Rivero-Gonzalez et al. (2012)

red: HI
blue: HeI
green: HeII
orange: NIV
magenta: NV

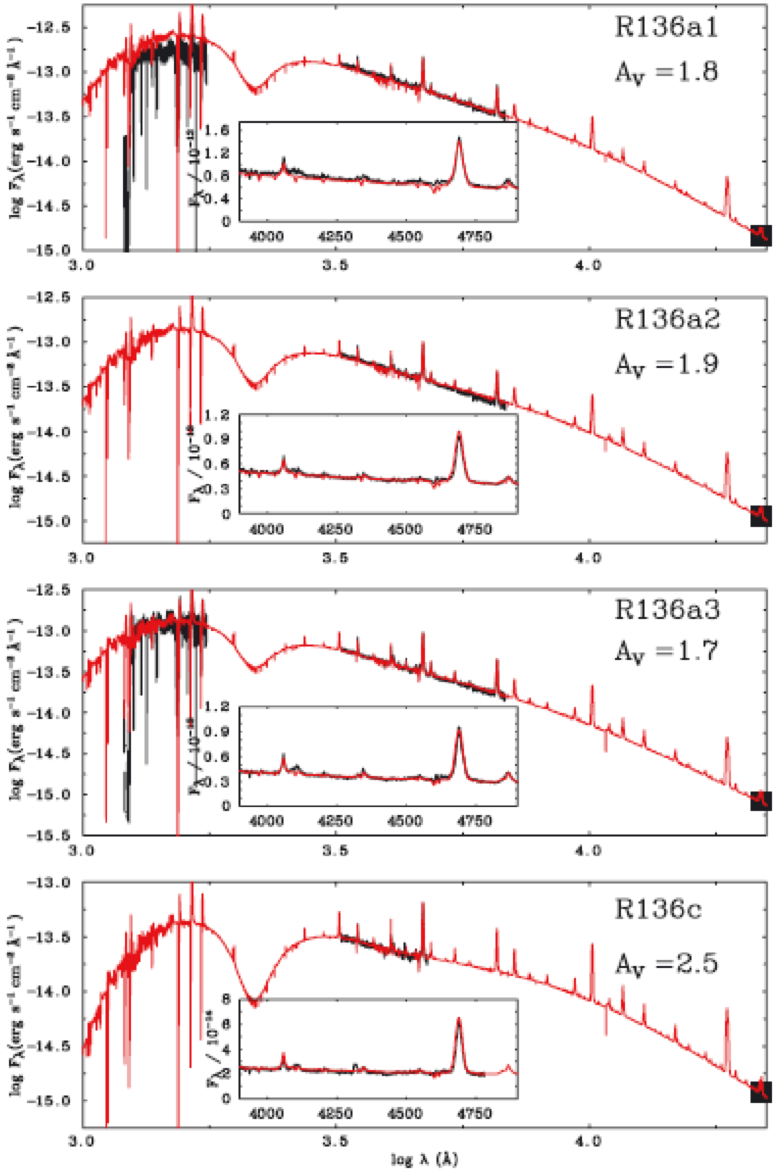




- Tarantula Nebula (30 Dor) in the LMC
- Largest starburst region in Local Group
- Target of VLT-FLAMES Tarantula survey (‘FLAMES II’, PI: Chris Evans)
- Cluster R136 contains some of the *most massive, hottest, and brightest* stars known
- Crowther et al. (2010): 4 stars with initial masses from 165-320 (!!!) $M_{\odot}$

Spectral energy distribution of the most massive stars in our "neighbourhood"

from Crowther et al. 2010



initial mass (Msun)	current mass (Msun)
320	265
240	195
165	135
220	175
typical uncertainty ± 40 Msun	

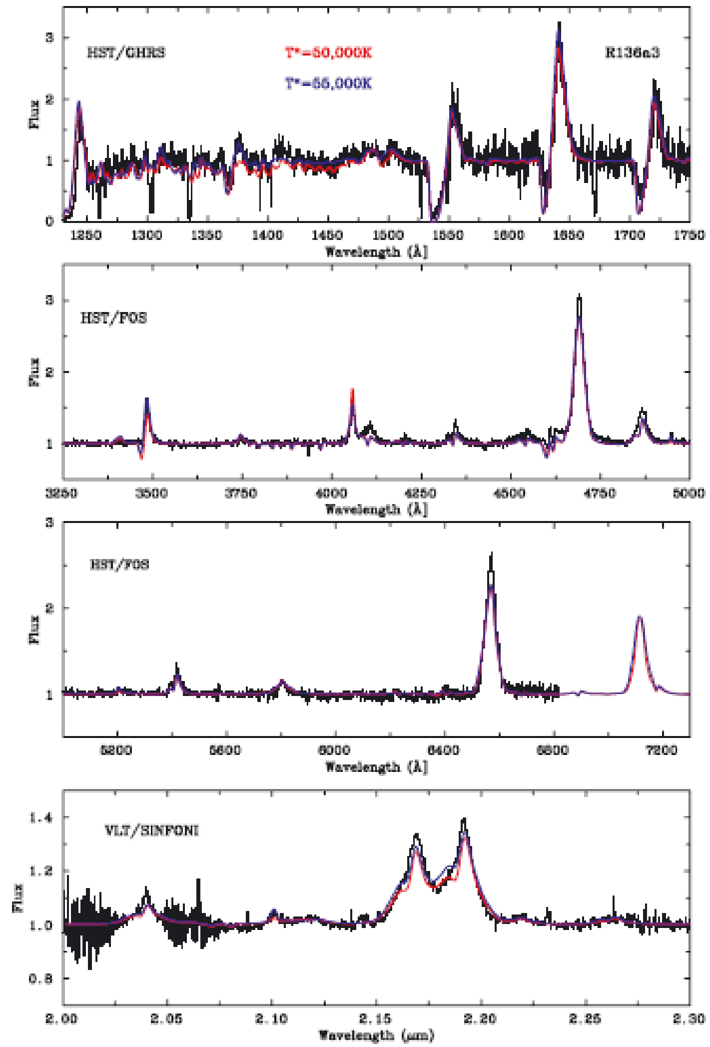


Figure 5. Rectified, ultraviolet (*HST*/GHRS), visual (*HST*/FOS) and near-IR (VLT/SINFONI) spectroscopy of the WN 5h star R136a3 together with synthetic UV, optical and near-IR spectra, for $T_* = 50\,000$ K (red) and $T_* = 55\,000$ K (blue). Instrumental broadening is accounted for, plus an additional rotational broadening of 200 km s^{-1}.

Figure 4. Spectral energy distributions of R136 WN 5h stars from *HST*/FOS together using K_s photometry from VLT/SINFONI calibrated with VLT/MAD imaging. Reddened theoretical spectral energy distributions are shown as red lines.

Number of particles in $(\mathbf{r},\mathbf{r}+d\mathbf{r})$ with momenta $(\mathbf{p},\mathbf{p}+d\mathbf{p})$ at time t

$$\delta N(\mathbf{r},\mathbf{p},t) = f(\mathbf{r},\mathbf{p},t)\, d^3\mathbf{r}\, d^3\mathbf{p}$$

distribution function f

For a detailed derivation and discussion, see, e.g., Cercignani, C., "The Boltzmann Equation and Its Applications", Appl. Math. Sciences 67, Springer, 1987

i) $f(\mathbf{r},\mathbf{p},t)$ is Lorentz-invariant

ii) $\delta N_0 = f(\mathbf{r}_0,\mathbf{p}_0,t_0)\, d^3\mathbf{r}_0\, d^3\mathbf{p}_0$

evolution

$$\delta N = f(\mathbf{r}_0+d\mathbf{r},\mathbf{p}_0+d\mathbf{p},t_0+dt)\, d^3\mathbf{r}\, d^3\mathbf{p}$$

$$(\dot{p}=F) = f(\mathbf{r}_0+\mathbf{v}dt,\mathbf{p}_0+\mathbf{F}dt,t_0+dt)\, d^3\mathbf{r}\, d^3\mathbf{p}$$

Theoretical mechanics: If no collisions, conservation of phase space volume:

$$d^3\mathbf{r}_0\, d^3\mathbf{p}_0 = d^3\mathbf{r}\, d^3\mathbf{p}$$

and

$\delta N_0 = \delta N$ (particles do not "vanish", again no collisions supposed)

$$\Rightarrow \boxed{f(\mathbf{r},\mathbf{p},t) = \text{const}, \quad \text{if no collisions}}$$

$$\Rightarrow \frac{\partial f}{\partial t} + \sum \frac{\partial f}{\partial r_i}\frac{\partial r_i}{\partial t} + \sum \frac{\partial f}{\partial p_i}\frac{\partial p_i}{\partial t} =$$

$$= \underbrace{\frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla) f}_{} + (\mathbf{F}\cdot\nabla_p) f = \begin{cases} 0 & \text{Vlasov} \\ \left(\frac{\delta f}{\delta t}\right)_{\text{coll}} & \text{Boltzmann if collisions} \end{cases}$$

D/Dt f, Lagrangian derivative
total derivative of *f* measured in fluid frame, at times t, t+Δt and positions **r**, **r** + **v** Δt

- implications for photon gas

$$\mathbf{p} = \frac{h\nu}{c}\mathbf{n}$$

$$d^3\mathbf{p} = p^2 dp d\Omega \quad \leftarrow \text{ solid angle with respect to } \mathbf{n}$$

absolute value

$$= \left(\frac{h\nu}{c}\right)^2 \frac{h}{c} d\nu d\Omega = \frac{h^3}{c^3}\nu^2 d\nu d\Omega$$

$$\Rightarrow f(\mathbf{r},\mathbf{p},t)\, d^3\mathbf{r}\, d^3\mathbf{p} = \frac{h^3}{c^3}\nu^2 f(\mathbf{r},\mathbf{n},\nu,t)\, d^3\mathbf{r}\, d\nu d\Omega =$$

$$\boxed{= \Psi(\mathbf{r},\mathbf{n},\nu,t)\, d^3\mathbf{r}\, d\nu d\Omega}$$

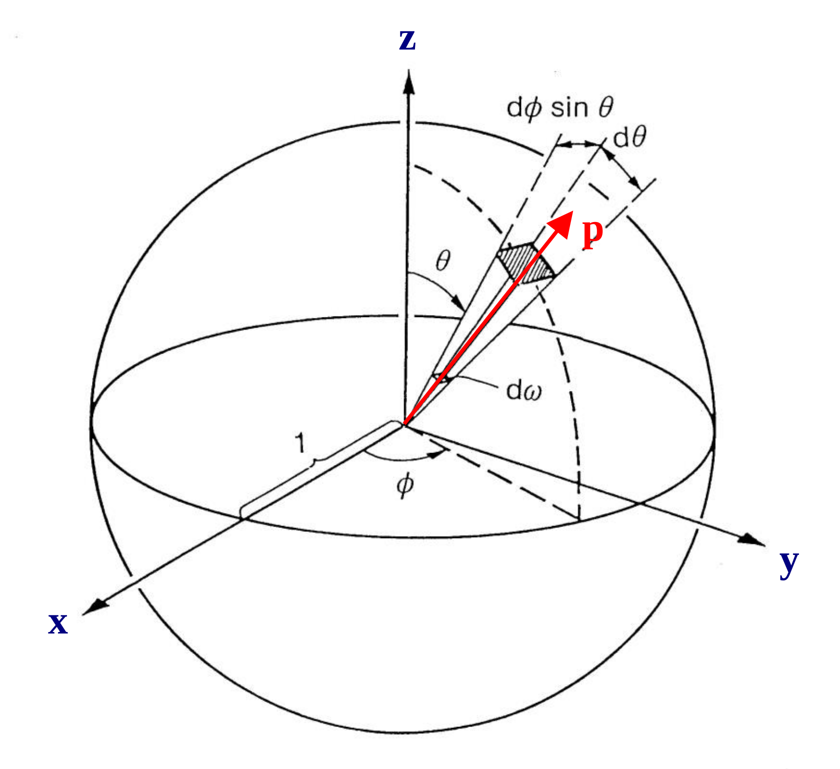
$$d^3\mathbf{p} = J(\mathbf{p},\mathbf{p}')\, d^3\mathbf{p}', \qquad \mathbf{p}' = (p,\theta,\phi)$$

cartesian Jacobi-det. spherical

$$\mathrm{p}_x = p \sin\theta \cos\phi$$
$$\mathrm{p}_y = p \sin\theta \sin\phi$$
$$\mathrm{p}_z = p \cos\theta$$

$$J = \det\begin{pmatrix} \frac{\partial p_x}{\partial p} & \frac{\partial p_x}{\partial \theta} & \frac{\partial p_x}{\partial \phi} \\ \frac{\partial p_y}{\partial p} & \frac{\partial p_y}{\partial \theta} & \frac{\partial p_y}{\partial \phi} \\ \frac{\partial p_z}{\partial p} & \frac{\partial p_z}{\partial \theta} & \frac{\partial p_z}{\partial \phi} \end{pmatrix} = \det\begin{pmatrix} \sin\theta\cos\phi & \mathrm{p}\cos\theta\cos\phi & -\mathrm{p}\sin\theta\sin\phi \\ \sin\theta\sin\phi & \mathrm{p}\cos\theta\sin\phi & \mathrm{p}\sin\theta\cos\phi \\ \cos\theta & -\mathrm{p}\sin\theta & 0 \end{pmatrix}$$

$$= \text{(exercise)} \; \boxed{p^2 \sin\theta}$$

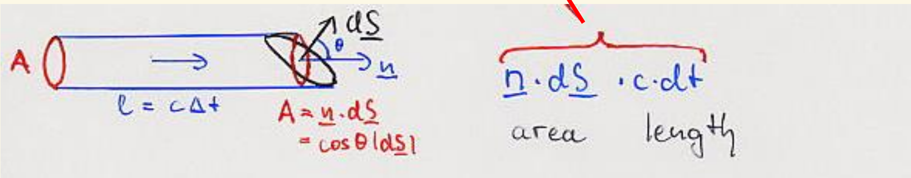


$$\Rightarrow d^3\mathbf{p} = dp_x dp_y dp_z = p^2 dp \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega}$$

The specific intensity

Number of photons with ν, $\nu + d\nu$ which propagate through surface element $d\mathbf{S}$ into direction $\mathbf{n}$ and solid angle $d\Omega$, at time t and with velocity c:

$$\delta N = \frac{h^3\nu^2}{c^3} f(\mathbf{r},\mathbf{n},\nu,t)\, d^3\mathbf{r}\, d\nu\, d\Omega$$



$$= \frac{h^3\nu^2}{c^3} f(\mathbf{r},\mathbf{n},\nu,t) \cos\theta\; c dt\; dS\; d\nu\, d\Omega$$

$$\sphericalangle(\mathbf{n}, d\mathbf{S})$$

- corresponding energy transport

$$\delta E = h\nu\; \delta N = \underbrace{\frac{h^4\nu^3}{c^2} f(\mathbf{r},\mathbf{n},\nu,t)}_{I(\mathbf{r},\mathbf{n},\nu,t)} \cos\theta\; dS\; d\nu\; dt\; d\Omega$$

specific intensity
[erg cm^{-2} Hz^{-1} $s^{-1}sr^{-1}$]

<u>summarized</u>

$$\boxed{I = ch\nu\;\Psi = \frac{h^4\nu^3}{c^2} f}$$ function of $\mathbf{r},\mathbf{n},\nu,t$

specific intensity is radiation energy, which is transported into direction $\mathbf{n}$ through surface $d\mathbf{S}$, per frequency, time and solid angle.

basic quantity in theory of radiative transfer

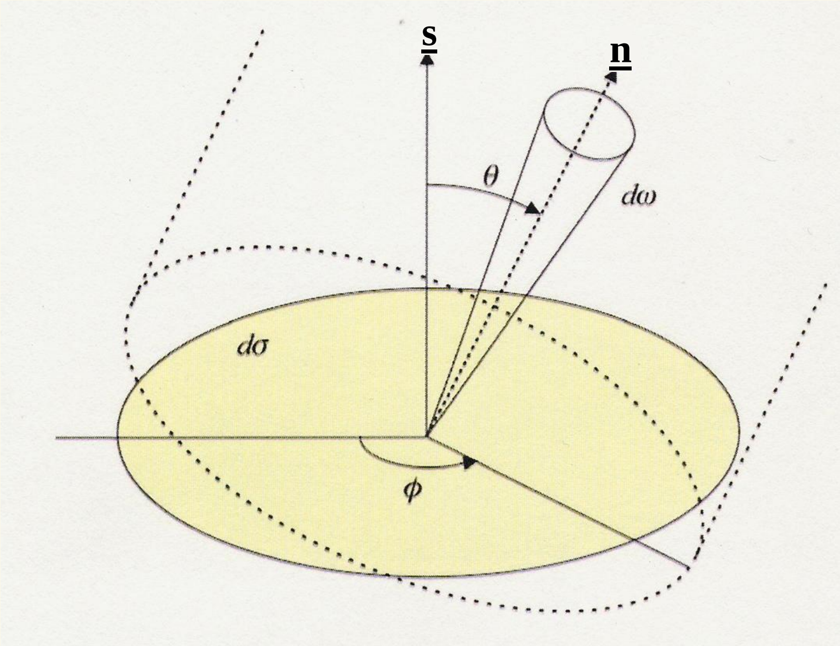
<u>invariance of specific intensity</u>

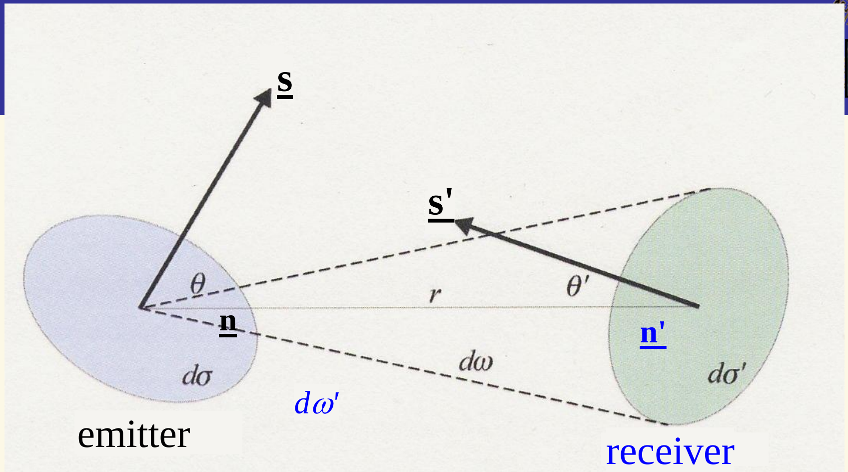
since $\frac{\mathrm{D}f}{\mathrm{D}t} = 0$ without collisions (Vlasov equation) and without GR (i.e., $\mathbf{F} \equiv \mathbf{0}$), we have

$I \sim f$

$\Rightarrow$ I = const in fluid frame, as long as no interaction with matter!

If stationary process, i.e. $\partial/\partial t = 0$, then $\underline{\mathbf{n}\nabla}\, I = d/ds\, I = 0$,
where ds is path element, i.e.
I = const also spatially!
(this is the major reason for working with specific intensities)





Invariance of specific intensity

Consider pencil of light rays which passes through both area elements δσ (emitter) and δσ' (receiver).

If no energy sinks and sources in between, the amount of energy which passes through both areas is given by

$$\delta E = I_\nu \cos\theta \, d\sigma \, dt \, d\omega =$$

$$\delta E' = I'_\nu \cos\theta' \, d\sigma' \, dt \, d\omega', \text{ and, cf. figure,}$$

$$d\omega = \frac{\text{projected area}}{\text{distance}^2} = \frac{\cos\theta' d\sigma'}{r^2}$$

$$d\omega' = \frac{\cos\theta d\sigma}{r^2}$$

$$\Rightarrow I_\nu = I'_\nu, \text{ independent of distance}$$

... but energy/unit area dilutes with r^{-2}!

specific intensity is **radiation energy** with frequencies (ν, ν + dν), which is transported through *projected* area element dσcosθ into direction **n**, per time interval dt and solid angle dω.

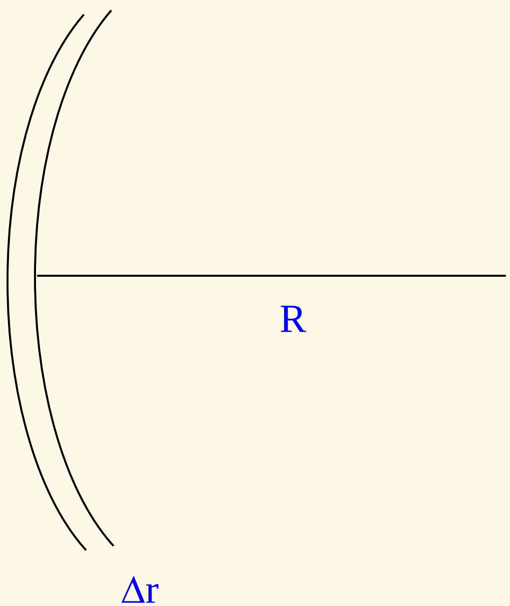
$$\delta E = I(\vec{r}, \vec{n}, \nu, t) \cos\theta \, d\sigma \, d\nu \, dt \, d\omega$$

Plane-parallel and spherical symmetries

stars = gaseous spheres => spherical symmetry

BUT: rapidly rotating stars (e.g., Be-stars, $v_{rot} \approx 300 \ldots 400$ km/s) rotationally flattened, only axis-symmetry can be used

AND: atmospheres usually very thin, i.e. Δr / R << 1



example: the sun

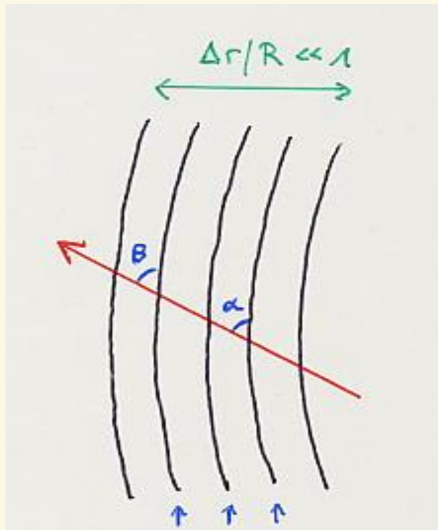
$R_{sun} \approx 700{,}000$ km
Δr (photo) ≈ 300 km

=> $\Delta r / R \approx 4\ 10^{-4}$

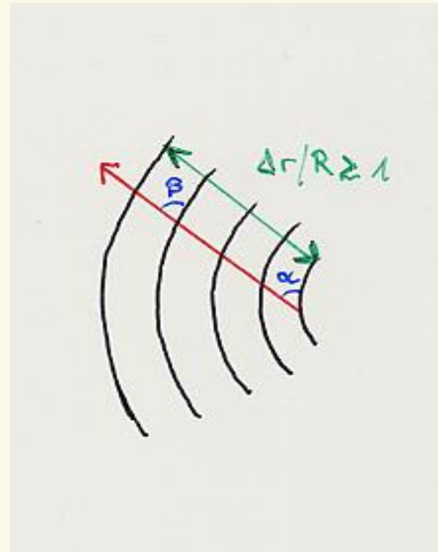
BUT corona
$\Delta r / R$ (corona) ≈ 3

as long as $\Delta r / R \ll 1$ => plane-parallel symmetry

light ray through atmosphere



lines of constant temperature
and density (isocontours)



curvature of atmosphere insignificant for photons' path : $\alpha = \beta$

significant curvature : $\alpha \neq \beta$,
spherical symmetry

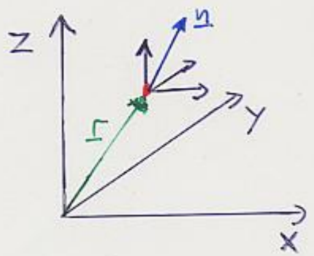
examples

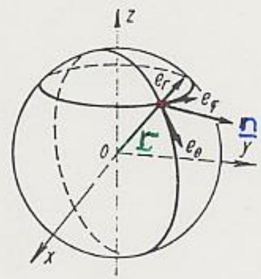
solar photosphere / cromosphere	solar corona
atmospheres of	atmospheres of
main sequence stars	supergiants
white dwarfs	expanding envelopes (stellar winds)
giants (partly)	of OBA stars, M-giants and supergiants

Co-ordinate systems / symmetries

Carthesian

spherical





$\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$ $\qquad$ $r = \Theta\underline{e}_\Theta + \Phi\underline{e}_\Phi + r\underline{e}_r$

$\underline{e}_x, \underline{e}_y, \underline{e}_z$ $\qquad$ right-handed orthonormal $\qquad$ $\underline{e}_\Theta, \underline{e}_\Phi, \underline{e}_r$

$I(x, y, z, \underline{n}, \nu, t)$ $\qquad$ $I(\Theta, \Phi, r, \underline{n}, \nu, t)$

important symmetries

plane-parallel $\qquad$ spherically symmetric

physical quantities depend only on z, e.g. $\qquad$ .... depend only on r, e.g.

$I(\underline{r}, \underline{n}, \nu, t)$ $\qquad$ $I(\underline{r}, \underline{n}, \nu, t)$

$\rightarrow I(z, \underline{n}, \nu, t)$ $\qquad$ $\rightarrow I(r, \underline{n}, \nu, t)$

intensity has direction $\underline{n}$ into $d\Omega$

$\underline{n}$ requires additional angles θ, ϕ

with respect to

$\underline{e}_x, \underline{e}_y, \underline{e}_z$ $\qquad$ $\underline{e}_\Theta, \underline{e}_\Phi, \underline{e}_r$

with

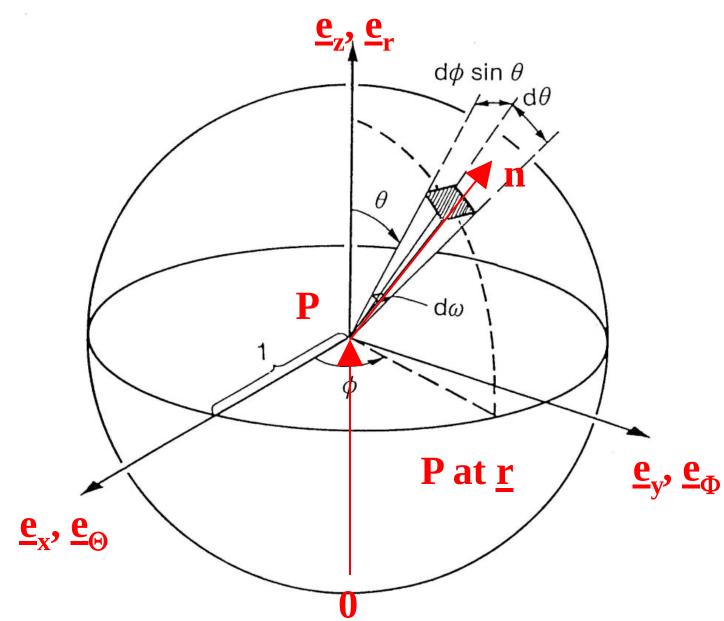
$\theta = \sphericalangle(\underline{e}_z, \underline{n})$ $\qquad$ $\theta = \sphericalangle(\underline{e}_r, \underline{n})$

p.p. symmetry $\qquad$ spherical sym.

$I_\nu(x, y, z, \theta, \phi)$ $\qquad$ $I_\nu(\Theta, \Phi, r, \theta, \phi)$

$\rightarrow I_\nu(z, \theta, \phi)$ $\qquad$ $\rightarrow I_\nu(r, \theta, \phi)$

$\rightarrow I_\nu(z, \theta)$ $\qquad$ $\rightarrow I_\nu(r, \theta)$



Moments of the specific intensity

1. mean intensity

$$J(\underline{r},\nu,t) = \frac{1}{4\pi}\oint I(\underline{r},\underline{n},\nu,t)\,d\Omega$$

specific intensity, averaged over solid angle

def. of solid angle

solid angle = ratio of area of sphere to radius^2

total solid angle $= \frac{4\pi R^2}{R^2} = 4\pi$

$d\Omega$ with $r=1$ $= dA$

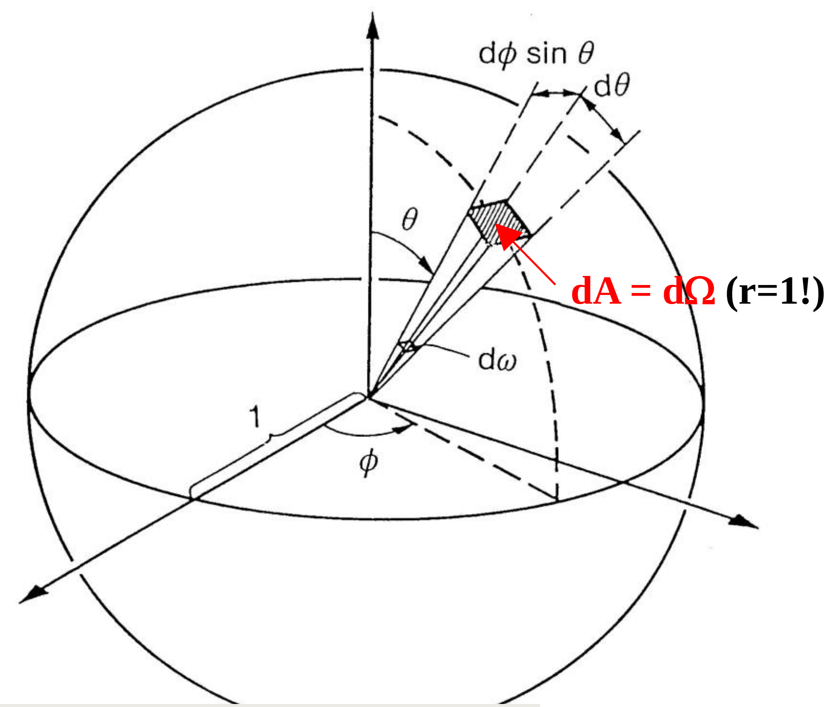
area $= d\theta \times \sin\theta\, d\phi$

def: $\mu =: \cos\theta$

$d\mu = -\sin\theta\, d\theta \Rightarrow d\Omega = -d\mu\, d\phi$

THUS

$$J(\underline{r},\nu,t) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{0\to +1}^{\pi \to -1} \underbrace{I(\underline{r},\underline{n},\nu,t)}_{\text{usually } f(\theta,\phi)}\underbrace{\sin\theta\, d\theta}_{-d\mu}$$



In plane-parallel or spherical symmetry:

$$J_\nu(r/z,t) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{+1} I_\nu(r/z,\mu,t)\,d\mu =$$

$$= \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\,d\mu \qquad \text{"0th" moment}$$

The Planck function

... on the other hand

energy density (i.e., per Volume $d^3\underline{r}$) per $d\nu$ (i.e., spectral) = $h\nu \oint$ (distr. function) $d\Omega$

$$u_\nu(\underline{r},t) = h\nu \oint \Psi_\nu(\underline{r},\mu,t)\, d\Omega$$

$$\overset{\text{def.}}{=} \frac{1}{c} \oint I_\nu(\underline{r},\mu,t)\, d\Omega = \frac{4\pi}{c} J_\nu(r,t)$$

$\dim[u_\nu] = \text{erg cm}^{-3}\,\text{Hz}^{-1}$

$\dim[J_\nu] = \text{erg cm}^{-2}\,\text{Hz}^{-1}\,\text{s}^{-1}$

- from thermodynamics, we know spectral energy density of a cavity or black body radiator (in thermodynamic equilibrium, "TE", with isotropic radiation, independent of material)

$$u_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT}-1}$$

$$\Rightarrow J_\nu = \frac{c}{4\pi} u_\nu \quad \text{and} \quad J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \, d\mu = I_\nu$$

isotropic

specific intensity of a cavity / black body radiator at temperature T

$$\boxed{I_\nu^{*} \overset{\text{T.E.}}{=} B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT}-1}}$$ "Planck-Function"

properties of Planck function

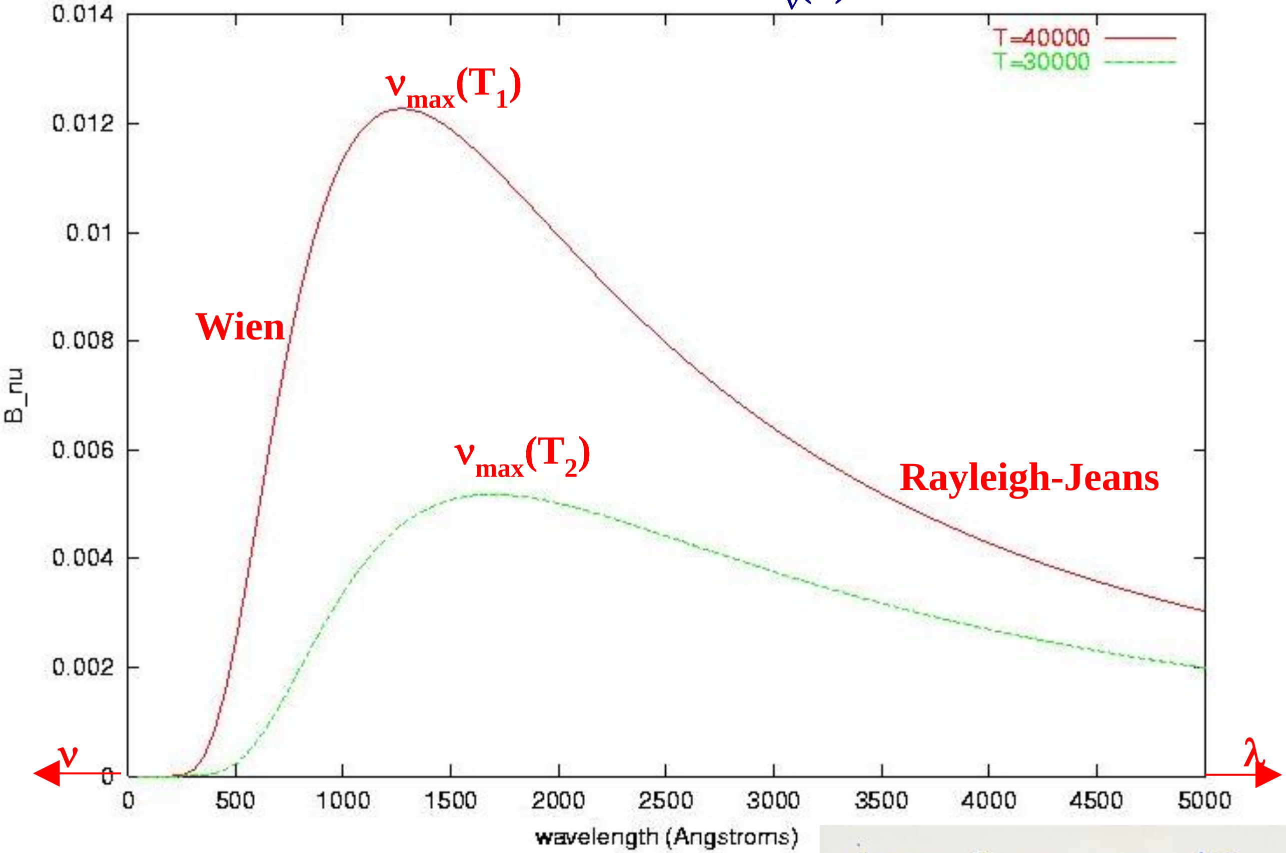
- $B_\nu(T_1) > B_\nu(T_2) \quad \forall \nu,$ if $T_1 > T_2$
  i.e., Planck functions do not cross each other!
- maximum is shifted towards higher wavelengths with decreasing temperature
  $\frac{\nu_{max}}{T}$ = const, Wien's displacement law
- Wien regime $\frac{h\nu}{kT} \gg 1 \Rightarrow B_\nu \approx \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$
- Rayleigh Jeans regime $\frac{h\nu}{kT} \ll 1 \Rightarrow B_\nu \approx \frac{2h\nu^3}{c^2}\frac{kT}{h\nu} = \frac{2\nu^2}{c^2} kT$

NOTE: in a number of cases one finds $B_\lambda \neq B_\nu$ since $B_\lambda d\lambda = B_\nu d\nu$

$$\Rightarrow B_\lambda = B_\nu \left|\frac{d\nu}{d\lambda}\right| = B_\nu \frac{c}{\lambda^2} = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/kT\lambda}-1}$$

$$\Rightarrow \text{Max}(B_\lambda) \neq \text{Max}(B_\nu)\,!$$

Planck function $B_\nu(\lambda)$



Max B_ν : $\nu_{MAX}/T = const^1 \Rightarrow \lambda^1_{MAX} = \frac{5.0995\cdot10^7}{T(K)}$ Å

Max B_λ : $\lambda_{MAX}\cdot T = const^2 \Rightarrow \lambda^2_{MAX} = \frac{2.898\cdot10^7}{T(K)}$ Å

- Stefan-Boltzmann law

$$\boxed{\int_0^\infty B_\nu(T)\,d\nu = \int_0^\infty B_\lambda(T)\,d\lambda = \frac{\sigma}{\pi}T^4} \quad \text{with}$$

$$\sigma = 5.67\cdot10^{-5}\ \frac{erg}{cm^2\,s\,K^4}, \quad \sigma/\pi = \frac{2k_B^4}{c^2h^3}\ \frac{\pi^4}{15}$$

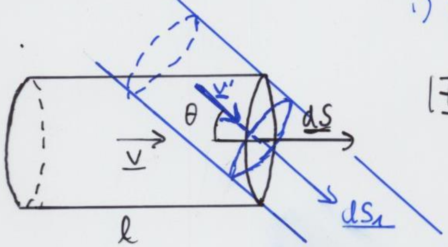
1st moment: radiative flux

a) general definition

flux: rate of flow of a quantity across a given surface

flux-density: flux / unit area, also called flux
vector quantity

i) mass flux $\underline{v} \parallel d\underline{S}$



$$|\underline{F}| = \frac{m}{\Delta t\, |d\underline{S}|} = \frac{m}{\mathrm{Vol}} \, \frac{l}{\Delta t} = \rho |\underline{v}|$$

mass flux = mass density · velocity

ii) $\underline{v}'$ arbitrarily oriented with respect to $d\underline{S}$

$$|\underline{F}| = \frac{m}{\Delta t\, |d\underline{S}|} = \frac{m}{\Delta t\, |d\underline{S}_\perp|} \, \frac{|d\underline{S}_\perp|}{|d\underline{S}|} = \frac{m}{\mathrm{Vol}} \, |\underline{v}'| \, \frac{|d\underline{S}| \cos\theta}{|d\underline{S}|}$$

$\mathrm{Vol} = |\underline{v}'| \, \Delta t \, |d\underline{S}_\perp|$

$$= \rho |\underline{v}'| \cos\theta$$

$\Rightarrow$ mass flux through $d\underline{S}$ $= \underline{F} \cdot d\underline{S} = \rho\, \underbrace{\underline{v}' \cdot d\underline{S}}_{|\underline{v}'|\,|d\underline{S}| \cdot \cos\theta}$

is reduced by factor $\cos\theta$, since less material is transported across smaller effective area ⊥ flow (in same Δt)

iii) mass-loss rate for spherically sym. outflow

$$\dot{M} = \underbrace{(\rho v)(r)}_{\text{mass flux}} \cdot \underbrace{4\pi r^2}_{\text{surface area}}$$

transported mass/unit time across surface with radius r

$\cos\theta = 1$!

b) application to radiation field

- photon flux through surface $d\underline{S}$ into direction $\underline{n}$ and solid angle $d\Omega$ ("radiation pencil")

$$\frac{\delta N}{dt\, d\nu} = \Big(\underbrace{\psi(\underline{r}, \underline{n}, \nu, t)\, d\Omega}_{\text{number DENSITY}} \cdot \underbrace{c \cdot \underline{n}}_{\text{velocity}} \Big) \cdot d\underline{S}$$

- net rate of total photon flow across $d\underline{S}$ (i.e., contribution of all pencils)

$$\frac{N}{dt\, d\nu} = \left(c \oint \psi(\underline{r}, \underline{n}, \nu, t)\, \underline{n}\, d\Omega \right) \cdot d\underline{S}$$

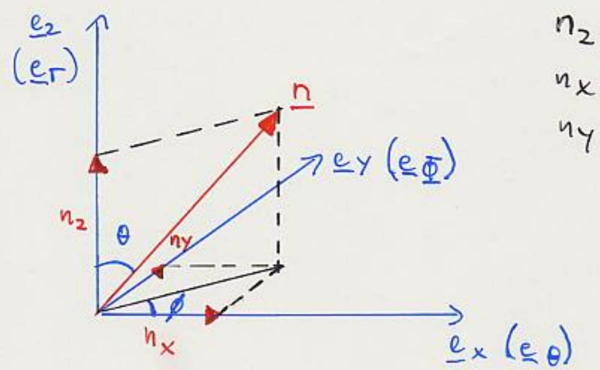
- net rate of radiant energy flow across $d\underline{S}$

$$\frac{E}{dt\, d\nu} = \left(c h\nu \oint \psi(\underline{r}, \underline{n}, \nu, t)\, \underline{n}\, d\Omega \right) d\underline{S} \overset{\text{def.}}{=} \left(\oint I(\underline{r}, \underline{n}, \nu, t)\, \underline{n}\, d\Omega \right) d\underline{S}$$

$$= \underline{F}_\nu(\underline{r}, t) \cdot d\underline{S}$$

$$\boxed{\underline{F}_\nu(\underline{r}, t) = \oint I_\nu(\underline{r}, \underline{n}, t)\, \underline{n}\, d\Omega}$$ radiative flux

$$\dim[\underline{F}_\nu] = \frac{\text{erg}}{\text{cm}^2\, \text{s}\, \text{Hz}} = \dim[J_\nu]$$



$n_z = \cos\theta = \mu$
$n_x = \sin\theta\cos\phi$
$n_y = \sin\theta\sin\phi$

Note: Carthesian / spherical co-ordinate system

$$\begin{pmatrix}\underline{e}_\theta\\ \underline{e}_\Phi\\ \underline{e}_r\end{pmatrix} \;\hat{=}\; (\text{locally}) \begin{pmatrix}\underline{e}_x\\ \underline{e}_y\\ \underline{e}_z\end{pmatrix},$$

θ, ϕ defined similarly

$$\Rightarrow \underline{\mathcal{F}} = \begin{pmatrix}\mathcal{F}_{x,\theta}\\ \mathcal{F}_{y,\Phi}\\ \mathcal{F}_{z,r}\end{pmatrix} = \oint \begin{pmatrix} I n_x d\Omega\\ I n_y d\Omega\\ I n_z d\Omega\end{pmatrix} = \int_0^{2\pi} d\phi \begin{pmatrix}\cos\phi\\ \sin\phi\\ 1\end{pmatrix} \int_{-1}^{1} d\mu \begin{pmatrix} I(1-\mu^2)^{\frac{1}{2}}\\ I(1-\mu^2)^{\frac{1}{2}}\\ I\mu\end{pmatrix}$$

p.p. / spherical symmetric

$I(\underline{r}, \underline{n}, \nu, t) \Rightarrow I\left(\begin{smallmatrix} r\\ z\end{smallmatrix}, \mu, \nu, t\right)$ independent of ϕ,

$x(\Theta)$, $y(\Phi)$ comp. cancel each other

(math: $\cos\phi$, $\sin\phi$ integrals $= 0$)

$$\Rightarrow \underline{\mathcal{F}} = \left(0, 0, 2\pi \int_{-1}^{+1} I\left(\begin{smallmatrix} r\\ z\end{smallmatrix}, \mu, \nu, t\right)\mu\, d\mu\right)$$

- in analogy to mean intensity $J_\nu = \frac{1}{2}\int_{-1}^{+1} I(\mu)\, d\mu$

  we define the Eddington flux

$$\boxed{H_\nu\left(\begin{smallmatrix} r\\ z\end{smallmatrix}, t\right) = \frac{1}{2}\int_{-1}^{+1} I_\nu\left(\begin{smallmatrix} r\\ z\end{smallmatrix}, \mu, t\right)\mu\, d\mu = \frac{1}{4\pi}\mathcal{F}_\nu\left(\begin{smallmatrix} r\\ z\end{smallmatrix}, t\right)}$$

  "first moment"

- "flux" from a cavity radiator

  small opening

$$\mathcal{F}_\nu = 2\pi\int_{-1}^{+1} I(\mu)\mu\, d\mu = 2\pi\int_0^1 I(\mu)\mu\, d\mu - 2\pi\int_0^1 I(-\mu)\mu\, d\mu$$

$$= \mathcal{F}^+ - \mathcal{F}^-$$

  only photons escaping from radiator

$$I(\mu),\ \mu = 0\ldots 1 = B_\nu(T) \quad \text{isotropic radiation}$$
$$I(-\mu) = 0$$

$$\Rightarrow \mathcal{F} = \int_0^\infty \pi B_\nu(T)\, d\nu = \pi\cdot\frac{\sigma_B}{\pi}T^4 = \sigma_B T^4$$

REMEMBER Black Body

frequ. integrated	specific and mean intensity	$\frac{\sigma_B}{\pi}T^4$
"	energy density	$\frac{4\sigma_B}{c}T^4$
"	flux	$\sigma_B T^4$

Effective temperature

- total radiative energy loss is
  flux (outwards directed) times surface area of star =

  luminosity $L = \mathscr{F}^+ 4\pi R^2$

  dim[L] = erg/s (units of power), $L_{sun}=3.83\ 10^{33}$ *erg/s*

- definition: "effective temperature" is temperature
  of a star with luminosity L at radius R_*,
  if it *were* a black body (semi-open cavity?)

- $T_{\rm eff}$ corresponds roughly to stellar surface temperature
  (more precise → later)

$$\boxed{L =: \sigma_B T_{\rm eff}^{\,4}\, 4\pi R^2 \quad \textit{or} \quad T_{\rm eff} = (L/\ \sigma_B 4\pi R^2)^{1/4}}$$

Examples

i) spherical symmetry, isotropic radiation

$I_\nu(\mu) = I_0 \quad (\text{e.g., } B_\nu(T))$

$$\Rightarrow J_\nu = \frac{1}{2}\int_{-1}^{1} I_0 d\mu = I_0$$

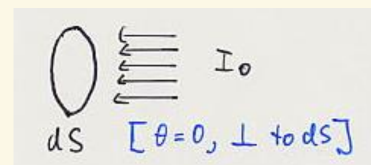
$$H_\nu = \frac{1}{2}\int_{-1}^{1} I_0 \mu d\mu = 0$$ [vanishing flux also in radial direction, since same number of photons from above and below surface $\perp$ radial direction]

THUS: $I_\nu = I_0 \;\Rightarrow\; J_\nu = I_0,\; H_\nu = 0$

ii) extremely anisotropic radiation (in carthesian co-ordinates)

$I_\nu(\mu,\phi) = I_0\delta(\mu-\mu_0)\delta(\phi-\phi_0)$, with Dirac δ-function [planar wave]

$$\Rightarrow J_\nu = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} I_0\delta(\mu-\mu_0)\delta(\phi-\phi_0)d\mu = \frac{I_0}{4\pi}$$



$$\mathbf{H}_\nu = \frac{\mathbf{F}_\nu}{4\pi} = \begin{pmatrix} \frac{1}{4\pi}\int_0^{2\pi}\cos\phi d\phi\int_{-1}^{1} I_0\delta(\mu-\mu_0)\delta(\phi-\phi_0)(1-\mu^2)^{1/2}d\mu \\ \frac{1}{4\pi}\int_0^{2\pi}\sin\phi d\phi\int_{-1}^{1} I_0\delta(\mu-\mu_0)\delta(\phi-\phi_0)(1-\mu^2)^{1/2}d\mu \\ \frac{1}{4\pi}\int_0^{2\pi} d\phi\int_{-1}^{1} I_0\delta(\mu-\mu_0)\delta(\phi-\phi_0)\mu d\mu \end{pmatrix} = \frac{1}{4\pi}\begin{pmatrix} I_0\cos\phi_0(1-\mu_0^2)^{1/2} \\ I_0\sin\phi_0(1-\mu_0^2)^{1/2} \\ I_0\mu_0 \end{pmatrix} \xrightarrow{\mu_0=1} \begin{pmatrix} 0 \\ 0 \\ I_0/4\pi \end{pmatrix}$$

Generally: $$\left|\mathbf{H}_\nu\right| = \frac{I_0}{4\pi}\sqrt{\cos^2\phi_0(1-\mu_0^2)+\sin^2\phi_0(1-\mu_0^2)+\mu_0^2} = \frac{I_0}{4\pi}$$

THUS: uni-directional radiation $\Rightarrow J_\nu = \left|\mathbf{H}_\nu\right|$ (independent of co-ordinate system)

iii) $\mathcal{F}_\nu^+ = 2\pi \int_0^1 I(\mu)\mu\, d\mu$ is stellar radiation energy, emitted into ALL directions (per $dS, d\nu, dt$)

$= \frac{d^2}{R_*^2} f_\nu$, if f_ν is the energy recieved on earth (per $dS, d\nu, dt$), d is the distance and $d \gg R_*$ [no extinction!]

proof if no extinction, totally emitted stellar energy remains conserved

$$L = \text{const} = \mathcal{F}_\nu^+(R_*) \cdot 4\pi R_*^2 \overset{!}{=} f_\nu^{obs}(d)\, 4\pi d^2$$

$$\Rightarrow f_\nu^{obs}(d) = \mathcal{F}_\nu^+(R_*) \frac{R_*^2}{d^2}$$ q.e.d.

("quadratic dilution")

iv) solar constant

total solar flux, measured on earth

$$\int_0^\infty f_\nu\, d\nu = f = 1.36 \cdot 10^6 \frac{\text{erg}}{\text{cm}^2\,\text{s}}$$

distance earth sun $\approx 1.5 \cdot 10^{13}$ cm

$R_\odot = 6.96 \cdot 10^{10}$ cm

$$\Rightarrow \mathcal{F}_\odot^+(R_\odot) = 6.3 \cdot 10^{10} \frac{\text{erg}}{\text{cm}^2\,\text{s}}$$

with def. of T_{eff}

$\sigma_B T_{eff}^4 = \mathcal{F}^+ \longrightarrow T_{eff} = 5777\,K$

$\longrightarrow B_\nu^{max}$ at $\lambda = 8826$ Å

B_λ^{max} at $\lambda = 5020$ Å

v) exercise

How many $L_\odot$ is emitted by a typical O-supergiant with $T_{eff} = 40{,}000$ K and $R_* = 20 R_\odot$? Where is its spectral maximum?

2^{nd} moment: radiation pressure (stress) tensor

P_{ij} is net flux of momentum, in the j-th direction, through a unit area oriented perpendicular to the ith direction (per unit time and frequency)

- this is just the general definition of "pressure" in any fluid

$$P_{ij}(\underline{r},\nu,t) = \oint \underbrace{\psi(\underline{r},\underline{n},\nu,t)\left(\frac{h\nu}{c}n_j\right)}_{\substack{\text{transported quantity}\\ = \text{distrib. function} \cdot \text{momentum}}}\underbrace{(c\cdot n_i)}_{\text{velocity}}d\Omega$$

$$\stackrel{\text{def}}{=} \frac{1}{c}\oint I(\underline{r},\underline{n},\nu,t)\, n_i n_j\, d\Omega$$

- $P_{ij} = P_{ji}$ generally
- NOW p-p/sph. symmetry
  from def. of n_i, $i=1,3$ $\quad P_{ij}=0$ for $i\neq j$

$$P = \begin{pmatrix} p_R & 0 & 0 \\ 0 & p_R & 0 \\ 0 & 0 & p_R \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 3p_R-u & 0 & 0 \\ 0 & 3p_R-u & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to

$(\underline{e}_x, \underline{e}_y, \underline{e}_z)$ or $(\underline{e}_\Theta, \underline{e}_\Phi, \underline{e}_r)$

$p_R = \frac{4\pi}{c}K$ radiation pressure scalar

$u = \frac{4\pi}{c}J$ radiation energy density

$K_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\underline{r},\mu,t)\,\mu^2 d\mu$ "2nd moment"

Note: In p-p/spherical symmetry the radiation pressure tensor is described by only two scalar quantities!

isotropic radiation ($\rightarrow$ stellar interior) cavity radiation

$I_\nu(r,\mu,t) \rightarrow I_\nu(r,t)$

$$\left.\begin{aligned} K &= \frac{I}{2}\int_{-1}^{+1}\mu^2 d\mu \\ J &= \frac{I}{2}\int_{-1}^{+1} d\mu \end{aligned}\right\} \quad K = \frac{1}{3}J \quad \text{or} \quad p_R = \frac{1}{3}u$$

$$\Rightarrow \quad P_\nu = \begin{pmatrix} p_R & 0 & 0 \\ 0 & p_R & 0 \\ 0 & & p_R \end{pmatrix}$$

ONE quantity sufficient

Advanced Reading

divergence of radiation pressure tensor

gas pressure → pressure force $\sim -\nabla p$

here: radiative acceleration = volume forces exerted by radiation field

$$(\underline{\nabla}\cdot\underline{\underline{P}})_i = \sum_j \frac{\partial}{\partial x_j} P_{ij}$$ ith component of divergence (Cartesian)

- p-p symmetry $\quad p_R, u = f(z)$

  only $\frac{\partial}{\partial z} \neq 0 \Rightarrow$

  $$(\underline{\nabla}\cdot\underline{\underline{P}})_z = \frac{\partial p_R(z,\nu,t)}{\partial z}$$

- spherical symmetry

  only $(\underline{\nabla}\cdot\underline{\underline{P}})_r$ has non-vanishing component

  $$(\underline{\nabla}\cdot\underline{\underline{P}})_r = \frac{\partial p_R}{\partial r} + \frac{1}{r}(3p_R - u)$$

  so far, this is the only expression which is different in p-p and spherical symmetry!

For symmetric tensors T^{ij} $(i, j = \Theta, \Phi, r)$ one can prove the following relations (e.g., Mihalas & Weibel Mihalas, "Foundations of Radiation Hydrodynamics", Appendix)

$$(\nabla\cdot T)_r = \frac{1}{r^2}\frac{\partial(r^2 T^{rr})}{\partial r} + f(T^{r\Theta}) + f(T^{r\Phi}) - \frac{1}{r}(T^{\Theta\Theta} + T^{\Phi\Phi})$$

$$(\nabla\cdot T)_\Theta = \frac{1}{r}\left\{ f(T^{r\Theta}) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta T^{\Theta\Theta})}{\partial\theta} + f(T^{\Theta\Phi}) + \frac{1}{r}(T^{r\Theta} - \cot\theta T^{\Phi\Phi})\right\}$$

$$(\nabla\cdot T)_\Phi = \frac{1}{r\sin\theta}\left\{ f(T^{r\Phi}) + f(T^{\Theta\Phi}) + \frac{1}{r\sin\theta}\frac{\partial T^{\Phi\Phi}}{\partial\phi} + f(\cot\theta T^{\Theta\Phi})\right\}$$

where f are (different) functions of the tensor-elements which are not relevant here.

Since in spherical symmetry the radiation pressure tensor P is diagonal (i.e., symmetric), and since p_R and u are functions of r alone, we have

$$(\nabla\cdot P)_r = \frac{1}{r^2}\left(2rP^{rr} + r^2\frac{\partial P^{rr}}{\partial r}\right) - \frac{1}{r}(P^{\Theta\Theta} + P^{\Phi\Phi}) = \frac{\partial P^{rr}}{\partial r} + \frac{1}{r}(2P^{rr} - P^{\Theta\Theta} - P^{\Phi\Phi})$$

(which in the isotropic case would yield $(\nabla\cdot P)_r = \frac{\partial P^{rr}}{\partial r} = \frac{\partial p_R}{\partial r}$)

$$(\nabla\cdot P)_\Theta = \frac{1}{r^2\sin\theta}\left(\cos\theta P^{\Theta\Theta} + \sin\theta\frac{\partial T^{\Theta\Theta}}{\partial\theta}\right) - \frac{1}{r^2}\cot\theta P^{\Phi\Phi} \to 0 \text{ (in spherical symmetry)}$$

$(\nabla\cdot P)_\Phi \to 0$ (in spherical symmetry).

Finally, we obtain

$$(\nabla\cdot P) \to (\nabla\cdot P)_r = \mathbf{e_r}\cdot\left\{\frac{\partial p_R}{\partial r} + \frac{1}{r}\left(2p_R - 2\left(p_R - \frac{1}{2}(3p_R - u)\right)\right)\right\} =$$

$$= \mathbf{e_r}\cdot\left(\frac{\partial p_R}{\partial r} + \frac{1}{r}(3p_R - u)\right), \text{ q.e.d.}$$

The equation of radiative transfer

- had Boltzmann eq. for particle distrib. function f

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla + \underline{F}\cdot\nabla_p\right) f = \left(\frac{\delta f}{\delta t}\right)_{coll}$$

for photons $v = c\cdot\underline{n}$, $\underline{F} \equiv 0$ without GR

$$\Rightarrow \left(\frac{\partial}{\partial t} + c\underline{n}\cdot\nabla\right)\Psi_\nu = \left(\frac{\delta\Psi_\nu}{\delta t}\right)_{coll} \leftarrow$$ photon creation/destr. along path in phase space

with

$$\Psi_\nu(\underline{r},\underline{n},t)\, d^3\underline{r}\, d\nu\, d\Omega = f(\underline{r},\underline{p},t)\, d^3\underline{r}\, d^3\underline{p}$$

and

$$\left(\frac{\partial}{\partial t} + c\cdot\underline{n}\cdot\nabla\right)\frac{I_\nu}{ch\nu} = \frac{1}{ch\nu}\left(\frac{\delta I_\nu}{\delta t}\right)_{\text{"coll"}}$$

$$\Rightarrow \left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\nabla\right) I_\nu = \left(\frac{\delta I_\nu}{ds}\right)_{\text{"coll"}} = \frac{\delta I_\nu^{em} - \delta I_\nu^{abs}}{ds}$$

with

$$I_\nu = ch\nu\,\Psi_\nu, \quad ds = c\cdot\delta t$$

gain/loss by interaction with matter

Equation of radiative transfer for specific intensity

Emissivity and opacity

a) vacuum

→ no "collisions" → Vlasov equation

$$\Rightarrow \left[\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\nabla\right] I = 0$$

stationary

$$(\underline{n}\cdot\nabla)\, I = \frac{d}{ds} I = 0 \Rightarrow I = \text{const} \quad (\text{cf. Chap 3})$$

directional derivative

b) energy gain by emission

add energy to ray (matter in dV radiates) by emission / photon creation

$$\delta E_\nu^{+} = \delta E_\nu^{em} \stackrel{def}{=} \eta_\nu(\underline{r},\underline{n},t)\, dV\, d\Omega\, d\nu\, dt$$

$$= \eta_\nu(\underline{r},\underline{n},t)\, \underbrace{\underbrace{\underline{n}\cdot d\underline{S}}_{\cos\theta\, dS}\cdot ds}_{dV}\, d\Omega\, d\nu\, dt$$

compare with def. of specific energy

$$\delta E_\nu = I_\nu(\underline{r},\underline{n},t)\cos\theta\, dS\, d\Omega\, d\nu\, dt$$

$$\Rightarrow \delta I_\nu^{em} = \eta_\nu\, ds$$ macroscopic emission coefficient

$$\dim[\eta_\nu] = \text{erg cm}^{-3}\,\text{sr}^{-1}\,\text{Hz}^{-1}\,\text{s}^{-1}$$

c) energy loss by absorption

remove energy from ray (matter in dV absorbs) by absorption / photon distruction

NOTE i) energy gain / emission property of interacting matter

ii) BUT: energy loss must depend on properties of matter and radiation, since

no radiation field $\Rightarrow$ no loss

no matter $\Rightarrow$ no loss

THUS following definition

$$\delta E_\nu^- = \delta E_\nu^{abs} = (\chi_\nu I_\nu)(\underline{r}, \underline{n}, t) \cos\theta \, dS \, ds \, d\Omega \, d\nu \, dt$$

$$\delta I_\nu^{abs} = \chi_\nu I_\nu ds$$

χ_ν absorption coefficient or opacity

$\dim[\chi_\nu] = cm^{-1}$

d) optical depth

define $d\tau_\nu = \chi_\nu ds \rightarrow \tau_\nu(s) = \int_0^s \chi_\nu(s) ds$

$\delta I_\nu^{abs} = I_\nu d\tau_\nu$ the higher τ_ν the more is absorbed

$\dim[\tau_\nu]$ dimensionless

interpretation later

e) emission and absorption in parallel

$$\left(\frac{\delta I_\nu}{ds}\right)_{cou} = \frac{\delta I_\nu^{em} - \delta I_\nu^{abs}}{ds} = \eta_\nu - \chi_\nu I_\nu$$

$\Rightarrow$ finally

$$\boxed{\left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n} \cdot \underline{\nabla}\right) I_\nu = \eta_\nu - \chi_\nu I_\nu}$$

η_ν, χ_ν depend on microphysics of interacting matter

NOTE • in static media η_ν, χ_ν (mostly) isotropic

• in moving media: Dopplereffect

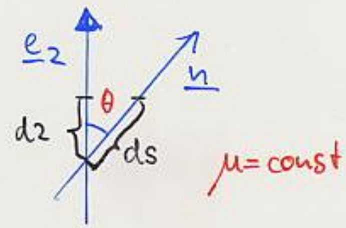
matter "sees" light at frequencies different than the observer $\Rightarrow$ dependency on angle

The equation of transfer for specific geometries

a) plane-parallel symmetry

$$dz = \mu \, ds$$

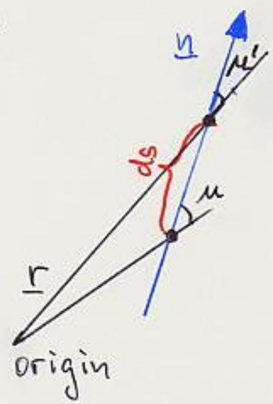
$$\rightarrow (\underline{n} \cdot \underline{\nabla}) = \frac{d}{ds} = \mu \frac{d}{dz}$$



$$\boxed{\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial z}\right) I_\nu(z, \mu, t) = \eta_\nu - \chi_\nu I_\nu}$$

b) spherical symmetry

along ds, $\mu \neq$ const



without proof

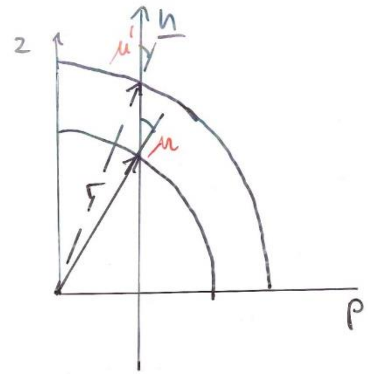
$$(\underline{n} \cdot \underline{\nabla}) = \frac{d}{ds} = \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu}$$

$$\boxed{\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\right) I_\nu(r, \mu, t) = \eta_\nu - \chi_\nu I_\nu}$$

c) in general

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \Theta}, \frac{\partial}{\partial \Phi}, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \phi}\right] I_\nu(\Theta, \Phi, r, \mu, \phi, t)$$

The equation of transfer (cont'd)



so-called p-z geometry

$$\frac{d}{ds} = \left.\frac{d}{dz}\right|_p$$

mit $r^2 = z^2 + p^2$

$\mu = \frac{z}{r}$

$$\Rightarrow \frac{d}{ds} = \left.\frac{d}{dz}\right|_p = \left.\frac{\partial r}{\partial z}\right|_p \frac{\partial}{\partial r} + \left.\frac{\partial \mu}{\partial z}\right|_p \frac{\partial}{\partial \mu}$$

$$r^2 = z^2 + p^2 \rightarrow \left.\frac{\partial r}{\partial z}\right|_p = \frac{z}{r} = \mu$$

$$\mu = \frac{z}{(z^2+p^2)^{\frac{1}{2}}} \rightarrow \left.\frac{\partial \mu}{\partial z}\right|_p = \frac{1}{r} - \frac{z^2}{r^3} = \frac{1}{r}(1-\mu^2)$$

$$\Rightarrow \underline{n}\cdot\underline{\nabla} = \frac{d}{ds} = \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}$$

$$\left\| \left(\frac{1}{c}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\right) I_\nu(r,\mu,t) = \eta_\nu - \chi_\nu I_\nu \right\|$$

General (without proof) for θ, Φ, r

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial r} + \frac{\gamma}{r}\frac{\partial}{\partial \theta} + \frac{\sigma}{r\sin\theta}\frac{\partial}{\partial \Phi} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu} - \frac{\sigma\cot\theta}{r}\frac{\partial}{\partial \phi}\right) I_\nu(\theta,\Phi,r,\mu,\phi,\nu,t) = \eta_\nu - \chi_\nu I_\nu$$

mit $\gamma = \cos\phi\sin\theta$

$\sigma = \sin\phi\sin\theta$

Source function and Kirchhoff-Planck law

Source function

transfer equation

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\underline{\nabla}\right) I_\nu = \eta_\nu - \chi_\nu I_\nu \quad \Big| \frac{1}{\chi_\nu}$$

now: stationary, $d\tau_\nu = \chi_\nu ds$, $\frac{\partial}{\partial s} = \underline{n}\cdot\underline{\nabla}$

$$\Rightarrow \frac{d}{\chi_\nu ds} I_\nu = \frac{d}{d\tau_\nu} I_\nu = \frac{\eta_\nu}{\chi_\nu} - I_\nu \stackrel{\text{def}}{=} S_\nu - I_\nu$$

compact form of transfer equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu \quad \text{with source function } S_\nu$$

- valid in any geometry, if stationary + $\frac{d}{d\tau_\nu} = \frac{\underline{n}\cdot\underline{\nabla}}{\chi_\nu}$

physical interpretation

- later we will show that mean free path of photons corresponds to $\tau_\nu = 1$

$$\Rightarrow 1 \approx \chi_\nu \Delta s, \quad \Delta s = \frac{1}{\chi_\nu}$$

$$\Rightarrow S_\nu = \frac{\eta_\nu}{\chi_\nu} = \eta_\nu \Delta s$$

source function corresponds to emitted intensity δI_ν^{em} over mean free path

Kirchhoff-Planck law

- assume thermodynamic equilibrium (TE)

$\rightarrow$ radiation field homogeneous, stationary

$$\rightarrow \left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\underline{\nabla}\right) =: 0$$

intensity Planck-function

$$\rightarrow 0 = I_\nu - S_\nu = B_\nu - S_\nu$$

TE: $\boxed{S_\nu^* = \frac{\eta_\nu^*}{\chi_\nu^*} = B_\nu(T)}$ $\leftarrow$ Kirchhoff-Planck law

or other way round

TE: $\eta_\nu^* = \chi_\nu^* B_\nu(T)$ [only one quantity to be specified]

True absorption and scattering

"true" absorption processes:

radiation energy => thermal pool
if not TE, temperature T(r) is changed
examples: photo-ionization
bound-bound absorption with subsequent collisional de-excitation

scattering:

no interaction with thermal pool
absorbed photon energy is directly reemitted (as photon)
no influence on T(r)
But direction $\underline{n}$ -> $\underline{n}'$ is changed (change in frequency mostly small)

examples: Thomson scattering at free electrons
Rayleigh scattering at atoms and molecules
resonance line scattering

ESSENTIAL POINT

true processes:

localized interaction with thermal pool,
drive physical conditions into local equilibrium
often (e.g., in LTE – page 116): $\eta_\nu(\text{true}) = \kappa_\nu B_\nu(T)$

scattering processes:

(almost) no influence on local thermodynamic properties of plasma
propagate information of radiation field (sometimes over large distances)
η_ν (Thomson) = $\sigma_{TH} J_\nu$ (-> next page)

Thomson scattering

- limiting case for long wavelengths of Klein-Nishima scattering
- almost freq. independent
- major source of scattering opacity in hot stars (as long as enough free electrons and hydrogen ionized)
- dipol characteristics not important, isotropic approximation sufficient

$$\sigma_\nu(\underline{r}, \mu) \rightarrow \sigma(\underline{r}) = n_e(\underline{r})\,\sigma_e,$$

$$\sigma_e = \frac{8\pi e^4}{3 m_e^2 c^4} = 6.65 \cdot 10^{-25}\,\mathrm{cm}^2$$

$$\eta^{TH} = \sigma_e\, n_e(\underline{r}) \cdot J_\nu(\underline{r})$$ (without proof)

"coherent scattering", $\nu_{abs} = \nu_{em}$

Total continuum opacity/source function

$$\chi_\nu = \kappa_\nu^t + \sigma_\nu \qquad (t = \text{true})$$

$$\eta_\nu = \kappa_\nu^t B_\nu(T) + \sigma_\nu J_\nu$$

$$\rightarrow S_\nu^{cont} = \frac{\kappa_\nu^t B_\nu + \sigma_\nu J_\nu}{\kappa_\nu^t + \sigma_\nu} \xrightarrow{\text{Th.scat}} (1 - \xi_\nu^{TH}) B_\nu + \xi_\nu^{TH} J_\nu$$

$$\xi_\nu^{TH} = \frac{\sigma_e n_e}{\kappa_\nu^t + \sigma_e n_e}$$

Moments of the transfer equation

transfer equation ($\equiv$ Boltzmann equation with $\underline{F} \equiv 0$)

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\cdot\underline{\nabla}\right) I_\nu = \eta_\nu - \chi_\nu I_\nu$$

0th moment: $\oint d\Omega$

note: $\underline{n}$ commutes with $\frac{\partial}{\partial t}$, $\underline{\nabla}$, since (t, $\underline{r}$, $\underline{n}$ independent variables here)

- integrate transfer equation over $d\Omega$

$$\frac{4\pi}{c}\frac{\partial}{\partial t} J_\nu + \underline{\nabla}\cdot\underline{\mathcal{F}}_\nu = \oint(\eta_\nu - \chi_\nu I_\nu)\, d\Omega$$

- if χ_ν, η_ν isotropic, i.e., no velocity fields $\rightarrow \; = 4\pi(\eta_\nu - \chi_\nu J_\nu)$

- Now frequency integration

$$\frac{4\pi}{c}\frac{\partial}{\partial t} J(\underline{r},t) + \underline{\nabla}\cdot\underline{\mathcal{F}}(\underline{r},t) = \int_0^\infty d\nu \oint(\eta_\nu - \chi_\nu I_\nu)\, d\Omega$$

total rad. energy added and removed

- IF energy transported by radiation alone (i.e., no convection) and no energy is created (which is true for stellar atmospheres)

$\Rightarrow$

$$\int_0^\infty d\nu \oint(\eta_\nu - \chi_\nu I_\nu)\, d\Omega = 0 \qquad \text{"radiative equilibrium"}$$

static atm. $\Rightarrow$

$$\boxed{\int_0^\infty d\nu\,(\eta_\nu - \chi_\nu J_\nu) = \int_0^\infty d\nu\, \chi_\nu (S_\nu - J_\nu) = 0}$$

- if radiation field time independent

$$\boxed{\underline{\nabla}\cdot\underline{\mathcal{F}} = 0} \qquad \text{"flux conservation"}$$

pp: $\frac{L}{4\pi R_*^2} = \mathcal{F}(z) = \text{const}$

spherical: $r^2\mathcal{F}(r) = \text{const} = \frac{L}{4\pi}$

- radiative equilibrium and flux conservation equivalent formulations, are used to calculate $T(r)$

- frequency dependent equations, stationary and static

$$\frac{\partial H_\nu}{\partial z} = \eta_\nu(z) - \chi_\nu J_\nu(z) \qquad \text{p-p}$$

$$\frac{1}{r^2}\frac{\partial(r^2 H_\nu)}{\partial r} = \eta_\nu(r) - \chi_\nu J_\nu(r) \qquad \text{spherical}$$

1st moment: $\oint \underline{n}\, d\Omega / c$

$$\oint \frac{d\Omega}{c}\left(\underline{n}\frac{1}{c}\frac{\partial}{\partial t} + \underline{n}\ \underline{n}\cdot\underline{\nabla}\right) I_\nu = \frac{1}{c}\oint (\eta_\nu - \chi_\nu I_\nu)\underline{n}\, d\Omega$$

$\Rightarrow$

$$\frac{1}{c^2}\frac{\partial}{\partial t}\underline{\mathcal{F}}_\nu + \underline{\nabla}\cdot \underset{\uparrow}{\mathbb{P}_\nu} = \frac{1}{c}\oint(\eta_\nu - \chi_\nu I_\nu)\underline{n}\, d\Omega$$

Tensor, cf. Chap. 3

frequency integrated analogous

- can be shown

$\frac{1}{c}\int_0^\infty d\nu \oint \chi_\nu I_\nu \underline{n}\, d\Omega$ is force/Volume, by radiation on matter

$= \underline{f}_{rad}(\underline{r})$ (momentum transfer photons $\rightarrow$ matter via absorption)

"radiation force"

$$\frac{\text{force}}{\text{volume}}\cdot\frac{1}{\rho} = \frac{\text{force}}{\text{mass}} = \underline{g}_{rad} \quad \text{"radiative acceleration"}$$

and

$\int d\nu \oint \eta_\nu \underline{n}\, d\Omega = 0$ because of fore/aft symmetry of emission process (even in v-fields)

- in total

$$\frac{1}{c^2}\frac{\partial}{\partial t}\underline{\mathcal{F}}(\underline{r},t) + \underline{\nabla}\cdot\mathbb{P}(\underline{r},t) = -\frac{1}{c}\int d\nu \oint \chi_\nu I_\nu \underline{n}\, d\Omega$$
$$= -\rho\, \underline{g}_{rad}(\underline{r})$$

- stationary

$$\underline{\nabla}\cdot\mathbb{P}(\underline{r}) = -\rho(\underline{r})\underline{g}_{rad}(\underline{r}) = -\frac{1}{c}\int_0^\infty d\nu \oint d\Omega (\chi_\nu I_\nu)\underline{n}$$

- static $\rightarrow -\frac{1}{c}\int_0^\infty d\nu\, \chi_\nu \underline{\mathcal{F}}_\nu(\underline{r})$

1-D $\rightarrow$
$$\boxed{g_{rad}\binom{r}{z} = \frac{4\pi}{c\rho\binom{r}{z}}\int_0^\infty d\nu\, \chi_\nu\binom{r}{z} H_\nu\binom{r}{z}}$$

- frequency dependent equations, stationary and static

$$\underline{\nabla}\cdot\mathbb{P}_\nu = -\frac{1}{c}\chi_\nu \underline{\mathcal{F}}_\nu$$

p-p $\rightarrow$ $\frac{\partial p_R(z,\nu)}{\partial z} = -\frac{1}{c}\chi_\nu(z)\mathcal{F}_\nu(z)$ or $\boxed{\frac{\partial K_\nu(z)}{\partial z} = -\chi_\nu(z)H_\nu(z)}$

sph. $\rightarrow$ $\frac{\partial p_R(r,\nu)}{\partial r} + \frac{3p_R - u_\nu}{r} = -\frac{1}{c}\chi_\nu(r)\mathcal{F}_\nu(r)$ or

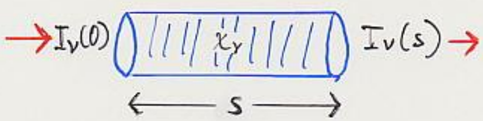
$$\frac{\partial K_\nu(r)}{\partial r} + \frac{3K_\nu(r) - J_\nu(r)}{r} = -\chi_\nu(r)H_\nu(r)$$

The change in radiative pressure drives the flux!

Pure absorption and optical depth

- from here on, stationary description (→ stellar atmospheres)
- radiative transfer without emission

$$\frac{dI_\nu}{ds} = -\chi_\nu I_\nu$$



$$\frac{dI_\nu}{I_\nu} = -\chi_\nu(s)\,ds$$

$$\ln I_\nu(s) - \ln I_\nu(0) = -\int_0^s \chi_\nu(s')\,ds'$$

$$I_\nu(s) = I_\nu(0)\, e^{-\int_0^s \chi_\nu(s')ds'} = I_\nu(0)\, e^{-\tau_\nu(s)}$$

optical depth, central quantity (more precisely: optical thickness)

or

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu}$$

- since $I_\nu \sim e^{-\tau_\nu}$, we look only until $\tau_\nu = 1$ (freq. dep.!)
- Question: What is the average distance over which photons travel?

Answer: $\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu \, p(\tau_\nu)\, d\tau_\nu$

expectation value — probability density function

$p(\tau_\nu)\,d\tau$ gives probability, that photon is absorbed in interval $\tau_\nu, \tau_\nu + d\tau_\nu$

- is probability, that photon is NOT absorbed between $0, \tau_\nu$ and then absorbed between $\tau_\nu, \tau_\nu + d\tau_\nu$

a) prob., that photon is absorbed

$$P(0,\tau_\nu) = \frac{\Delta I(\tau)}{I_0} = \frac{I_0 - I(\tau_\nu)}{I_0} = 1 - \frac{I(\tau_\nu)}{I_0}$$

b) prob, that photon is not absorbed

$$1 - P(0,\tau_\nu) = \frac{I(\tau_\nu)}{I_0} = e^{-\tau_\nu}$$

c) prob., that photon is absorbed in $\tau_\nu, \tau_\nu + d\tau_\nu$

$$P(\tau_\nu, \tau_\nu + d\tau_\nu) = \left|\frac{dI(\tau_\nu)}{I(\tau_\nu)}\right| = d\tau_\nu$$

d) total probability is $e^{-\tau_\nu}\, d\tau_\nu$

THUS

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu e^{-\tau_\nu}\, d\tau_\nu = \underline{1}$$

mean free path $\bar{s}$ corresponds to $\langle \tau_\nu \rangle = 1$

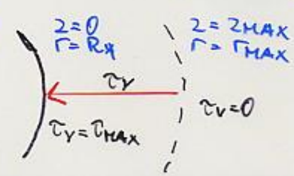
$$\Delta\tau_\nu = \chi_\nu \Delta s \quad \rightarrow \quad \underbrace{\Delta s}_{=\bar{s}} = \frac{1}{\chi_\nu}, \text{ q.e.d.}$$

USUAL convention

- Since we "measure" from outside to inside, $\tau_\nu = 0$ is defined at outer "edge" of atmosphere

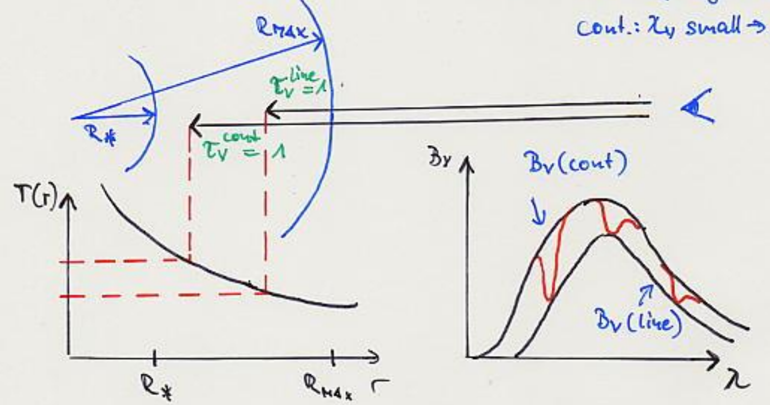
$$\Rightarrow ds = -\,dz \quad (\text{or } -dr)$$

$$\Rightarrow \underbrace{d\tau_\nu}_{>0!} = -\chi_\nu \underbrace{\begin{pmatrix} dz \\ dr \end{pmatrix}}_{<0!}$$



Formation of spectral lines: the principle

- look always down to $\tau_\nu \approx 1$
- BUT line: χ_ν large $\rightarrow \bar{s}$ small
  cont.: χ_ν small $\rightarrow \bar{s}$ large



$$T(r_{cont}) > T(r_{line})\ !$$

"Formal solution"

solve eq. of RT with known source function

- pp geometry

$$\mu \frac{dI_\nu}{dz} = \eta_\nu - \chi_\nu I_\nu$$

$$\Rightarrow \mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \qquad (\tau_\nu = 0 \text{ outside!})$$

- solution with integrating factor $e^{-\tau_\nu/\mu}$

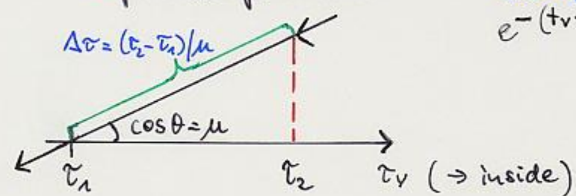
multiply equation, integrate between τ_1 and τ_2
τ_2 (inside) $> \tau_1$ (outside)

$\Rightarrow$

$$I_\nu(\tau_1,\mu) = \underbrace{I_\nu(\tau_2,\mu)\, e^{-(\tau_2-\tau_1)/\mu}}_{>0\ \checkmark} + \underbrace{\int_{\tau_1}^{\tau_2} S_\nu(t_\nu)\, e^{-(t_\nu-\tau_1)/\mu} \frac{dt_\nu}{\mu}}_{>0\ \checkmark}$$

intensity "emitted" at τ_2, loss (abs) by factor $e^{-\Delta\tau}$
$\hat{=}$ pure absorption case

gain by emission with subsequent absorption
$e^{-(t_\nu-\tau_1)/\mu}$



Boundary conditions

a) incident intensity from inside
$\mu > 0$ at $\tau_2 = \tau_{max}$

- either $I_\nu(\tau_2 = \tau_{max}, \mu) = I_\nu^+(\mu)$ (e.g., from diffusion approx)
- or "semi-infinite" atmosphere
  $\tau_2 = \tau_{max} \rightarrow \infty$ with $\lim_{\tau_\nu \to \infty} I_\nu(\tau_\nu,\mu)\, e^{-\tau_\nu/\mu} = 0$
  ($I_\nu(\tau_\nu,\mu)$ increases slower than exp.)

$$\Rightarrow I_\nu(\tau_\nu,\mu) = \int_{\tau_\nu}^{\infty} S_\nu(t)\, e^{-(t-\tau_\nu)/\mu} \frac{dt}{\mu} \qquad \mu > 0$$

b) incident intensity from outside
$\mu < 0$ at $\tau_\nu = 0$

- usually $I_\nu(0,\mu) = 0$ no irradiation from outside (however, binaries!)

$$\Rightarrow I_\nu(\tau_\nu,\mu) = \int_{\tau_\nu}^{0} S_\nu(t)\, e^{-(t-\tau_\nu)/\mu} \frac{dt}{\mu} \quad \mu < 0$$

$$= \int_{0}^{\tau_\nu} S_\nu(t)\, e^{-(\tau_\nu - t)/(-\mu)} \frac{dt}{(-\mu)} \quad (-\mu) > 0$$

c) emergent intensity = observed intensity (if no extinction)

$\tau_\nu = 0$, $\mu > 0$

$$I_\nu^{em}(\mu) = \int_0^\infty S_\nu(t)\, e^{-t/\mu} \frac{dt}{\mu}$$

emergent intensity is Laplace-transformed of source function!

NOW: suppose that S_ν is linear in τ_ν, i.e.,

$S_\nu(\tau_\nu) = S_{\nu 0} + S_{\nu 1} \cdot \tau_\nu$ (Taylorexpansion around $\tau_\nu = 0$)

$$\rightarrow I_\nu^{em}(\mu) = \int_0^\infty (S_{\nu 0} + S_{\nu 1} \cdot t)\, e^{-t/\mu} \frac{dt}{\mu} = \dots\dots$$

$$= S_{\nu 0} + S_{\nu 1} \cdot \mu = S_\nu(\tau_\nu = \mu)$$

Eddington-Barbier-relation

$$I_\nu^{em}(\mu) \approx S_\nu(\tau_\nu = \mu)$$

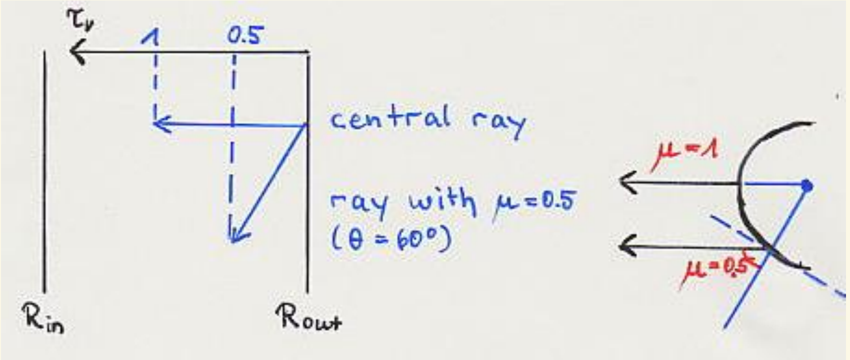
We "see" source function at location $\tau_\nu = \mu$ (remember: τ_ν radial quantity)
(corresponds to optical depth along path $\tau_\nu / \mu = 1!$)

Generalization of principle that we can see only until $\Delta\tau_\nu = 1$

i) spectral lines (as before)

for fixed μ, $\tau_\nu / \mu = 1$ is reached further out in lines (compared to continuum)

=> $S_\nu^{line}(\tau_\nu^{line}/\mu = 1) < S_\nu^{cont}(\tau_\nu^{cont}/\mu = 1)$ => "dip" is created



ii) limb darkening

for $\mu = 1$ (central ray), we reach maximum in depth (geometrical)
temperature / source function rises with τ
=> central ray: largest source function, limb darkening

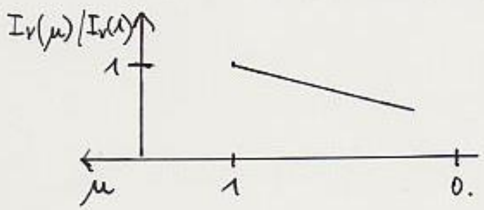
iii) "observable" information only from layers with $\tau_\nu \leq 1$
deepest atmospheric layers can be analyzed only indirectly

Application: solar limb-darkening

Had $I_\nu^{em}(\mu) = S_{\nu 0} + \mu S_{\nu 1}$

$\rightarrow$ LTE $\quad S_\nu \approx B_\nu, \quad I_\nu^{em} = B_\nu(0) + \mu \left.\frac{dB_\nu}{d\tau_\nu}\right|_0$

$$\rightarrow \frac{I_\nu(\mu)}{I_\nu(1)} = \frac{B_\nu(0) + \mu\, dB_\nu/d\tau_\nu}{B_\nu(0) + dB_\nu/d\tau_\nu}$$



measurement

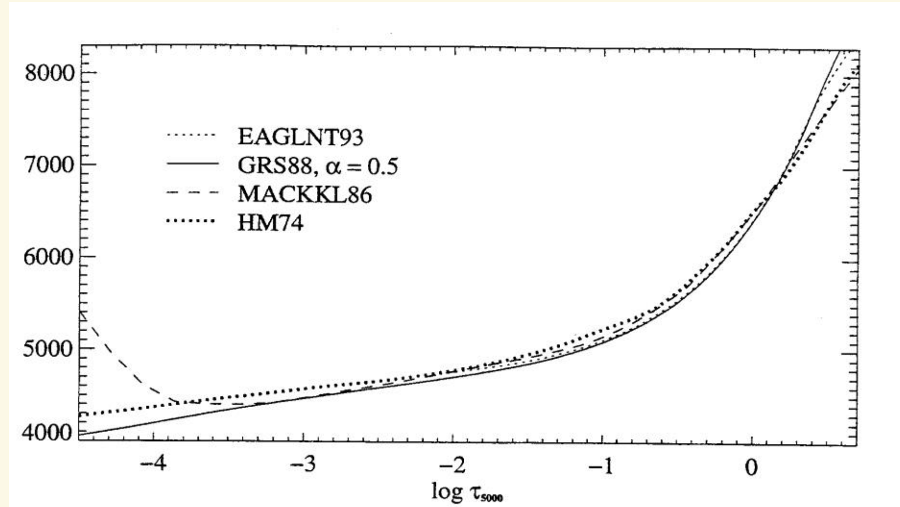
$\Rightarrow B_\nu(0), \left.\frac{dB_\nu}{d\tau_\nu}\right|_0$

(one absolute measurement required, e.g., $B_\nu(0)$)

$$\Rightarrow B_\nu(\tau) = B_\nu(0) + \left.\frac{dB_\nu}{d\tau_\nu}\right|_0 \cdot \tau =: \frac{2h\nu^3}{c^2}\,\frac{1}{e^{h\nu/kT(\tau)} - 1}$$

$\Rightarrow T(\tau)$, empirical temperature stratification of solar photosphere

empirical temperature structure of solar photosphere by Holweger & Müller (1974)



Lambda operator

The Lambda operator

had mean intensity

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\,d\mu = \frac{1}{2}\int_0^1 \left[I^+_\nu(\mu) + I^-(-\mu)\right]d\mu \xrightarrow[\text{infinite atm.}]{\text{semi}}$$

$$\frac{1}{2}\left\{\int_0^1 d\mu\left[\underbrace{\int_{\tau_\nu}^{\infty} S_\nu(t)\,e^{-(t-\tau_\nu)/\mu}\frac{dt}{\mu}}_{\text{outwards}} + \underbrace{\int_0^{\tau_\nu} S_\nu(t)\,e^{-(\tau_\nu-t)/\mu}\frac{dt}{+\mu}}_{\text{inwards } (I(-\mu))}\right]\right\}$$

$$= \left(x = \frac{1}{\mu},\ \frac{dx}{x} = -\frac{d\mu}{\mu}\right)$$

$$\frac{1}{2}\int_{\tau_\nu}^{\infty} dt\, S_\nu(t)\int_1^{\infty} e^{-(t-\tau_\nu)x}\frac{dx}{x} + \frac{1}{2}\int_0^{\tau_\nu} dt\, S_\nu(t)\int_1^{\infty} e^{-(\tau_\nu-t)x}\frac{dx}{x}$$

$$\left(\int_1^{\infty} e^{-t\cdot x}\frac{dx}{x} = \int_t^{\infty}\frac{e^{-x}}{x}\,dx = E_1(t)\right)$$

1st Exponential integral

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^{\infty} S_\nu(t)\,E_1(|t-\tau_\nu|)\,dt$$ Karl Schwarzschild

with $\Lambda_\tau[f] = \frac{1}{2}\int_0^{\infty} f(t)\,E_1(|t-\tau|)\,dt$ "Lamba Operator"

$$J_\nu(\tau_\nu) = \Lambda_{\tau_\nu}(S_\nu) \quad \text{or} \quad J = \Lambda(S)$$

Diffusion approximation

The diffusion approximation

- for large optical depths $S_\nu \to B_\nu$
- Question: what is response of radiation field?
- expansion

$$S_\nu(t_\nu) = \sum_{n=0}^{\infty} \left.\frac{d^n B_\nu}{d\tau_\nu^n}\right|_{\tau_\nu} (t_\nu - \tau_\nu)^n / n!$$

- put into formal solution

$$\Rightarrow I_\nu^+(\tau_\nu, \mu) = \sum_{n=0}^{\infty} \mu^n \frac{d^n B_\nu}{d\tau_\nu^n} = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu} + \mu^2 \frac{d^2 B_\nu}{d\tau_\nu^2} + \ldots$$

I_ν^- analogous, difference $O(e^{-\tau_\nu/\mu})$

$$\Rightarrow J_\nu(\tau_\nu) = \sum_{n=0}^{\infty} (2n+1)^{-1} \frac{d^{2n} B_\nu}{d\tau_\nu^{2n}} = B_\nu(\tau_\nu) + \frac{1}{3} \frac{d^2 B_\nu}{d\tau_\nu^2} + \text{even}$$

$$H_\nu(\tau_\nu) = \sum_{n=0}^{\infty} (2n+3)^{-1} \frac{d^{2n+1} B_\nu}{d\tau_\nu^{2n+1}} = \frac{1}{3} \frac{dB_\nu}{d\tau_\nu} + \ldots \quad \text{odd}$$

$$K_\nu(\tau_\nu) = \sum_{n=0}^{\infty} (2n+3)^{-1} \frac{d^{2n} B_\nu}{d\tau_\nu^{2n}} = \frac{1}{3} B_\nu + \frac{1}{5} \frac{d^2 B_\nu}{d\tau_\nu^2} + \ldots \quad \text{even}$$

$\Rightarrow$ diffusion approx. for radiation field

$\tau_\nu \gg 1$, use only first order

$I_\nu = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu}$ required to obtain $H_\nu \neq 0$

$$\left.\begin{aligned} J_\nu &= B_\nu(\tau_\nu) \\ H_\nu &= \frac{1}{3} \frac{dB_\nu}{d\tau_\nu} = -\frac{1}{3} \frac{1}{\chi_\nu} \frac{\partial B_\nu}{\partial T} \frac{dT}{dz} \\ K_\nu &= \frac{1}{3} B_\nu(\tau_\nu) \end{aligned}\right\} \quad f_\nu = \frac{K_\nu}{J_\nu} = \frac{1}{3} \quad (\tau_\nu \gg 1)$$

"Eddington factor"

- $H_\nu = -\frac{1}{3} \frac{1}{\chi_\nu} \underbrace{\frac{\partial B_\nu}{\partial T}}_{>0} \frac{dT}{dz}$

$\Rightarrow$ in order to transport flux $H_\nu > 0$, $\frac{dT}{dz} < 0$, i.e., temperature must decrease!

Thermalization

From approximate solution of moments equations accounting for true plus scattering continuum opacity (Milne-Eddington model $\rightarrow$ advanced reading), it turns out that the difference between mean intensity and Planck-function (as a function of optical depth) can be written as

$$J_\nu - B_\nu \approx f(\varepsilon_\nu)\exp\left[-(3\varepsilon_\nu)^{1/2}\tau_\nu\right],$$

with *thermalization parameter*

$$\varepsilon_\nu = \frac{\kappa_\nu^{\mathrm{t}}}{\kappa_\nu^{\mathrm{t}} + \sigma_\nu}$$

given by the ratio of true and total opacity.

Thus, only for large arguments of the exponent we achieve $J_\nu \rightarrow B_\nu$, namely if

$$\tau_\nu \geq \frac{1}{\sqrt{\varepsilon_\nu}}$$

with $\frac{1}{\sqrt{\varepsilon_\nu}}$ the so-called thermalization depth [$\sqrt{3}$ in denominator neglected]

a) for $\sigma_\nu \ll \kappa_\nu^{\mathrm{t}}$ (negligible scattering) $\rightarrow J_\nu(\tau_\nu \geq 4...5) \rightarrow B_\nu$

b) SN remnants: scattering dominated, very large thermalization depth

The Milne-Eddington model

The Milne-Eddington model for continua with scattering

- allows understanding of emergent (continuum) fluxes from stellar atmospheres
- can be extended to include lines
- required for curve of growth method (➔ Chap. 7)

assume source function (→ page 72)

$$S_\nu = (1-\varepsilon_\nu)B_\nu + \varepsilon_\nu J_\nu \quad \text{with } \varepsilon_\nu = \frac{\sigma_e n_e}{\kappa_\nu^+ + \sigma_e n_e}$$

$$=: \varepsilon_\nu B_\nu + (1-\varepsilon_\nu) J_\nu, \quad \varepsilon_\nu = 1-\varepsilon_\nu$$

and

$$B_\nu = a_\nu + b_\nu \cdot \tau_\nu \quad + \text{ plane-parallel symmetry}$$

- 0th moment

$$\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu, \qquad d\tau_\nu = -(\kappa_\nu^+ + n_e\sigma_e)dz$$

$$= J_\nu - (\varepsilon_\nu B_\nu + (1-\varepsilon_\nu)J_\nu) = \varepsilon_\nu (J_\nu - B_\nu)$$

- 1st moment

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu$$

in diffusion approximation, we had

$$K_\nu = \frac{1}{3} J_\nu \quad (\tau_\nu \to \infty)$$

- Eddington's approximation (1929, 'The formation of absorption lines')

use $K_\nu / J_\nu = \frac{1}{3}$ everywhere ... not so wrong

$$\Rightarrow \frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu \rightarrow \frac{1}{3}\left(\frac{\partial J_\nu}{\partial \tau_\nu}\right) = H_\nu$$

$\Rightarrow$ (with 0th moment)

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial \tau_\nu^2} = \varepsilon_\nu (J_\nu - B_\nu) = \frac{1}{3}\frac{\partial^2 (J_\nu - B_\nu)}{\partial \tau_\nu^2},$$

since B_ν linear in τ_ν!

assume $\varepsilon_\nu = \text{const}$ (otherwise similar solution)

$$J_\nu - B_\nu = \text{const}' \cdot \exp(-(3\varepsilon_\nu)^{\frac{1}{2}}\tau_\nu) \quad \left[\text{with lower b.c. } J_\nu \to B_\nu \text{ for } \tau \to \infty\right]$$

- Eddington's approximation implies also

a) $J_\nu(0) = \sqrt{3}\, H_\nu(0)$ (without proof)

b) $\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu \rightarrow \frac{1}{3}\frac{\partial J_\nu}{\partial \tau_\nu}\Big|_0 = H_\nu(0)$

Thus $\frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau_\nu}\Big|_0 = J_\nu(0)$

$\Rightarrow$ insert in above equation

$$\text{const}' = \frac{b_\nu/\sqrt{3} - a_\nu}{(1+\varepsilon_\nu^{\frac{1}{2}})}$$

$$\Rightarrow J_\nu = a_\nu + b_\nu \tau_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{1+\varepsilon_\nu^{\frac{1}{2}}}\, e^{-(3\varepsilon_\nu)^{\frac{1}{2}}\tau_\nu}$$

$$J_\nu = a_\nu + b\tau_\nu + \frac{b/\sqrt{3} - a_\nu}{1+\varepsilon_\nu^{1/2}} e^{-(3\varepsilon_\nu)^{1/2}\tau_\nu}$$

$$J_\nu(0) = a_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{1+\varepsilon_\nu^{1/2}}$$

$$H_\nu(0) = \frac{1}{\sqrt{3}} J_\nu(0)$$

- assume isothermal atmosphere, $b_\nu = 0$ (possible, if gradient not too strong)

$$\Rightarrow J_\nu(0) = \frac{\varepsilon_\nu^{1/2}}{1+\varepsilon_\nu^{1/2}} a_\nu \quad \begin{cases} \nearrow B_\nu/2 & \text{for } \varepsilon_\nu = 1, \text{ i.e. } \sigma = 0 \\ \searrow \varepsilon_\nu^{1/2} B_\nu \ll B_\nu & \text{for } \varepsilon_\nu \ll 1 \end{cases}$$

$\rightarrow J_\nu(0) < B_\nu(0)$!!!

- Thermalization

  only for large arguments of the exponent, we have $J_\nu \approx B_\nu$

  $\Rightarrow \tau_\nu \gtrsim \frac{1}{\varepsilon_\nu^{1/2}}$ thermalisation depth

  a) $\sigma \ll \kappa^+ \Rightarrow J_\nu(\tau_\nu \gtrsim 1) \rightarrow B_\nu$

  b) SN remnants: scattering dominated, very large thermalization depth

- pure scattering (test case)

  $\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu = 0$ for $\varepsilon_\nu = 0$ Flux conservation

  $+ \; H_\nu \approx \frac{1}{3}\frac{\partial B_\nu}{\partial \tau_\nu}$ from diffusion limit

in Milne Eddington model

$$H_\nu(0) = \frac{1}{\sqrt{3}}\left(a_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{1+\varepsilon_\nu^{1/2}}\right) \xrightarrow{\varepsilon_\nu \to 0} \frac{b_\nu}{3} \;\hat{=}\; \frac{1}{3}\frac{\partial B_\nu}{\partial \tau_\nu}$$

consistent result

- Question: why $J_\nu(0) \ll B_\nu(0)$?
  - remember: $J_\nu(0)$ determined by $S_\nu(\tau_\nu = 1)$
  - $J_\nu(1)$ might fall significantly below $B_\nu(1)$, since many photons can escape from photosphere (into interstellar medium)
  - minimum value is given by incident flux, if no thermal emission
- interesting possibility

  if ε_ν small, $H_\nu(0)$ can become larger than $H_\nu(0)\,(\varepsilon_\nu = 1)$, if

$$\underbrace{a_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{2}}_{J_\nu(0,\,\varepsilon_\nu = 1)} < \underbrace{\frac{b_\nu}{\sqrt{3}}}_{J_\nu(0,\,\varepsilon_\nu \ll 1)}, \quad \text{i.e. } \frac{b_\nu}{a_\nu} > \sqrt{3}$$

  i.e. for large temperature gradients

  (information is transported from hotter regions to outer boundary by scattering dominated stratifications)

- further consequences later

Basic assumptions

1. Geometry
plane-parallel or spherically symmetric (→ Chap. 3)

2. Homogeneity
atmospheres assumed to be homogenous (both vertical and horizontal)

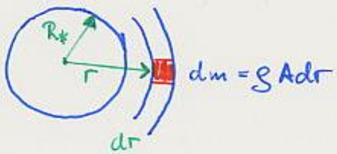
BUT: **sun** with spots, granulation, non-radial pulsations ...
white dwarfs with depth dependent abundances (diffusion)
stellar winds of hot stars (partly) with clumping **($\langle\rho^2\rangle \neq \langle\rho\rangle^2$)**

HOPE: "mean" = homogenous model describes non-resolvable phenomena in a reasonable way
[attention for (magnetic) Ap-stars: ***very*** strong inhomogeneities!]

3. Stationarity
vast majority of spectra time-independent => $\partial/\partial t = 0$

BUT: explosive phenomena (supernovae)
pulsations
close binaries with mass transfer ...

Density stratification



mass element dm
in (spherically sym.) atmosphere

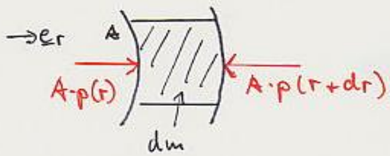
assume (at first) no velocity-fields, i.e. hydrostatic stratification

$\sum_i df_i = 0$, if f_i are forces acting on dm

- $df_{grav} = -G \frac{M_r \, dm}{r^2} = -g(r)\, dm$ with grav. accel.

  $g(r) = \frac{G M_r}{r^2}$ and M_r mass within r

- df_p pressure forces



gas pressure causes forces on surfaces $\perp e_r$. Forces on surfaces $\parallel e_r$ compensate each other in spherical (or p-p) symmetry

$$df_p = A \cdot p(r) - A\, p(r+dr) = -A \frac{dp}{dr} dr$$

- df_{Rad} (radiation force) $= g_{rad}(r)\, dm$

$$\sum dk_i = -g(r)\, dm + g_{rad}(r)\, dm - A \frac{dp}{dr} dr = 0$$

$$dm = A \cdot \varrho(r)\, dr$$

$$\Rightarrow \boxed{\frac{1}{\varrho} \frac{dp}{dr} = -g(r) + g_{rad}(r)} \quad \text{or}$$

$$\frac{dp}{dr} = -\varrho(r)\left[g(r) - g_{rad}(r)\right]$$

Hydrostatic equilibrium

Approximation $\quad \| \; g(r) = \frac{G M_r}{r^2} \rightarrow \frac{G M_*}{r^2} \; \|$

since mass within atmosph: $M(r) - M(R_*) \ll M(R_*)$

example: The sun

$$\Delta M_{phot} = \varrho \frac{4\pi}{3}\left((R+\Delta r)^3 - R^3\right) \approx \varrho \, 4\pi R^2 \Delta r$$

$R \approx 7 \cdot 10^{10}$ cm, $\Delta r \approx 3 \cdot 10^7$ cm (later), $\varrho \approx m_H \bar{N}$,
with $\bar{N} = 10^{15}$ cm^{-3} and $m_H \approx 1.7 \cdot 10^{-24}$ g

$\Rightarrow \Delta M_{phot} \approx 3 \cdot 10^{21}$ g $\ll M_\odot \approx 2 \cdot 10^{33}$ g

(same argument holds also if atmosphere is extended)

in plane-parallel geometry, we have additionally
$\Delta r \ll R_*$, thus $\| \; g(r) = g_* = \frac{G M_*}{R_*^2} \; \|$

examples		log g [cgs]
	main seq. stars (O → A)	≈ 4
	supergiants	3.5 … 0.8
	white dwarfs	8!
	sun	4.44
	earth	3.0

- if stellar wind present, hydrodynamic description

$\dot{M} = 4\pi r^2 \varrho(r)\, v(r)$ equation of continuity

$\rightarrow v(r) = \frac{\dot{M}}{4\pi} \frac{1}{r^2 \varrho(r)} \neq 0$ (everywhere)

Question When are velocity fields important, i.e. induce significant deviations from hydrostatic equilibrium?

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{v}) = 0$$

Equation of momentum

("Euler equation")

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \underbrace{\nabla\cdot(\rho \mathbf{v}\mathbf{v})}_{\mathbf{v}[\nabla\cdot(\rho\mathbf{v})]+[\rho\mathbf{v}\cdot\nabla]\mathbf{v}} = -\nabla p + \rho \mathbf{g}^{\mathrm{ext}}$$

$\Rightarrow$ stationarity, i.e., $\frac{\partial}{\partial t}=0$ and spherical symmetry, i.e., $\nabla\cdot\mathbf{u} \to \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r)$

$$r^2 \rho v = \mathrm{const} = \frac{\dot{M}}{4\pi} \quad \text{(I)}$$

with $\nabla\cdot(\rho\mathbf{v}) = 0$

$$\underline{\rho v \frac{\partial v}{\partial r}} = -\frac{\partial p}{\partial r} + \rho g_r^{\mathrm{ext}} \quad \text{(II)}$$

"advection term", (from inertia)

I: Conservation of mass-flux (Chap. 3)

II: "Equation of motion"

$$\Rightarrow \frac{\partial p}{\partial r} = \rho(r)\left(-\frac{GM}{r^2} + g_{\mathrm{Rad}}(r)\right) - \rho(r)v(r)\frac{\partial v}{\partial r}$$

Exercise:
Show, by using the cont. eq., that the Euler eq. can be alternatively written as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g}^{\mathrm{ext}}$$

$p = \frac{k_B}{\mu m_H}\rho T$ equation of state for ideal gas, μ mean molecular weight

$$\rightarrow \frac{dp}{dr} = \frac{k_B}{\mu m_H}\left(T\frac{d\rho}{dr} + \rho\frac{dT}{dr}\right) \qquad \frac{k_B T}{\mu m_H} = v_{\mathrm{sound}}^2,$$

$$= v_{\mathrm{sound}}^2\left(\frac{d\rho}{dr} + \rho\frac{d\ln T}{dr}\right) \qquad \text{isothermal speed of sound}$$

- with $\dot{M} = 4\pi r^2 \rho v = \mathrm{const}$

$$r^2 \rho \frac{dv}{dr} = -2r\rho v - r^2 v\frac{d\rho}{dr}$$

$$\rho v\frac{dv}{dr} = -\frac{2\rho v^2}{r} - v^2\frac{d\rho}{dr}$$

$\Rightarrow$

alternative formulation of equation of motion

$$\left(v_s^2 - v^2\right)\frac{d\rho}{dr} = -\rho\left(g - g_{\mathrm{rad}} + \frac{dv_s^2}{dr} - \frac{2v^2}{r}\right) \quad \text{hydrodyn.}$$

$$v_s^2 \frac{d\rho}{dr} = -\rho\left(g - g_{\mathrm{rad}} + \frac{dv_s^2}{dr}\right) \quad \text{hydrostatic}$$

- conclusion: For $v \ll v_{\mathrm{sound}}$, density stratification becomes ('quasi'-) hydrostatic
- is reached in deeper photospheric layers (well below "sonic point", defined by $v(r_s) = v_s$)

example v_{sound} (solar photosphere) $\approx$ 6 km/s
" (O-stars) 20 km/s

- density stratification for stars with wind

a) deep layers $\rho(r)$ hydrostatic ($\rightarrow$ next sect.)

$$\rightarrow v(r) = \frac{\dot{M}}{4\pi}\frac{1}{r^2\rho} \quad (v \ll v_{\mathrm{sound}})$$

c) outer layers $v(r)$, from obs. or theory

$$\rightarrow \rho(r) = \frac{\dot{M}}{4\pi}\frac{1}{r^2 v}$$

b) intermediate (transonic) regions (smooth) transition from a) to c)

Barometric formula

The barometric formula

had hydrostatic equation ($v(r) \ll v_s$)

$$v_s^2 \frac{d\rho}{dr} = -\rho\left(g - g_{rad} + \frac{dv_s^2}{dr}\right) \quad \text{and} \quad v_s^2 = \frac{k_B T}{\mu m_H}$$

$\rightarrow$ for given $T(r)$, $g_{rad}(r)$: $\rho(r)$ by num. integration

NOW analytic approximation

Neglect photospheric extension $\rightarrow g(r) = g_* = \text{const}$ ✓

radiative acceleration $\rightarrow$ main seq. etc

$\frac{dv_s^2}{dr}$, shall be small against other terms

$\rightarrow$ neglect of $\frac{dT}{dr}$

$$\Rightarrow \quad v_s^2 \frac{d\rho}{dr} = -\rho g_*$$

$$\frac{d\rho}{\rho} = -g_*/v_s^2 \qquad \text{barometric formula}$$

$$\boxed{\rho(r) = \rho(r_0)\, e^{-\frac{(r-r_0)g_*}{v_s^2}} = \rho(r_0)\, e^{-\frac{r-r_0}{H}}}$$

$$\left(\rho(z) = \rho(0)\, e^{-z/H}\right)$$

with pressure scale height $H = \frac{kT}{m_H \cdot \mu \cdot g_*}$

- extension no longer negligible, if H significant fraction of R_*

$$H/R_* = \frac{kTR_*}{m_H \mu G M} = \frac{v_s^2}{g R_*} = \frac{2 v_s^2}{v_{esc}^2}$$

with v_{esc} photospheric esc. velocity

$$= \left(\frac{2GM}{R_*}\right)^{\frac{1}{2}} = (2 g R_*)^{1/2} \qquad \left[\text{from } \frac{m}{2}v^2 = \frac{GmM}{R_*}\right]$$

example sun $\quad v_s \approx \left(\frac{1.38\cdot 10^{-16}\cdot 5700}{1.7\cdot 10^{-24}}\right)^{\frac{1}{2}} \approx 6.8\ \text{km/s}$

$$v_{esc} \approx (2\cdot 10^{4.44}\cdot 7\cdot 10^{10})^{\frac{1}{2}} \approx 620\ \text{km/s}$$

$$\Rightarrow H/R_* \approx 2.5\cdot 10^{-4}, \quad H \approx 170\ \text{km}$$

Total pressure

Alternative solution

had also

$$\frac{1}{\rho}\frac{dp}{dr} = -g + g_{rad}$$

$$g_{rad} = -\frac{1}{\rho}\,\nabla\cdot P \qquad (\rightarrow \text{Chap. 4})$$

$$\Rightarrow \frac{1}{\rho}\frac{dP_{tot}}{dr} = -g\,, \qquad P_{tot} = p_{gas} + P_{Rad}\,,$$

$\nabla\cdot P$ only comp. in rad. direct.

define column density $dm = -\rho\, dr$

in analogy to $d\tau = -\chi\, dr$ optical depth

$$\Rightarrow \frac{dP_{tot}}{dm} = g\,, \qquad P_{tot} = g\cdot m \quad \text{exact}$$

or

$$\frac{dp_{gas}}{dm} = g - g_{Rad} = g - \frac{4\pi}{c\rho}\int_0^\infty \chi_\nu H_\nu \, d\nu$$

- solution by numerical integration
- analytic approx: neglect ... as before

$$\rightarrow p_{gas} = g_* \cdot m$$

$$\rho = \frac{g_* \mu m_H}{k\cdot T}\cdot m = \frac{1}{H}\cdot m$$

or $\log\rho = \log m - \log H$

Example

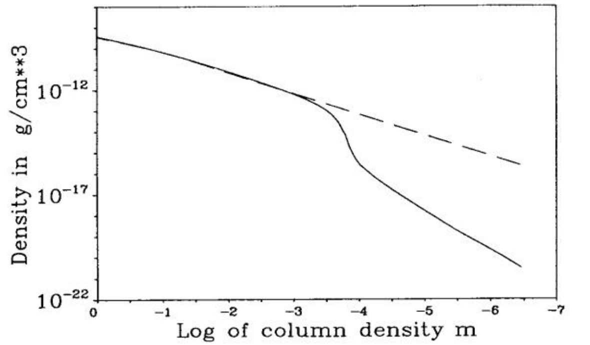


Fig. 16. Mass density ρ as function of logarithm of atmospheric column density m for a typical unified model (solid) and a hydrostatic model (dashed) with similar T_{eff} and $\log g$

Exercise: derive H directly from above figure
compare with result from
calculation of H (T_{eff} = 40,000 K, log g = 3.6)

Eddington limit

The Eddington limit

$$-\frac{1}{\rho}\frac{dp}{dr} = g - g_{rad} = g_{eff} \qquad \text{(without rotation)}$$

inwards outwards

- $g_{rad} = \frac{4\pi}{c\rho}\int_0^\infty \chi_\nu H_\nu \, d\nu$ in static atmospheres (χ_ν isotropic)

- minimum value ($\hat{=}$ main part of total continuum rad. acceleration in outer atmospheres of hot stars)

  Thomson scattering

$$g_{rad}^{\text{Thomson}} = \frac{4\pi}{c\rho}\int_0^\infty \sigma^{TH} H_\nu \, d\nu = \frac{4\pi}{c}\underbrace{\frac{n_e\sigma_e}{\rho}}_{s_e} H(r)$$

$n_e\sigma_e$, freq. independent

s_e: 0.34 H/He fully ionized; 0.4 pure H, fully ionized

Define $\Gamma_e = \frac{g_{rad}^{TH}}{g_{grav}} = \frac{\frac{4\pi}{c} s_e \frac{L}{16\pi^2 r^2}}{\frac{GM}{r^2}} =$ const (for s_e = const)

$$= \frac{L}{4\pi c GM} s_e = 7.64\cdot 10^{-5} \cdot s_e \cdot \frac{L/L_\odot}{M/M_\odot}$$

- $\Gamma_e = 1$ defines "Eddington limit": unstable atmosphere

- $g_{eff} = g - g_{rad} = g(1-\Gamma_e)$ $(- g_{rad}^{rest})$

  defines "effective" gravity

NOTE
- bound-free + free-free absorption has similar contribution (in intermediate layers)
- bound-bound absorption dominates the radiative acceleration in hot, luminous stars $\Rightarrow$ "line driven winds"

Summary: stellar atmospheres – the solution principle

THUS problem of stellar atmospheres solved (in principle, without convection, p-p geometry, static)

Given $\log g_*$, T_{eff}, abundances

(A) hydrostatic equilibrium

$$\frac{dp_{gas}}{dz} = -\rho(g_* - g_{rad}); \quad g_{rad} = \frac{4\pi}{c\rho}\int_0^\infty \chi_\nu H_\nu \, d\nu = \frac{4\pi}{c\rho}\left(\sigma^{TH} H(z) + \int_0^\infty \chi_\nu^{rest} H_\nu \, d\nu\right)$$

$$H = \frac{1}{4\pi}\sigma_B T_{eff}^4 \quad \left(= \frac{1}{4\pi}\mathcal{F}\right)$$

$$\Rightarrow \frac{dp_{gas}}{dz} = -\rho g_* + \sigma^{TH}\frac{\sigma_B T_{eff}^4}{c} + \frac{4\pi}{c}\int_0^\infty \chi_\nu^{rest} H_\nu \, d\nu$$

(B) equation of rad. transfer

$$\mu \frac{dI_\nu}{dz} = \chi_\nu(S_\nu - I_\nu) \quad \forall \nu,\mu \quad \Rightarrow J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\, d\mu; \quad H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\,\mu \, d\mu$$

(C) a) radiative equilibrium

scattering terms cancel, since conservative

$$\int_0^\infty (\eta_\nu - \chi_\nu J_\nu)\, d\nu = \int_0^\infty \{(\sigma^{TH} J_\nu + \chi_\nu^{rest} S_\nu^{rest}) - (\sigma^{TH} + \chi_\nu^{rest}) J_\nu\}\, d\nu = \int_0^\infty \chi_\nu^{rest}(S_\nu^{rest} - J_\nu)\, d\nu \overset{?}{=} 0$$

b) flux-conservation: $4\pi \int_0^\infty H_\nu(z)\, d\nu = 4\pi H(z) \overset{?}{=} \sigma_B T_{eff}^4$

$\Rightarrow \Delta T(z)$

$\rightarrow \Delta \chi_\nu(z)$ etc

(D) equation of state $p_{gas}(z) = \frac{k_B}{\mu m_H}\rho(z) T(z)$

solution by iteration!

Solution of differential equations A and B by discretization
differential operators => finite differences
all quantities have to be evaluated on suitable grid

Eq. of radiative transfer (B)
usually solved by the so-called
Feautrier and/or Rybicki scheme

Grey temperature stratification

- for iteration, we need initial values
- analytic understanding

$\Rightarrow$ "grey" approximation

assume $\chi_\nu = \chi$, freq. independent opacities
(corresponds to suitable averages)

$\Rightarrow \mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu$ $\qquad$ $\neq$ radiative eq.

$$\left.\begin{aligned} \frac{dH_\nu}{d\tau} &= J_\nu - S_\nu \\ \frac{dK_\nu}{d\tau} &= H_\nu \end{aligned}\right\} \text{(freq. integr.)} \quad J = \int_0^\infty J_\nu d\nu \text{ etc} \qquad \frac{dH}{d\tau} = J - S \; (=0) \qquad \frac{dK}{d\tau} = H$$

$\Rightarrow \frac{dK}{d\tau} = H$, i.e. $K = H \cdot \tau + C$

For large $\tau \gg 1$, we know from diff. approx. that $K_\nu / J_\nu = \frac{1}{3}$

Eddington's approx. $K/J = \frac{1}{3}$ everywhere

$\Rightarrow J = 3H(\tau + c)$

- From rad. equilibrium

$J = S$, $\quad S = 3H(\tau + c)$

- remember Λ-operator

$J = \Lambda_\tau(S)$

- analogous

$H = \Phi_\tau(S)$, in particular

$H(0) = \frac{1}{2} \int_0^\infty S(t) E_2(t) dt$ $\qquad$ E_2 2nd Exp. integral

$\Rightarrow H(0) = \frac{1}{2} \int_0^\infty (3H(t + c)) E_2(t) dt = \ldots$

$\ldots H\left(\frac{1}{2} + c\frac{3}{4}\right)$

But $H(0) = H$, i.e., $\left(\frac{1}{2} + c\frac{3}{4}\right) = 1$

$c = \frac{2}{3}$ in Eddington approx

Exact sol. $c = q(\tau)$, "Hopf function",

$0.51 < q(\tau) < 0.71$

- $J = 3H(\tau + 2/3)$

$H = \frac{\sigma T_{\rm eff}^4}{4\pi}$; $\quad J \xrightarrow{\rm LTE} B = \frac{\sigma_B T^4}{\pi}$

Finally

$$\boxed{T^4 = \frac{3}{4} T_{\rm eff}^4 (\tau + 2/3)}$$

grey temp. in Eddington approx!

Consequences

- $T = T_{\rm eff}$ at $\tau = 2/3$
- $T(0)/T_{\rm eff} = \left(\frac{1}{2}\right)^{1/4} = 0.841$

grey temp. in spherical symmetry

basic difference

$J, H \sim \frac{1}{r^2}$ for $r \gg R_*$ quadratic dilution

$J/K = 1$ for $r \gg R_*$

result

$$T^4(r) = T_{eff}^4 \left(W + \frac{3}{4} \tau' \right)$$

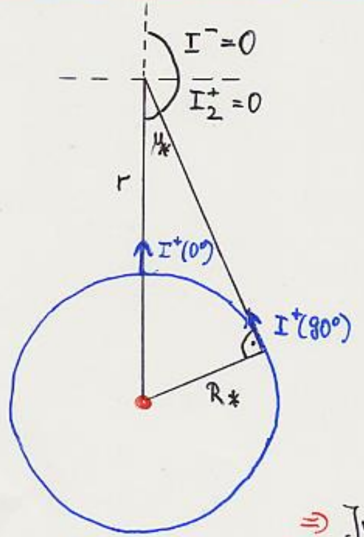
W dilution factor, $\frac{1}{2}\left[1 - \left(1 - \left(\frac{R_*}{r}\right)^2\right)^{\frac{1}{2}}\right]$

$$\tau' = \int_r^\infty \chi(r) \left(\frac{R_*}{r}\right)^2 dr$$

NOTE

$$T^{sph}(r) \xrightarrow{r \to R_*} T^{PP}(\tau)$$

Radiation field in optically thin envelopes



assume

- envelope optically thin $\Rightarrow I$ = const
- radiation field leaving photosphere isotropic $\Rightarrow I^+_{phot}(\mu)$ = const

$$\Rightarrow J_\nu(r) = \frac{1}{2}\int_{-1}^{+1} I_\nu(r)\, d\mu \longrightarrow$$

$$= \frac{1}{2}\int_{\mu_*}^{1} I_\nu^+(R_*)\, d\mu + \frac{1}{2}\int_0^{\mu^*} \underset{0}{I_2^+}\, d\mu + \frac{1}{2}\int_{-1}^{0} \underset{0}{I^-}\, d\mu$$

$$= \frac{1}{2} I_\nu^+(R_*)(1 - \mu_*)$$

$$\sin\theta = \frac{R_*}{r} \Rightarrow \mu_* = \cos\theta = \sqrt{1 - \left(\frac{R_*}{r}\right)^2}$$

$$J_\nu(r) = W \cdot I_\nu^+(R_*), \quad W = \frac{1}{2}\left(1 - \left(1 - \frac{R_*^2}{r^2}\right)^{\frac{1}{2}}\right)$$

"Dilution factor"

exercise: show that for $r \gg R_*$,

$$J_\nu(r) \approx H_\nu(r) \approx K_\nu(r)$$

Rosseland opacities

Rosseland opacities

grey approximation $\chi_\nu \equiv \chi$

BUT ionization edges, lines, bf-opacities $\sim \nu^{-3}$, ...

Question: can we define suitable means which might replace the grey opacity?

answer: not generally, but in specific cases

most important: Rosseland mean

($\rightarrow$ T-stratification, stellar structure, ...)

$$\frac{dK_\nu}{dz} = -\chi_\nu H_\nu \quad \text{exact}$$

- require, that freq. integration results in correct flux

$$\rightarrow -\int_0^\infty \frac{1}{\chi_\nu}\frac{dK_\nu}{dz}\,d\nu = \int_0^\infty H_\nu\, d\nu = H \equiv -\frac{1}{\bar\chi}\frac{dK}{dz}$$

Problem: to calculate $\bar\chi$, we have to know K_ν

- thus, use additionally diffusion approximation

$$K_\nu = \frac{1}{3}B_\nu \quad \text{and} \quad H_\nu = \frac{1}{3}\frac{dB_\nu}{d\tau_\nu}$$

$$\Rightarrow \frac{1}{\bar\chi_{\mathrm{R}}} = -\frac{H}{dK/dz} \rightarrow \frac{\int_0^\infty \frac{1}{3}\frac{1}{\chi_\nu}\frac{\partial B_\nu}{\partial T}\frac{dT}{dz}\,d\nu}{\int_0^\infty \frac{1}{3}\frac{\partial B_\nu}{\partial T}\frac{dT}{dz}\,d\nu} = \frac{\int_0^\infty \frac{1}{\chi_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,d\nu} = \frac{\int_0^\infty \frac{1}{\chi_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu}{\frac{4\sigma_{\mathrm{B}}}{\pi}T^3}$$

$$\left[\text{since } \int B_\nu\, d\nu = \frac{\sigma_{\mathrm{B}}}{\pi}T^4 \rightarrow \frac{\partial}{\partial T} = \frac{4\sigma_{\mathrm{B}}}{\pi}T^3\right]$$

$\Rightarrow$ Rosseland opacity

$$\boxed{\bar\chi_{\mathrm{R}} = \frac{\frac{4\sigma_{\mathrm{B}}}{\pi}T^3}{\int_0^\infty \frac{1}{\chi_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu}}$$

- can be calculated without radiative transfer
- harmonic weighting: maximum flux transport where χ_ν is small!

- alternatively, from construction (for $\tau_\nu \gg 1$)

$$\frac{1}{\bar{\chi}_\mathrm{R}} = -\frac{H}{dK/dz} \rightarrow -\frac{H}{\int_0^\infty \frac{1}{3}\frac{\partial B_\nu}{\partial z}d\nu} = -\frac{H}{\frac{1}{3}\frac{dT}{dz}\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu} = -\frac{H}{\frac{1}{3}\frac{4\sigma_\mathrm{B}}{\pi}T^3\frac{dT}{dz}}$$

$\Rightarrow$

i) $F = 4\pi H = \dfrac{16\sigma_\mathrm{B}}{3}T^3\dfrac{dT}{d\tau_\mathrm{R}}$

ii) in spherical geometry

$$\frac{L(r)}{4\pi r^2} = -\frac{16\sigma_\mathrm{B}}{3\bar{\chi}_\mathrm{R}}T^3\frac{dT}{dr} \quad \text{(used for stellar structure)}$$

iii) integrate i), + $F = \sigma_\mathrm{B}T_\mathrm{eff}^4$

$\rightarrow T^4 = T_\mathrm{eff}^4\,\dfrac{3}{4}(\tau_\mathrm{Ross} + const)$, as in grey case, but now with τ_Ross

THUS possibility to obtain initial (or approx.) values for temperature stratification ($\approx$ exact for large optical depths)

$$\left.\begin{array}{l}\text{calculate (LTE) opacities } \chi_\nu \\ \quad\text{calculate } \bar{\chi}_\mathrm{R},\ \tau_\mathrm{R} \\ \quad\text{calculate } T(\tau_\mathrm{R})\end{array}\right\}\ \text{again, iteration required}$$

Now we define the stellar radius via

$$\boxed{R_* = R(\tau_\mathrm{Ross} = 2/3)}$$

as the average layer ("'stellar surface") where the observed UV/optical radiation is created.

Furthermore, if we approximate $const = 2/3$ as in the (approx.) grey case, i.e.,

$$\boxed{T^4(\tau_\mathrm{Ross}) \approx T_\mathrm{eff}^4\,\frac{3}{4}(\tau_\mathrm{Ross} + 2/3)}$$

then we obtain $T(\tau_\mathrm{Ross} = 2/3) = T(R_*) = T_\mathrm{eff}$ and the definition $L = 4\pi R_*^2\sigma_\mathrm{B}T_\mathrm{eff}^4$ has also a physical meaning (at least for LTE conditions): "the effective temperature is the atmospheric temperature of a star at its surface".

Note: in reality, $T(\tau_\mathrm{Ross} = 2/3)$ deviates (slightly) from T_eff, since $const \neq 2/3$, and because of deviations from LTE

... back to Milne Eddington Model (page 83)

had $B_\nu(\tau_\nu) = a_\nu + b_\nu \tau_\nu$ linear approx

and $J_\nu(0) = \frac{b_\nu}{\sqrt{3}}$ for $\varepsilon_\nu = 0$ pure scattering

$= a_\nu + \frac{b_\nu/\sqrt{3} - a_\nu}{2}$ for $\varepsilon_\nu = 1$ purely thermal

$$\varepsilon_\nu = \frac{\kappa_\nu^+}{\kappa_\nu^+ + \sigma_e n_e}$$

- since temperature stratification known by now, can perform some estimates concerning continuum fluxes

$$\left.\begin{array}{l} \text{had} \quad T^4 \approx T_{eff}^4 \frac{3}{4}(\tau_R + 2/3) \\ T(0)^4 = T_{eff}^4 \frac{3}{4} \cdot 2/3 \end{array}\right\} T^4 = T^4(0)\left(1 + \frac{3}{2}\tau_R\right)$$

$$B_\nu(\tau_R) \approx B_\nu(T_0) + \left(\frac{\partial B_\nu}{\partial \tau_R}\right)_0 \tau_R = B_0 + B_1 \tau_R$$

$$\Rightarrow B_1 = \frac{\partial B_\nu}{\partial T}\Big|_{T_0} \cdot \frac{\partial T}{\partial \tau_R}\Big|_{T_0} = B_\nu \frac{h\nu/kT \cdot \frac{1}{T} e^{h\nu/kT}}{(e^{h\nu/kT} - 1)}\Big|_{T_0} \frac{\partial T}{\partial \tau_R}\Big|_{T_0}$$

$$= B_\nu \frac{u_0}{1 - e^{-u_0}} \frac{1}{T_0} \frac{\partial T}{\partial \tau_R}\Big|_0 \quad \text{with} \quad u_0 = \frac{h\nu}{kT_0}$$

$$4T^3 \frac{\partial T}{\partial \tau_R} = T^4(0)\frac{3}{2}, \quad \frac{\partial T}{\partial \tau_R}\Big|_{T_0} = \frac{3}{8} T_0$$

Thus $B_1 = B_0 \frac{u_0}{1 - e^{-u_0}} \frac{3}{8}$ → (Rayleigh-Jeans) $B_1 = \frac{3}{8} B_0$
→ (Wien) $B_1 = \frac{3}{8} u_0 B_0$

example $T_{eff} = 40{,}000$ K, $\lambda = 500, 912$ Å
[Hydrogen Balmer, Lyman continuum, $\varepsilon_\nu \ll 1$)

$T_0 = 33{,}600$ K

$u_0 = \frac{8.56}{4.70}$ → $B_1 \approx \frac{3.21}{1.76} B_0$

→ if $(\kappa_\nu^+ + \sigma_\nu) \approx \chi_R$ $\quad J_\nu(0, \varepsilon_\nu = 1) \approx \frac{1.42}{1.0} B_0$

$H_\nu(0) = \frac{1}{\sqrt{3}} J_\nu(0)$ $\quad J_\nu(0, \varepsilon_\nu \to 0) \approx \frac{1.85}{1.01} B_0$

can look down deeper into atm.

- additional effect 1

T-stratification with respect to τ_R ($\bar{\chi}_R$), but radiation transfer with respect to freq. τ_ν

$$J_\nu = B_\nu + \ldots = a_\nu + b_\nu \tau_\nu + \ldots$$

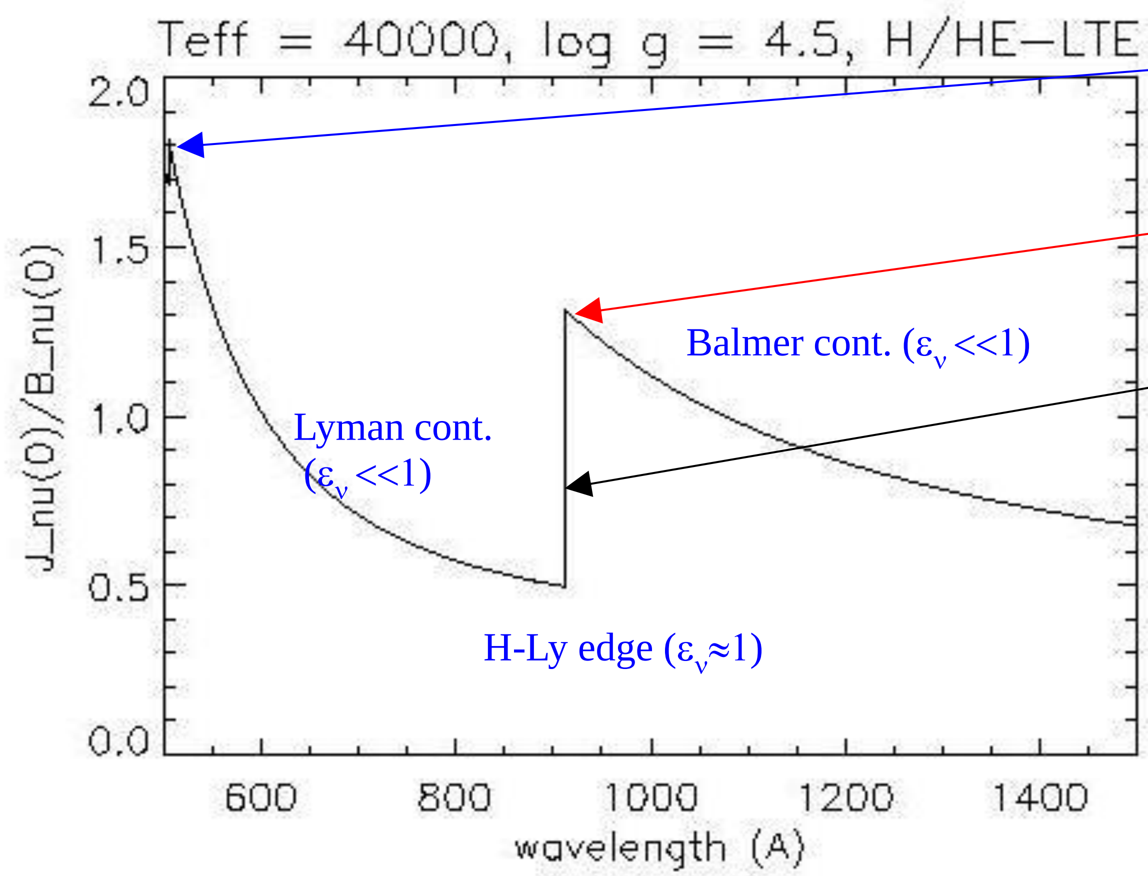
$$B_\nu = B_{\nu 0} + B_1 \tau_R = B_{\nu 0} + B_1 \tau_\nu \frac{\tau_R}{\tau_\nu} \approx B_{\nu_0} + \underbrace{B_1 \frac{\chi_R}{\chi_\nu}}_{b_\nu} \cdot \tau_\nu$$

effective gradient increased, if κ_ν small compared to χ_R

- additional effect 2

far away from ionization edges (where ε_ν is small, anyway), also χ_ν small ($\kappa_\nu^+ \sim \left(\frac{\nu_0}{\nu}\right)^3$, cf Chapter 5) → additional enhancement

H/He continuum of a hot star around 1000 Å



Predictions

Lyman cont: $J_\nu / B_\nu \geq 1.85$, OK
(at 500 Å) $(\chi_\nu \approx \chi_R)$

Balmer cont: $J_\nu / B_\nu \geq 1.01$, OK
(at 912 Å) $(\chi_\nu < \chi_R)$

Lyman edge: $J_\nu / B_\nu \leq 1.0$, OK
(911 Å) $(\chi_\nu > \chi_R)$

note: large opacity leads to very small effective T-gradient, minimum value $J_\nu / B_\nu = 0.5$, (cf. page 84)

Convection (simplified)

Convection

energy transport not only by radiation, however also by

- waves
- heat conduction
- convection

} not efficient in typical stellar atmospheres, but ... coronae, chromospheres, white dwarfs

Thus

total flux = const

$$\nabla \cdot (\underline{F}^{\rm rad} + \underline{F}^{\rm conv}) \overset{\downarrow}{=} 0 \quad \text{(in quasi-hydrostatic atmospheres)}$$

or

$$\frac{dF^{\rm conv}}{dz} = -\frac{dF^{\rm rad}}{dz} = -4\pi \int_0^\infty d\nu\, \chi_\nu (S_\nu - J_\nu)$$

energy transport by

radiation — convection

most efficient way is chosen

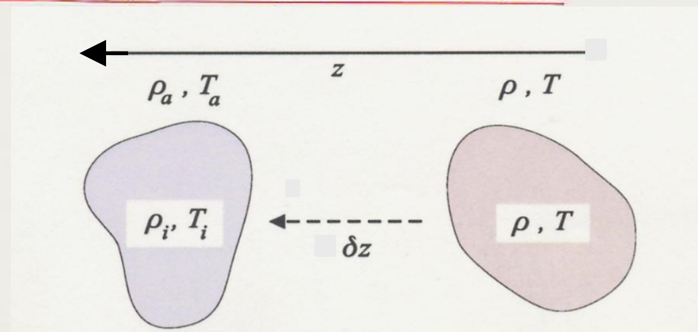
early — spectral type — late

O → (A) — M → F

convective core

outer convection zone

The Schwarzschild Criterion



assume mass element in photosphere, which moves upwards (by perturbation). Ambient pressure decreases, and "bubble" expands

Thus

$\rho \to \rho_i$, $T \to T_i$ in bubble ("i" = internal)

$\rho \to \rho_a$, $T \to T_a$ in ambient medium

two possibilities

$\rho_i > \rho_a$ bubble falls back — stable

$\rho_i < \rho_a$ bubble rises further — instable

buoyancy as long as $\rho_i(r+\Delta r) < \rho_a(r+\Delta r)$

since

$F_b = -g(\rho_i - \rho_a) > 0$, i.e., for $\Delta\rho = (\rho_i - \rho_a) < 0$

The Schwarzschild criterion

assumption 1

movement so slow, that pressure equilibrium ($\bar{v} < v_{sound}$)

$\Rightarrow$ $p_i = p_a$ and $(\rho T)_i = (\rho T)_a$ over Δr

$$\Rightarrow \quad \Delta\rho = \left[\frac{d\rho_i}{dr} - \frac{d\rho_a}{dr}\right]\Delta r = \left(\left|\frac{d\rho_a}{dr}\right| - \left|\frac{d\rho_i}{dr}\right|\right)\Delta r$$

Instability, if density inside bubble drops faster

$$\boxed{\left|\frac{d\rho_i}{dr}\right| > \left|\frac{d\rho_a}{dr}\right| \quad \text{or} \quad \left|\frac{dT_i}{dr}\right| < \left|\frac{dT_a}{dr}\right|}$$

assumption 2

no energy exchange between bubble and ambient medium (will be modified later)

$\Rightarrow$ adiabatic change of state in bubble

$$\rho_i = a \cdot p_i^{1/\gamma}, \quad \gamma = c_p/c_v$$

$$\rightarrow \quad \frac{d\rho_i}{dr} = a\frac{1}{\gamma}p_i^{1/\gamma - 1}\frac{dp_i}{dr} = \frac{1}{\gamma}\frac{\rho_i}{p_i}\frac{dp_i}{dr} = \frac{1}{\gamma}\rho_i\frac{d\ln p_i}{dr}$$

$\Rightarrow$ ambient medium ideal gas

$$\rho_a = a'\frac{p_a}{T_a}$$

$$\rightarrow \quad \frac{d\rho_a}{dr} = a'\left(\frac{1}{T_a}\frac{dp_a}{dr} - \frac{p_a}{T_a^2}\frac{dT_a}{dr}\right) = \rho_a\left(\frac{d\ln p_a}{dr} - \frac{d\ln T_a}{dr}\right)$$

$\Rightarrow$ instability for

$$\frac{1}{\gamma}\rho_i\frac{d\ln p_i}{dr} < \rho_a\left(\frac{d\ln p_a}{dr} - \frac{d\ln T_a}{dr}\right), \quad \rho_i(r_0) = \rho_a(r_0), \quad \frac{d\ln p_i}{dr} = \frac{d\ln p_a}{dr}$$

$$\frac{1}{\gamma}\frac{d\ln p}{dr} < \left(\frac{d\ln p}{dr} - \frac{d\ln T}{dr}\right)$$

$$\Rightarrow \left(\frac{d\ln p}{dr} < 0\right) \quad \frac{1}{\gamma} > 1 - \frac{d\ln T}{d\ln p}$$

$$\boxed{\nabla_a = \frac{d\ln T_a}{d\ln p} > 1 - \frac{1}{\gamma} = \nabla_{ad} \quad \text{"Schwarzschild criterion"}}$$

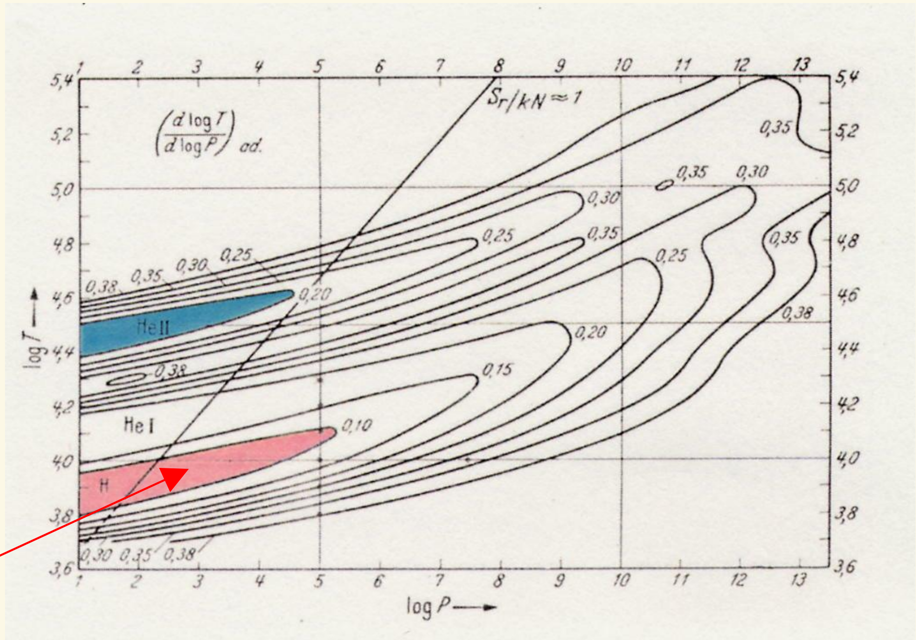
convection, if $\nabla_a > \nabla_{ad}$

- ∇_a : if no convection, radiative stratification

$$\nabla_a = \nabla_{rad} = \frac{d\ln T/dr}{d\ln p/dr} = \frac{3}{16}\,\frac{\bar{\chi}\cdot F_{rad}}{\sigma_B T^4} \Big/ \underbrace{\frac{g_{eff}\,\mu m_H}{kT}}_{1/H}$$

$$= \frac{3}{16}\left(\frac{T_{eff}}{T}\right)^4 \cdot \underbrace{(\bar{\chi} H)}_{\le 1 \text{ in photosphere}} \lesssim \frac{3}{16}\left(\frac{T_{eff}}{T}\right)^4$$

- $\nabla_{ad} = \left(\frac{d\ln T}{d\ln p}\right)_{ad} = \frac{\gamma - 1}{\gamma}$
- monoatomic gas $\nabla_{ad} = 0.4$
- must include ionization effects (number of particles!) and radiation pressure (weak influence in atmosph.)
- pure hydrogen, fully ionized
  $\nabla_{ad} = 0.4 \gg \nabla_{rad}$
  $\Rightarrow$ hot star atmospheres (convectively) stable!
- pure hydrogen: minimum for 50% ionization
  $\nabla_{ad} \approx 0.07 < \nabla_{rad}$ solar convection zone, $T = 9000$ K!



∇_{ad} as function of T and p

Mixing length theory

- most simplistic approach, however frequently used (reality is much too complex)
- suggested by Prandtl (1925)
- idea: - if atmosphere convective unstable at r_0, assume mass element rises until $r_0 + \ell$ (mixing length)
  - at $r_0 + \ell$, excess energy $\Delta E = c_p \rho \overline{\Delta T}$ (continued on next page) is released into ambient medium, and temperature is increased. Always valid $\nabla_{ad} \le \nabla_i < \nabla_a < \nabla_{rad}$
  - bubble cools, sinks down, absorbs energy, rises, etc...

⇒ Energy is transported, temperature gradient becomes smaller

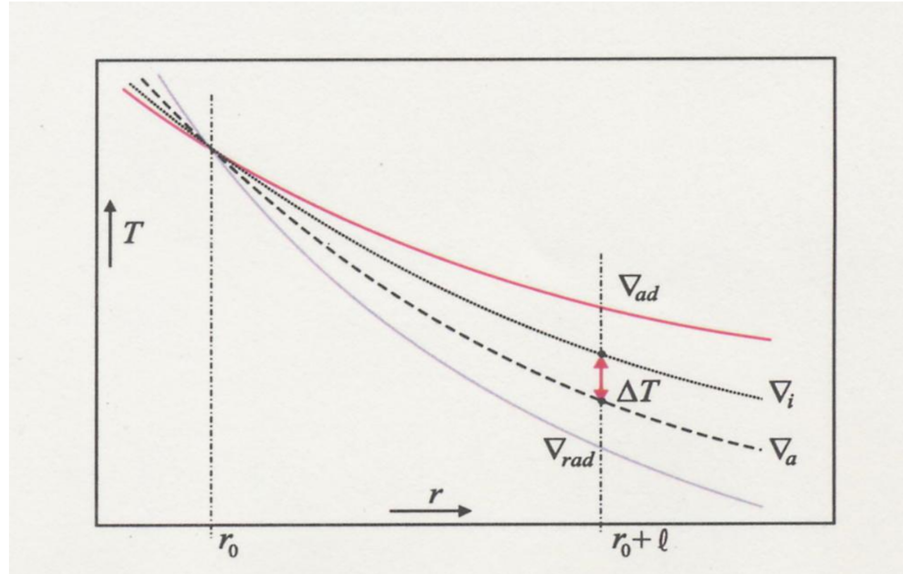
- Flux, temperature etc. calculated from simple arguments, $\ell = \alpha \cdot H$, $\alpha = 1, \dots 2$
- have to account for radiative losses during lifetime of element until energy is released

⇒ efficiency $\gamma = \frac{\text{excess energy lost}}{\text{radiative losses}}$

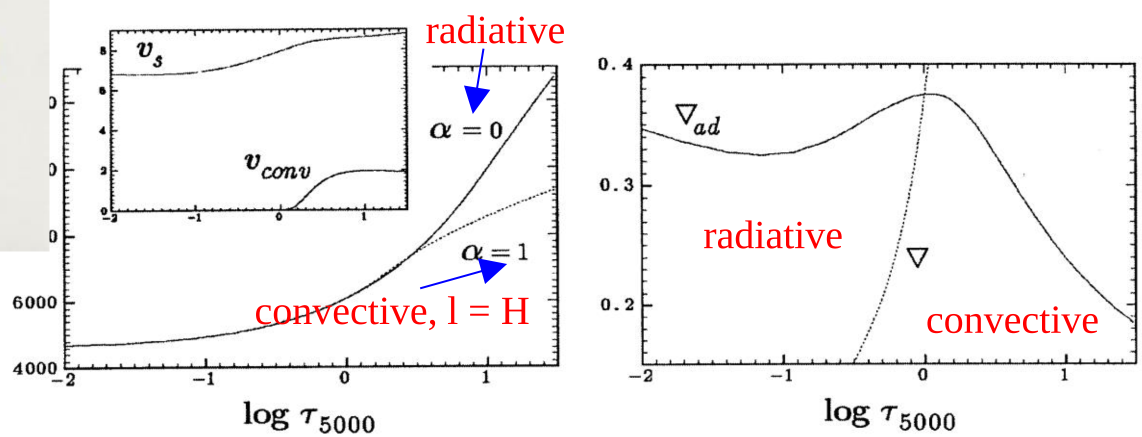
- γ large $\longrightarrow \nabla_a \approx \nabla_{ad}$; γ small $\longrightarrow \nabla_a \approx \nabla_{rad}$

Note:
- mixing length theory only 0th order approach
- modern approach: calculate consistent hydrodynamic solution (e.g., solar convective layer+photosphere, Asplund+, see 'Intermezzo')

radiative vs. adiabatic T-stratification



Model for solar photosphere



Mixing length theory – some details

$\Delta E = \rho C_p \delta T$ is excess energy density delivered to ambient medium when bubble merges with surroundings. C_p is specific heat per mass.

$\Rightarrow F_{conv} = \Delta E \bar{v} = C_p \delta T \rho \bar{v}$ is convective flux (transported energy) with $\bar{v}$ average velocity of rising bubble over distance Δr ($\rho \bar{v}$ mass flux).

δT is temperature difference between bubble and ambient medium.

$$\delta T = \left[\left(-\frac{dT}{dr}\bigg|_a\right) - \left(-\frac{dT}{dr}\bigg|_i\right)\right]\Delta r > 0$$ when convective instable,

since then $\left[(-\Delta T)_a - (-\Delta T)_i\right] > 0$

From the definiton of ∇,

$$-\frac{dT}{dr} = -\frac{T}{p}\frac{dp}{dr}\nabla = \frac{T}{H}\nabla,$$ with pressure scale height H, since

$$p = \frac{k\rho T}{\mu m_H}, \quad \frac{dp}{dr} = -g\rho \text{ and } \frac{1}{p}\frac{dp}{dr} = -\frac{\mu m_H g}{kT} = -\frac{1}{H}$$

(assuming hydrostatic equilibrium and neglecting radiation pressure; inclusion of p_{rad} possible, of course)

Defining l as the **mixing length** after which element dissolves, and averaging over all elements (distributed randomly over their paths), we may write $\Delta r = \frac{l}{2}$.

$$\Rightarrow F_{conv} = C_p \rho \bar{v} (\nabla_a - \nabla_i)\frac{T}{H}\frac{l}{2} = \boxed{\frac{1}{2} C_p \rho \bar{v} T (\nabla_a - \nabla_i)\alpha},$$ with

mixing length parameter $\alpha = \frac{l}{H}$ (from fits to observations, α=O(1))

The average velocity is calculated by assuming that the work done by the buoyant force is (partly) converted to kinetic energy, where the average of this work might be calculated via

$$\bar{w} = \int_0^{l/2} F_b(\Delta r) d(\Delta r),$$

and the upper limit results from averaging over elements passing the point under consideration. The buoyant force is given by (see page 93)

$$F_b = -g\delta\rho = -g(\rho_i - \rho_a) > 0$$

Using the equation of state, and accounting for pressure equilibrium ($p_i = p_a$),

we find $\frac{\delta\rho}{\rho} = -Q\frac{\delta T}{T}$ with $Q = \left(1 - \frac{\partial \ln \mu}{\partial \ln T}\bigg|_p\right)$, to account for ionization effects.

$$\Rightarrow F_b = -g\delta\rho = gQ\frac{\rho}{T}\delta T = gQ\frac{\rho}{T}\left[\left(-\frac{dT}{dr}\bigg|_a\right) - \left(-\frac{dT}{dr}\bigg|_i\right)\right]\Delta r =$$

$$gQ\frac{\rho}{H}(\nabla_a - \nabla_i)\Delta r := A\Delta r.$$ Thus, F_b is linear in Δr, and

$$\bar{w} = \int_0^{l/2} A\Delta r d(\Delta r) = A\frac{l^2}{8} = gQ\rho\frac{H}{8}(\nabla_a - \nabla_i)\left(\frac{l}{H}\right)^2$$

Mixing length theory – some details

Let's assume now that 50% of the work is lost to friction (pushing aside the turbulent elements), and 50% is converted into kinetic energy of the bubbles, i.e.,

$$\frac{1}{2}\bar{w} = \frac{1}{2}\rho\bar{v}^2 \quad \Rightarrow \quad \boxed{\bar{v} = \left(\frac{\bar{w}}{\rho}\right)^{1/2} = \left(\frac{gQH}{8}\right)^{1/2}(\nabla_a - \nabla_i)^{1/2}\alpha,}$$

and the convective flux is finally given by

$$\boxed{F_{conv} = \left(\frac{gQH}{32}\right)^{1/2}\left(\rho C_p T\right)\left(\nabla_a - \nabla_i\right)^{3/2}\alpha^2.}$$

NOTE : different averaging factors possible and actually found in different versions!

Remember that still $\nabla_{ad} \le \nabla_i < \nabla_a < \nabla_{rad}$.

The gradients ∇_i and ∇_a are calculated from the efficiency γ and the condition that the *total* flux remains conserved (outside the nuclear energy creating core), i.e.,

$$r^2(F_{conv} + F_{rad}) = r^2 F_{tot} = R_*^2 F_{rad}(R_*) = R_*^2 \sigma_B T_{eff}^4 = \frac{L}{4\pi}$$

or from the condition that

$(F_{conv} + F_{rad}) = \dfrac{L_r}{4\pi r^2}$ with L_r the luminosity at r.

Usually, a tricky iteration cycle is necessary.

Chap. 7 Microscopic theory

Absorption- and emission coefficients

- can calculate now a lot, if absorption- and emission-coefficients given, e.g.

$$\chi_\nu \sim \sigma_{lu} \bullet \phi(\nu) \bullet n_l(r)$$

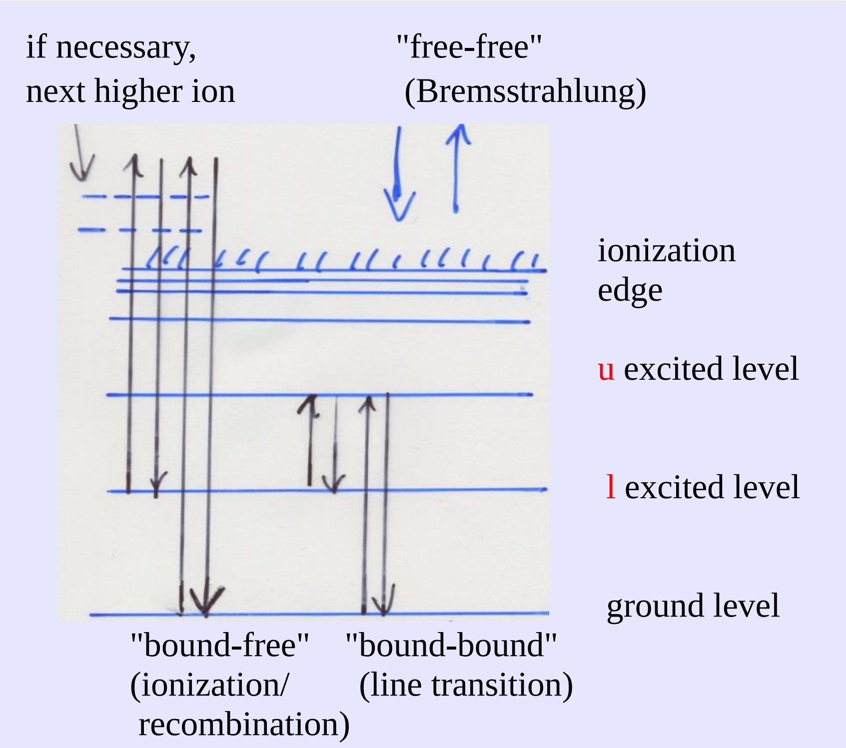
cross-section (quantum mech.)

profile function (normalized)

occupation numbers

LTE
Saha-Boltzmann

NLTE
detailed calculation



absorption

$h\nu + E_l \rightarrow E_u$

emission

- spontaneous ← isotropic

  $E_u \rightarrow E_l + h\nu$

- stimulated

  $h\nu + E_u \rightarrow E_l + 2\,h\nu$

same angular distribution

Line transitions

- Einstein coefficients

probability, that photon with energy $\sim[\nu, \nu+d\nu]$ is absorbed by atom in state E_l with resulting transition $l \to u$, per second

$$dW^{abs}(\nu,\Omega,l,u) = \underbrace{\underbrace{B_{lu}}_{\text{atomic property}} \cdot \underbrace{I_\nu(\Omega)}_{\text{prop. to number of incident photons}}}_{\text{prob. for } l\to u} \; \underbrace{\phi(\nu)d\nu}_{\text{probability, that } \nu \in [\nu,\nu+d\nu]} \; \underbrace{\frac{d\Omega}{4\pi}}_{\text{prob., that } \Omega \in [\Omega,\Omega+d\Omega]}$$

B_{lu} Einstein coefficient for absorption

analogously $\qquad \psi_\nu \neq \phi_\nu$ without further assumpt.

$$dW^{sp}(\nu,\Omega,u,l) = A_{ul}\,\psi(\nu)d\nu\,\frac{d\Omega}{4\pi}$$

$$dW^{stim}(\nu,\Omega,u,l) = B_{ul}\,I_\nu(\Omega)\,\psi(\nu)d\nu\,\frac{d\Omega}{4\pi}$$

(q.m.)

compare absorbed energy

$$dE_\nu^{abs} = n_l\,dW^{abs}\,h\nu\,dV - \underbrace{n_u\,dW^{stim}\,h\nu\,dV}_{\text{stimulated emission delivers back part of absorbed energy, with same angular distrib. as } I_\nu(\Omega)}$$

and emitted energy

$$dE_\nu^{em} = n_u\,dW^{sp}\,h\nu\,dV$$

with definition of opacity and emissivity

$$\Rightarrow \chi_\nu^{line} = \frac{h\nu}{4\pi}\phi(\nu)\left[n_l B_{lu} - n_u B_{ul}\frac{\psi(\nu)}{\phi(\nu)}\right]$$

$$\eta_\nu^{line} = \frac{h\nu}{4\pi}\psi(\nu)\,n_u A_{ul}$$

$\frac{\psi(\nu)}{\phi(\nu)} = 1$ for "complete redistribution"

- Einstein coefficients are atomic properties, must NOT depend on thermodynamic state of matter

Thus assume thermodynamic equilibrium

- from chap 4, we know $S_\nu^* = \frac{\eta_\nu^*}{\chi_\nu^*} = B_\nu(T)$ (and $\psi_\nu^* = \phi_\nu$)

$$\Rightarrow S_\nu^* = \frac{n_u A_{ul}}{n_l B_{lu} - n_u B_{ul}}$$

freq. independent (also valid in (N)LTE, if "complete redistribution")

$$= \frac{A_{ul}}{B_{ul}}\,\frac{1}{\left(\frac{n_l}{n_u}\right)^*\frac{B_{lu}}{B_{ul}} - 1}$$

- TE : Boltzmann excitation, $\left(\frac{n_u}{n_l}\right)^* = \frac{g_u}{g_l}e^{-h\nu_{ul}/kT}$

stat. weights

- $B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} = S_\nu^* = \frac{A_{ul}}{B_{ul}}\frac{1}{\left(\frac{g_l B_{lu}}{g_u B_{ul}}\right)e^{h\nu/kT}-1}$

$$\Rightarrow \boxed{g_l B_{lu} = g_u B_{ul}, \qquad A_{ul} = \frac{2h\nu^3}{c^2}B_{ul}}$$

ONLY ONE EINSTEIN COEFF. HAS TO BE CALCULATED!

- has to be calculated from quantum mechanics (from "dipoloperator")
- result

$$\boxed{\frac{h\nu}{4\pi} B_{lu} = \frac{\pi e^2}{m_e c} f_{lu}}$$

classical result, from electrodynamics

f "oscillator strength", dimensionless

"strong" transitions have $f \approx 0.1 \ldots 10$

and "selection rules", e.g. $\Delta l = \pm 1$

"forbidden transitions": magnetic dipole, electr. quadrupol: f very low, 10^{-5} and lower

- THUS $\chi_\nu = \frac{\pi e^2}{m_e c} f_{lu} \left(n_l - \frac{g_l}{g_u} \cdot n_u \right) \cdot \varphi_\nu$

$$= \frac{\pi e^2}{m_e c} (gf)_{lu} \cdot \left(\frac{n_l}{g_l} - \frac{n_u}{g_u} \right) \cdot \varphi_\nu$$

"gf-value" $= g_l \cdot f_{lu}$

with $\int_{-\infty}^{\infty} \varphi(\nu) d\nu = 1$

$$\frac{\pi e^2}{m_e c} \approx 0.02654 \frac{cm^2}{s}$$

Profile function?

1. Radiation damping ("natural" line broadening)

- QED effect
- heuristic: finite life time with respect to spontaneous emission

$$\tau = \frac{1}{A_{ul}} \quad (\text{e.g., } 10^{-8}\,\text{s for H } 2 \to 1)$$

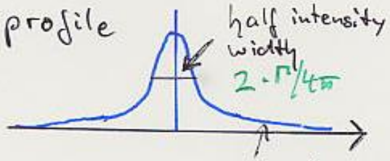
and uncertainty principle

$$\Delta E \cdot \tau \gtrsim \hbar$$

⇒ broadening (classical theory: damping by radiation)

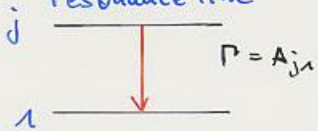
→ dispersion (Lorentzian) profile

$$\boxed{\varphi(\nu) = \frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2}}$$

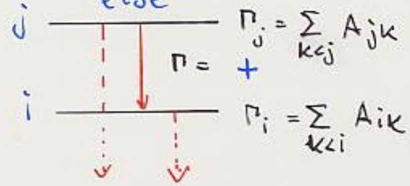


wing $\sim \frac{1}{(\Delta\nu)^2}$

resonance line



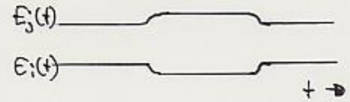
else



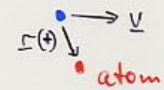
- of primary importance for strong lines (res. lines) in low density environment (no other broadening mechanisms), e.g. L_α in interstellar medium

2. Collisional broadening

- radiating atoms perturbed by passing particles
- brief perturbation, close perturbers



"impact theory"



$$\Delta E(t) \sim \frac{1}{r^n(t)}$$

n = 2 linear Stark effect

for levels with degenerate angular momentum, e.g., H I, He II

$$\Delta E \sim F = \frac{q}{r^2}$$

(F: field strength)

very important, if many electrons: photospheres of hot stars, $n_e \gtrsim 10^{12}\,\text{cm}^{-3}$

n = 3 resonance broadening

atom A is perturbed by atom A' of same species in "cool" stars, e.g. Balmer lines in sun

n = 4 quadratic Stark effect

metal ions in photospheres of hot stars

$$\Delta E \sim F^2$$

n = 6 van der Waals broadening

atom A perturbed by atom B in cool stars, e.g. Na perturbed by H in sun

resulting profiles are dispersion profiles!

- impact theory fails for (far) wings

$\Rightarrow$ statistical description (mean field of ensemble of perturbers) + q.m.

approximate behaviour for linear Stark broadening

$$\varphi(\Delta\nu \to \infty) \sim \frac{1}{(\Delta\nu)^{5/2}} \quad \left(\text{instead of } \frac{1}{(\Delta\nu)^2}\right)$$

3. Thermal velocities: Doppler broadening

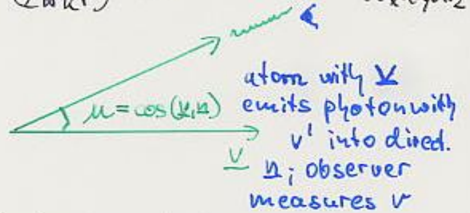
- radiating atoms have thermal velocity (so far assumed as zero)

Maxwellian distribution

$$P(v_x, v_y, v_z)\, dv_x\, dv_y\, dv_z = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}(v_x^2+v_y^2+v_z^2)}\, dv_x\, dv_y\, dv_z$$

+ Doppler effect

$$\underset{\text{observer's}}{\nu} \approx \underset{\text{atomic frame}}{\nu'} + \nu_0 \frac{\mathbf{n}\cdot\mathbf{v}}{c}$$



$\Rightarrow$ convolution; as long as isotropic emission:

$$\phi(\nu) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{+\infty} e^{-v^2}\, \underbrace{\varphi}_{\substack{\text{profile function}\\ \text{in atomic frame}}}(\nu - \nu_0 - \underbrace{\Delta\nu_D}_{\frac{\nu_0 v_{th}}{c}} v)\, dv$$

$\frac{\nu_0 v_{th}}{c}$ "Doppler width"

$v_{th} = \left(\frac{2kT}{m_A}\right)^{1/2}$ therm. velocity

i) assume sharp line, i.e. $f(\nu'-\nu_0) = \delta(\nu'-\nu_0)$

$$\rightarrow \phi(\nu) = \frac{1}{\Delta\nu_D}\frac{1}{\sqrt{\pi}}\, e^{-\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2}$$

Doppler profile, valid in line cores

ii) assume dispersion (Lorentzian) profile with Γ

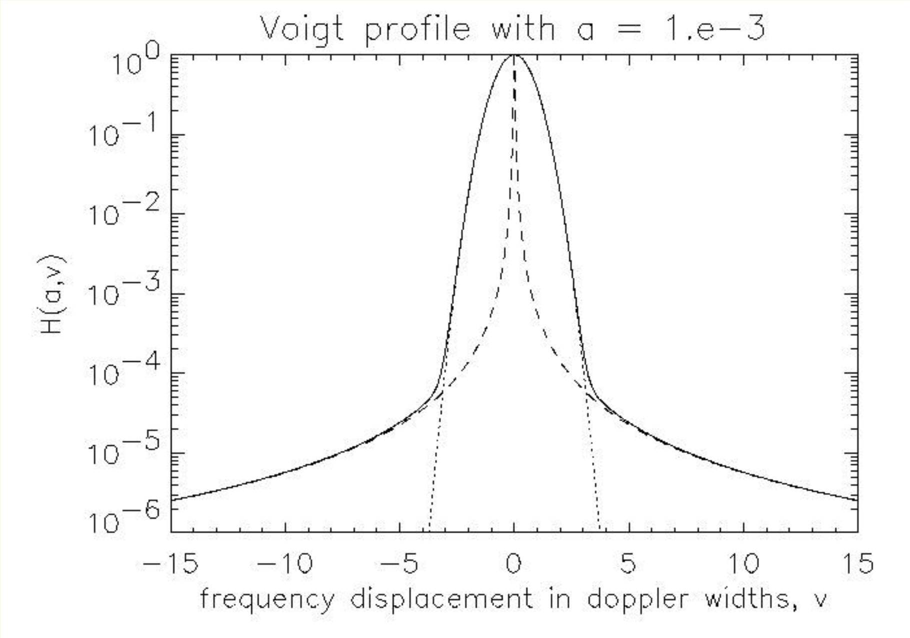
$$\rightarrow \phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}}\;\frac{a}{\pi}\int_{-\infty}^{+\infty}\frac{e^{-y^2}dy}{\left(\frac{\nu-\nu_0}{\Delta\nu_D}-y\right)^2+a^2}$$

$$= \frac{1}{\Delta\nu_D\sqrt{\pi}}\; H\left(a,\frac{\nu-\nu_0}{\Delta\nu_D}\right), \quad a = \frac{\Gamma}{4\pi\Delta\nu_D} \text{ damping parameter}$$

Voigt function, can be calculated numerically

NOTE $H\left(a,\frac{\nu-\nu_0}{\Delta\nu_D}\right) \approx \underbrace{e^{-\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2}}_{\text{line core}} + \underbrace{\frac{a}{\sqrt{\pi}\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2}}_{\text{wings}}$

iii) assume other "intrinsic" profile functions $\phi(\nu)$ from (numerical) convolution (e.g., with fast Fourier transformation)



fully drawn: Voigt profile $H(a,v)$

dotted : $\exp(-v^2)$, Doppler profile (core)

dashed: $a / (\sqrt{\pi}\, v^2)$, dispersion profile (wings)

Curve of growth method

Theoretical curve of growth

- standard diagnostic tool to determine metal abundances in cool stars in a simple way
- assumptions
  pure absorption line
  Milne Eddington model, LTE, $\varepsilon_\nu = 1$ (no scattering)

$$\chi_\nu = \chi_c + \underbrace{\bar{\chi}_L \phi_\nu}_{\chi_\nu^{Line}} = \chi_c(1+\beta_\nu), \quad \beta_\nu = \frac{\bar{\chi}_L}{\chi_c}\phi_\nu$$

↑ depth independent

$$B_\nu(\tau) = a + b\,\tau_c \quad \text{defined on continuum scale}$$

$$= a + b\frac{\chi_c}{\chi_\nu}\tau_\nu = a + \underbrace{b\frac{1}{1+\beta_\nu}}\tau_\nu$$

$\hat{=}\ b_\nu$ in Milne-Edd. model

- From Milne Edd. model we have (advanced reading, page 83)

$$H_\nu^{Line}(0),\ \varepsilon_\nu = 1 = \frac{1}{\sqrt{3}}J_\nu(0) = \frac{1}{\sqrt{3}}\left(a + \frac{\frac{1}{1+\beta_\nu}\frac{b}{\sqrt{3}} - a}{2}\right)$$

$$H_\nu^{cont}(0),\ \varepsilon_\nu = 1 = (\beta_\nu = 0) = \frac{1}{\sqrt{3}}\left(a + \frac{b/\sqrt{3} - a}{2}\right)$$

⇒ residual intensity ("line profile")

$$R_\nu = \frac{H_\nu^{Line}}{H_\nu^{cont}} = \frac{b\frac{1}{1+\beta_\nu} + \sqrt{3}a}{b + \sqrt{3}a}$$

$$\beta_\nu = \frac{\pi e^2}{m_e c} f_{lu}\frac{n_l}{\chi_c}\left(1 - e^{-h\nu/kT}\right)\phi(\nu) = \beta_0\phi(\nu)$$

↙ Voigt profile!

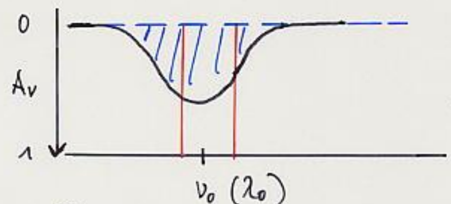
line depth

$$A_\nu = 1 - R_\nu = \frac{\beta_0\phi_\nu}{1+\beta_0\phi_\nu}\underbrace{\left(\frac{b}{b+\sqrt{3}a}\right)}$$

A_0 central depth of line with $\beta_0 \to \infty$

$$A_\nu = A_0\beta_0\frac{\phi_\nu}{1+\beta_0\phi_\nu}$$

equivalent width $W_\nu = \int_0^\infty A_\nu\,d\nu$ area below continuum (see also p. 8.3)



W_λ width of line in Å, if line would have depth "1"

$$\Rightarrow W_\nu = A_0\beta_0\int_0^\infty \frac{\phi_\nu}{1+\beta_0\phi_\nu}\,d\nu$$

$$W_\lambda = \int_0^\infty A(\lambda)\,d\lambda \approx \left(\int_0^\infty A_\nu\,d\nu\right)\frac{\lambda_0^2}{c} \qquad W_\lambda = \frac{\lambda_0^2}{c}\cdot W_\nu$$

with Voigt profile H (Doppler core + Lorentz wings)

$$W_\nu = A_0\beta_0\frac{1}{\sqrt{\pi}\Delta\nu_D}\int_0^\infty \frac{H\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)d\nu}{1+\frac{\beta_0}{\sqrt{\pi}\Delta\nu_D}H\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)} \qquad v = \frac{\nu-\nu_0}{\Delta\nu_D},\quad d\nu = dv\,\Delta\nu_D$$

$= \ldots$

$$W_\nu = \frac{A_0\beta_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{H(v)\,dv}{1+\frac{\beta_0}{\sqrt{\pi}\Delta\nu_D} H(v)}$$

3 regimes

a) linear regime: Doppler core not saturated,

$$H(a,v) = e^{-v^2}$$

$$\Rightarrow W_\nu \approx \frac{A_0\beta_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-v^2}dv}{1+\frac{\beta_0}{\sqrt{\pi}\Delta\nu_D}e^{-v^2}}$$

$$\rightarrow (\beta_0/\Delta\nu_D < 1) \quad \frac{A_0\beta_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-v^2}\left(1-\frac{\beta_0}{\Delta\nu_D\sqrt{\pi}}e^{-v^2}+\dots\right)dv$$

$\approx A_0\beta_0 \sim \beta_0$, independent on $\Delta\nu_D$

b) saturation part: line reaches maximum depth ($=A_0$), however wings still unimportant

as above, i.e. $\phi_\nu \sim e^{-v^2}$, however $\beta_0/\Delta\nu_D > 1$

$\Rightarrow$ (integration tricky)

$$W_\nu = 2A_0\Delta\nu_D\sqrt{\ln\beta^*}\left(1-(\pi^2/24(\ln\beta^*)^2-\dots\right)$$

with $\beta^* = \beta_0/\sqrt{\pi}\Delta\nu_D$

flat growth with $\sqrt{\ln\beta^*}$, $W_\nu \sim \Delta\nu_D$

c) damping (square-root) part

line wings dominate equivalent width

$$\Rightarrow W_\nu \approx \frac{A_0\beta_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{a/(\sqrt{\pi}v^2)\,dv}{1+\frac{\beta_0}{\sqrt{\pi}\Delta\nu_D}\frac{a}{\sqrt{\pi}v^2}}$$

a damping parameter

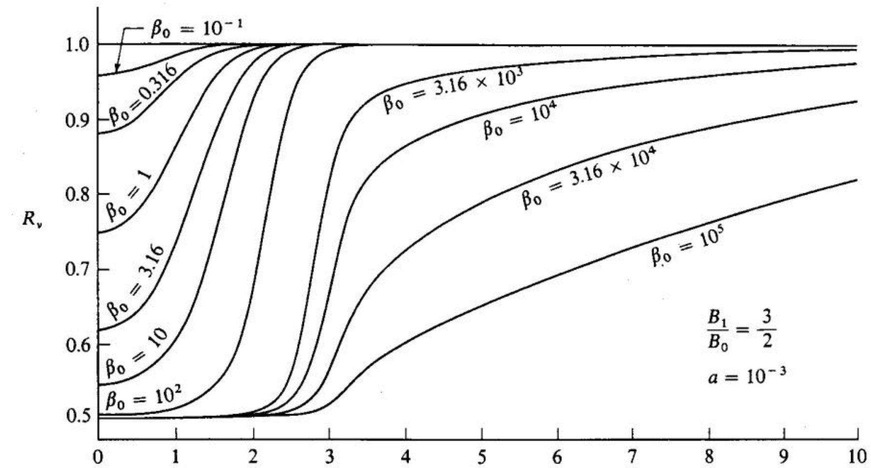
$$= \frac{A_0\beta_0}{\pi} a \int_{-\infty}^{+\infty} \frac{dv}{v^2+\frac{\beta_0 a}{\pi\Delta\nu_D}}$$

$$= A_0(a\pi\Delta\nu_D\beta_0)^{\frac{1}{2}}$$

(attention: typo in Mihalas)

growth with $\beta_0^{\frac{1}{2}}$

in total, we have $W_\nu = f(\beta_0)$ or $f\left(\frac{\beta_0}{\Delta\nu_D\sqrt{\pi}}\right) = f(\beta^*)$



Voigt profile with $A_O \hat{=} 0.5$, $\beta_o =: \beta^*$

Development of a spectrum line with increasing number of atoms along the line of sight. The line is assumed to be formed in pure absorption. For $\beta_0 \lesssim 1$, the line strength is directly proportional to the number of absorbers. For $30 \lesssim \beta_0 \lesssim 10^3$ the line is saturated, but the wings have not yet begun to develop. For $\beta_0 \gtrsim 10^4$ the line wings are strong and contribute most of the equivalent width.

Now:

$$\beta^* = \frac{\pi e^2}{m_e c} f_{lu} \frac{n_l}{\chi_c} \left(1 - e^{-h\nu/kT_e}\right) \frac{1}{\Delta\nu_D \sqrt{\pi}}$$

$$\chi_c = \chi_c^0 \left(1 - e^{-h\nu/kT_e}\right)$$ LTE, next section

$$n_l = n_1 \frac{g_l}{g_1} e^{-h\nu_{1l}/kT_e}$$ Boltzmann excitation, next section

$$\Delta\nu_D = \frac{\nu_0 v_{th}}{c} = \sqrt{\frac{2kT}{m}} \frac{1}{\lambda}$$

$$\Rightarrow \log \beta^* = \log\left(g_l f_{lu} \cdot \lambda\right) + \log\left(e^{-E_{1l}/kT_e}\right) + \log\left(\frac{n_1}{g_1 \chi_c^0} \frac{\sqrt{\pi} e^2}{m_e c} \sqrt{\frac{m}{2kT_e}}\right)$$

$$= \log\left(g_l f_{lu} \cdot \lambda\right) - \frac{5040 \cdot E_{1l}}{T_e} + \log C$$

in one ionization stage and if E in eV

- in one ionization stage, $C \approx$ const

→ lines belonging to one ionization stage should form curve of growth, since β^* varies as function of considered transition

→ if T_e and χ_c^0 known

→ shift "observed" $W_\nu(\beta^*_{lu})$ horizontally until curve matches theoretical curve

→ $n_1 \Rightarrow$ (using Saha-Boltzmann equation for ionization, next section)

abundances

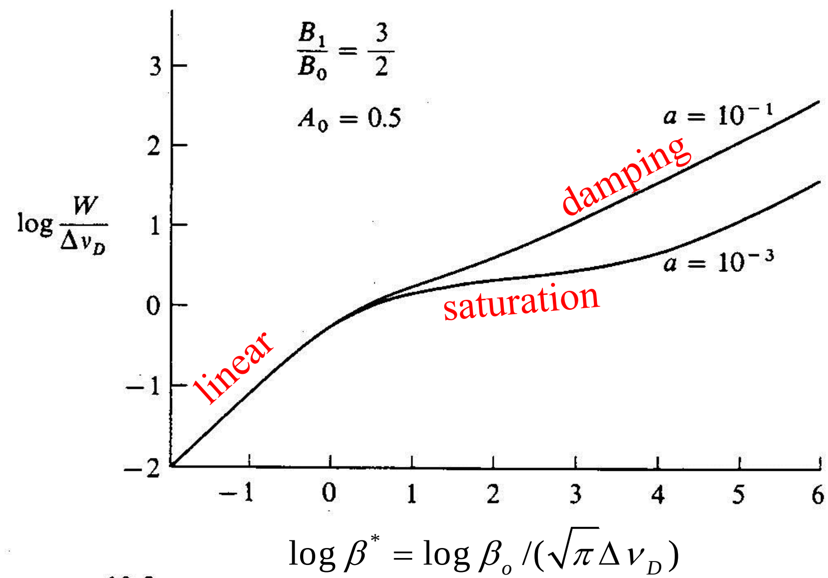
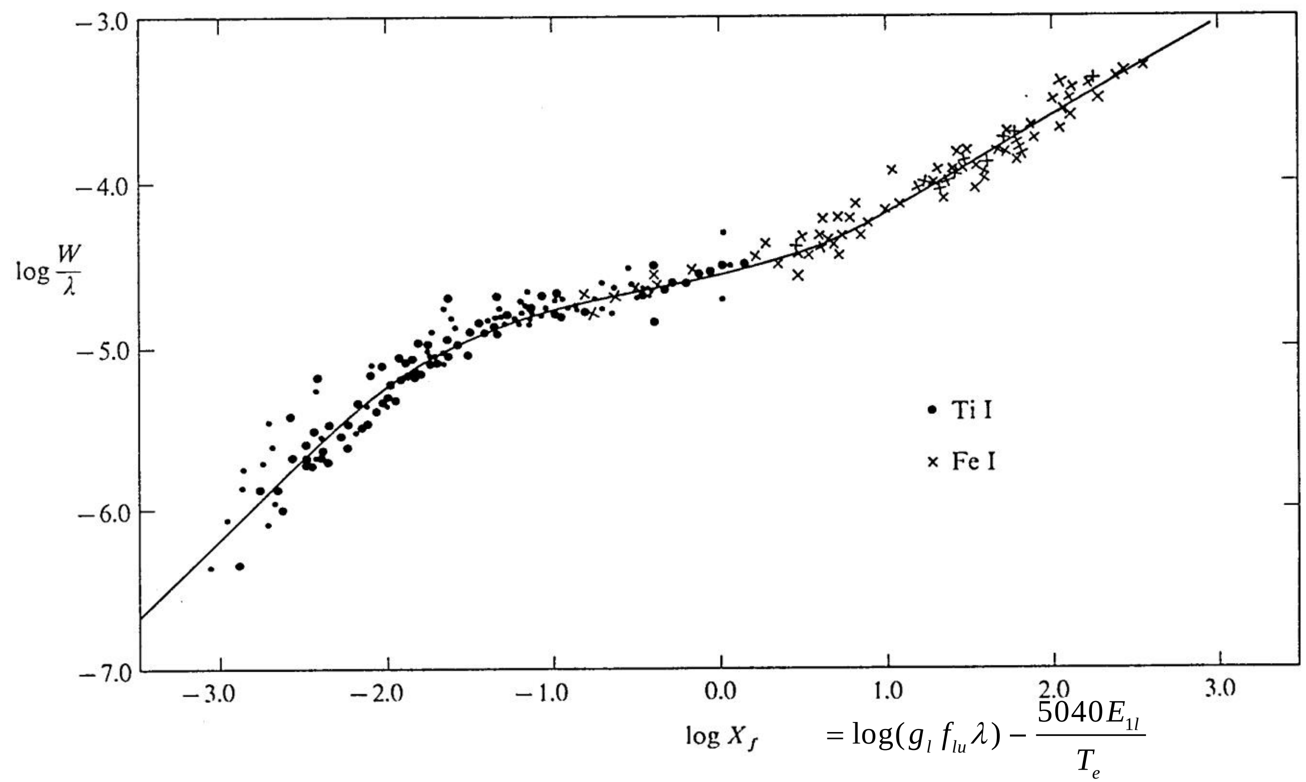


FIGURE 10-2
Curves of growth for pure absorption lines. Note that the larger the value of a, the sooner the square-root part of the curve rises away from the flat part.

measure W(λ) for different lines (with different strengths) of one ionization stage

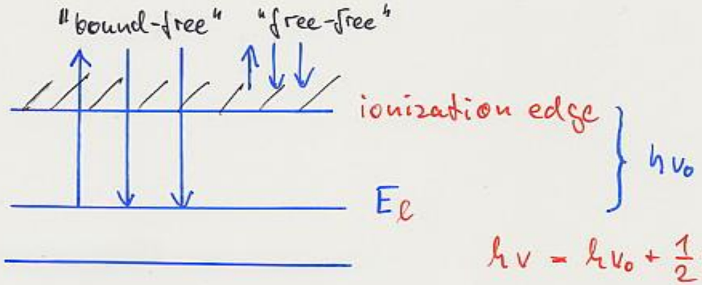
plot as function of $\log(g_l f_{lu} \lambda) - \frac{5040 E_{1l}}{T_e} + \log C$, with "C" fit-quantity

shift horizontally until *theoretical curve of growth* W(β*) is matched => log C => $\frac{n_1}{\chi_c^0}$ => n_1



Empirical curve of growth for solar Fe I and Ti I lines. Abscissa is based on laboratory *f*-values. From (686).
Ti I lines shifted horizontally to define a unique relation

Continous absorption/emission and scattering



$$h\nu = h\nu_0 + \frac{1}{2} m_e v^2$$

- bound free processes

"one" transition : $\chi_\nu^{bf} = n_\ell \, \sigma_{\ell k}(\nu)$, $\nu > \nu_0$

($\sigma_{\ell k}$: absorption cross section; ν_0: threshold)

in total : many processes at one frequency

$$\chi_\nu^{bf} = \sum_{\text{elements}} \sum_{\text{ions}} \sum_{\ell} n_\ell \, \sigma_{\ell k}(\nu)$$

hydrogenic ions $\qquad \sigma_{\ell k}(\nu) = \sigma_0(\ell) \left(\frac{\nu_0}{\nu}\right)^3 \cdot \underbrace{g_{bf}(\nu)}_{\text{"gaunt-factor"} \approx 1}$

EINSTEIN-MILNE relations

$$\chi_\nu^{bf} = \sum_{\text{elements, ions}} \sum_{\ell} \sigma_{\ell k}(\nu) \left(n_\ell - n_\ell^* e^{-h\nu/kT} \right)$$

← stim. emission

$$\eta_\nu^{bf} = \sum \sum_{\ell} \sigma_{\ell k}(\nu) \frac{2h\nu^3}{c^2} n_\ell^* e^{-h\nu/kT}$$

↑ spontaneous emission

$n_\ell^* =$ LTE value

NOTE : $n_\ell = n_\ell^* \Rightarrow S_\nu^{bf} = \frac{\eta_\nu^{bf}}{\chi_\nu^{bf}} = B_\nu(T)$!

free-free processes

(emission process: "bremsstrahlung", decelerated charges radiate!)

$$\chi_\nu^{ff} = n_e \, n_k^{ion} \, \sigma_{kk}(\nu) \left(1 - e^{-h\nu/kT}\right)$$

↑ stim. emission

$\sigma_{kk} \sim \frac{\lambda^3}{\sqrt{T}}$, important in IR and radio!

$$\eta_\nu^{ff} = n_e \, n_k^{ion} \, \sigma_{kk}(\nu) \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

NOTE $S_\nu^{ff} = B_\nu(T)$ always!

Scattering

1. electron scattering

- important for hot stars
- difference to f-f processes

f-f: photon interacts with e^- in ion's central field

$\Rightarrow$ absorption $\Rightarrow$ photon destruction, i.e. "true" process

scattering: without influence of central field, i.e., no "third" partner in collisional process

$\Rightarrow$ no absorption possible, since energy and momentum conservation cannot be fulfilled simultaneously

$\Rightarrow$ scattering

- very high energies (many MeVs)
  Klein Nishina (Q.E.D.)

- high energies
  Compton / inverse Compton scattering
  e^- has low / has high kinetical energy

- low energies (< 12.4 keV $\hat{=}$ 1 Å)
  Thomson scattering

$$\sigma^{TH} = n_e \sigma_T; \quad \sigma_T = \sigma_{class} = \frac{8\pi}{3} \overset{\text{classical } e^- \text{ radius}}{r_0^2} = \frac{8\pi}{3} \frac{e^4}{m_e^2 c^4}$$
$$= 6.65 \cdot 10^{-25}\,\text{cm}^2$$

2. Rayleigh-scattering

actually: line absorption/emission of atoms/molecules far from resonance frequency

$\Rightarrow$ from q.m., Lorentzprofile with $|\nu - \nu_0| \gg \nu_0$

$$\sigma(\nu) = f_{ln} \sigma_T \cdot \left(\frac{\nu}{\nu_0}\right)^4 \sim \lambda^{-4} \quad \text{for } \nu \ll \nu_0$$

- if line transition strong, λ^{-4} decrease of far wings can be of major importance

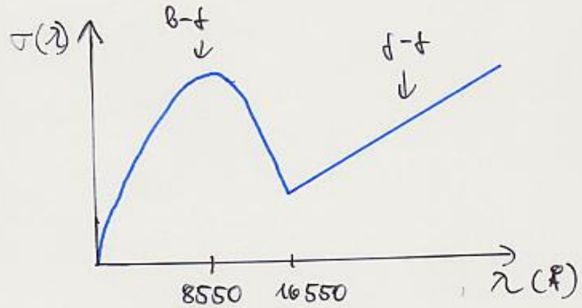
example: Rayleigh wings *of Ly-alpha* in metal-poor, cool stars (G/K-type, few electrons, thus few H^-, see next paragraph) become important opacity source, even in the optical

The H^- ion

- for cool stars (e.g., the sun), one bound state of H^- ($1p + 2e^-$)
  } 0.75 eV $\hat{=}$ 16550 Å

- dominant bf-opacity (also ff component)

- only by inclusion of H^- (Pannekoek + Wildt, 1939) the solar continuum could be explained



Total opacities and emissivities

$$\chi_\nu^{tot} = \chi^{line} \phi(\nu) + \sum \chi_\nu^{bf} + \sum \chi_\nu^{ff} + n_e \sigma_T$$

$$\eta_\nu^{tot} = \chi^{line} \phi(\nu) S_L + \sum \eta_\nu^{bf} + \sum \eta_\nu^{ff} + n_e \sigma_T J_\nu$$

NOTE: for LTE ($n_i = n_i^*$) and $J_\nu = B_\nu$ we have always

$$\frac{\eta_\nu^{tot}}{\chi_\nu^{tot}} = B_\nu(T), \quad \text{good test!}$$

Ionization and Excitation

Ionization and Excitation

had $$\chi_\nu^{\text{Line}} = \frac{\pi e^2}{m_e c} g f_{\ell u} \left(\frac{n_\ell}{g_\ell} - \frac{n_u}{g_u}\right) \phi(\nu)$$

$$\chi_\nu^{bf} = \sum_\ell \left(n_\ell - n_\ell^* \, e^{-h\nu/kT}\right) \sigma_{\ell k}(\nu)$$

$$\sigma^{TH} = n_e \sigma_T$$

How to determine occupation numbers and electron densities?

Local Thermodynamic Equilibrium (LTE)

- each volume element in TE, with temperature $T_e(\tau)$

Hypothesis: collisions ($e^- \Leftrightarrow$ ions) adjust equilibrium

problem: interaction with non-local photons

LTE valid, if

- influence of photons small or
- radiation field Planckian at $T_e(\tau)$ (and isotropic)

Excitation

- Fermi statistics → low density, high temperat.
  → Boltzmann statistics
- distribution of level occupation n_{ij} (per dV, ionizationstage j)

$$\frac{n_{ij}}{n_{1j}} = \frac{g_{ij}}{g_{1j}} e^{-E_{ij}/kT}$$

(if $E_1 = 0$)

- g_i statistical weights (number of degen. states)
- for hydrogen $g_i = 2i^2$, i = princ. quant. number
  " LS coupling $g = (2S+1)(2L+1)$
- if E_i excitation energy with resp. to ground state

$$\boxed{\frac{n_u}{n_\ell} = \frac{g_u}{g_\ell} e^{-E_{u\ell}/kT}}$$ with $E_{u\ell} = E_u - E_\ell$

Ionization

- from generalization of Boltzmann formula for ratio of two (neighbouring) ionic species j and $j+1$

$$n_{1j} \text{ with } g_{1j} \longrightarrow \underset{+\text{free } e^-}{n_{1j+1}} \text{ with } \underbrace{g_{1j+1} \cdot g_{el}}_{\text{weight of final state}}$$

g_{el}: Number of available elements in phase space for free e^-,

$$\frac{d^3r \, d^3p}{h^3} \cdot \underset{\text{spin}}{2} \quad , \quad d^3r = dV = \underset{1e^- \text{ per } dV}{\frac{1}{n_e}}$$

$$\Rightarrow \frac{n_{1j+1}}{n_{1j}} = \frac{1}{n_e} \, 2 \, \frac{g_{1j+1}}{g_1} \left(\frac{2\pi m kT}{h^2}\right)^{3/2} e^{-E_{ion}/kT}$$

Saha eq., 1920

- ratio (i.e., ionization) grows with T (clear!)
  falls with n_e (recomb.!)

- generalization for arbitrary levels:
  calculate n_{1j}, then $n_{ij} = n_{1j} \frac{g_{ij}}{g_{1j}} e^{-E_{i}/kT}$

- all levels

$$N_j = \sum_{i=1}^{\infty} n_{ij} \quad , \quad N_{j+1} = \sum_{i=1}^{\infty} n_{ij+1}$$

- Boltzmann excitation

$$\sum_{i=1}^{\infty} n_{ij} = \frac{n_{1j}}{g_{1j}} \underbrace{\sum_{i=1}^{\infty} g_{ij} \, e^{-E_{ij}/kT}}_{U_j(T) \text{ partition function}} = N_j$$

$$\Rightarrow \frac{n_{1j}}{g_{1j}} = \frac{N_j}{U_j(T)} \quad , \quad \frac{n_{1j+1}}{g_{1j+1}} = \frac{N_{j+1}}{U_{j+1}(T)}$$

$$\Rightarrow \boxed{\frac{N_{j+1} \cdot n_e}{N_j} = \left(\frac{2\pi m kT}{h^2}\right)^{3/2} 2 \, \frac{U_{j+1}(T)}{U_j(T)} \, e^{-E_{ion}/kT}}$$

Note: Summation in partition function until finite maximum, to account for extent of atom

$$\frac{4\pi}{3} r_{MAX}^3 = \Delta V = \frac{1}{N}$$

example hydrogen $r_i = a_0 i^2 = r_{MAX} \Rightarrow i_{MAX}$

An Example: Pure Hydrogen Atmosphere in LTE

given: temperature + density (here: total particle density)

- $N = n_p + n_e + \sum_{i=1}^{i_{MAX}} n_i$

  $= n_p + n_e + \frac{n_1}{g_1} U(T)$

- only hydrogen: $n_p = n_e$

  $\frac{n_e \cdot n_p}{n_1} = \left(\frac{2\pi m k T}{h^2}\right)^{3/2} \frac{2 \cdot \overset{=1}{g_p}}{g_1} e^{-E_{ion}/kT}$

  $\Rightarrow \frac{n_1}{g_1} = \frac{n_e^2}{2} \left(\frac{h^2}{2\pi m k T}\right)^{3/2} e^{E_{ion}/kT}$

- $N = 2 n_e + n_e^2 \underbrace{\frac{1}{2}\left(\frac{h^2}{2\pi m k T}\right)^{3/2} e^{E_{ion}/kT} \cdot U(T)}_{\alpha(T)}$

  $= 2 n_e + n_e^2 \alpha(T)$

$$\Rightarrow n_e = -\frac{1}{\alpha(T)} + \sqrt{\frac{1}{\alpha^2(T)} + \frac{N}{\alpha(T)}}$$

$= n_p \xrightarrow{\text{Saha}} n_1 \xrightarrow{\text{Boltzmann}} n_i$; finished!

- for mixture of elements, analogously!

LTE bf and ff opacities for hydrogen

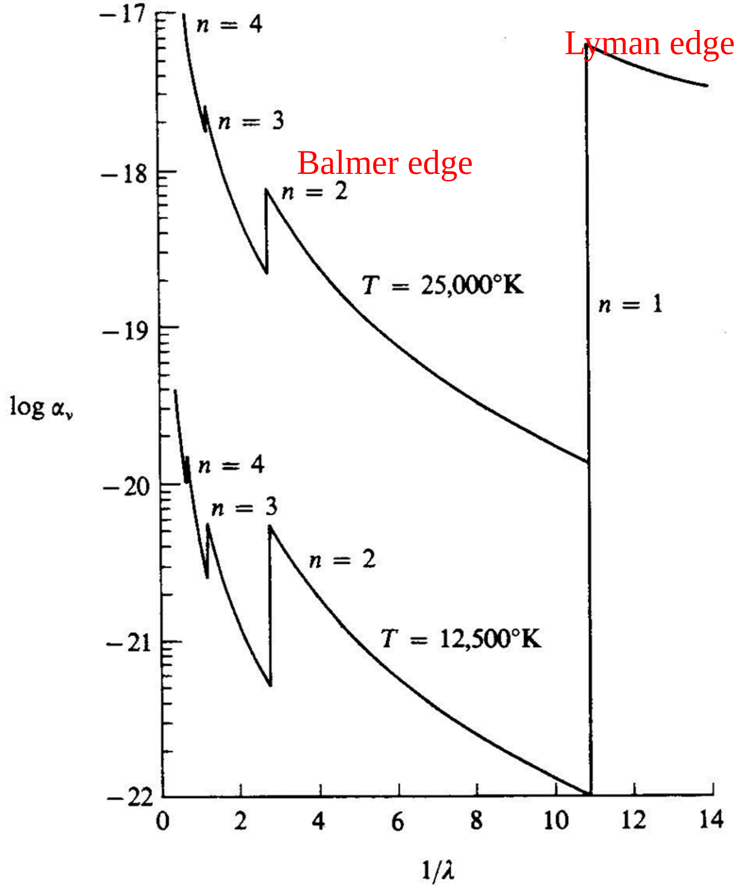


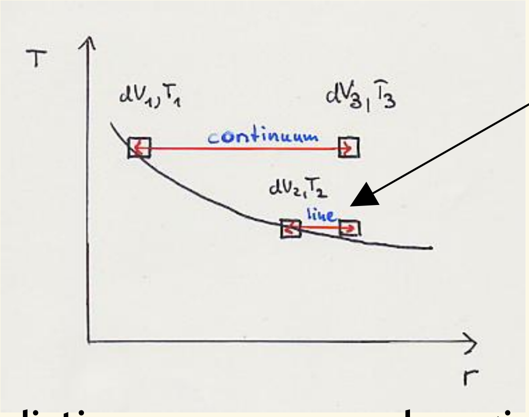
FIGURE 4-1
Opacity from neutral hydrogen at $T = 12{,}500$°K and $T = 25{,}000$°K, in LTE; photoionization edges are labeled with the quantum number of state from which they arise/neutral atom
Ordinate: sum of bound–free and free–free opacity in cm^2/atom; *abscissa*: $1/\lambda$ where λ is in microns.

(L)TE: for **each process**, there exists an inverse process with **identical transition rate**

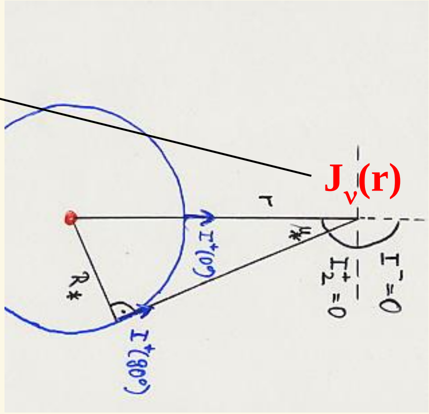
LTE = 'detailed balance' for all processes!

processes = radiative + collisional

- **collisional** processes (and those which are essentially collisional in character, e.g., radiative recombination, ff-emission) in detailed balance, if velocity distribution of colliding particles is **Maxwellian** (valid in stellar atm., see below)
- **radiative** processes: photoionization, photoexcitation (= bb absorption) in detailed balance only if radiation field **Planckian** and **isotropic** (approx. valid only in innermost atmosphere)



$J_\nu(r) = W(r)\, B_\nu$



radiative processes couple regions with different temperatures, as a function of frequency: $\Delta\tau_\nu \leq 1$

anisotropy

Question: **is f(v) dv Maxwellian?**

- elastic collisions -> establish equilibrium
- inelastic collisions/recombinations disturb equilibrium
  inelastic collisions: involve electrons only in certain velocity ranges, tend to shift them to lower velocities
  recombinations : remove electrons from the pool, prevent further elastic collisions

- can be shown: in *typical* stellar plasmas, $t_{el} / t_{rec} \approx 10^{-5} \ldots 10^{-7} \approx t_{el} / t_{inel}$
  => Maxwellian distribution

- under certain conditions (solar chromosphere, corona), certain deviations in high-energy tail of distribution possible

Question: **is T(electron) = T(atom/ion)?**

- equality can be proven for stellar atmospheres with 5,000 K < Te < 100,000 K

When is LTE valid???

roughly: electron collisions $\propto n_e T^{1/2}$ >> photoabsorption rates $\propto I_\nu(T) \propto T^x,\ x \geq 1$

LTE:	T low, n_e high	dwarfs (giants), late B and cooler
NLTE:	T high, n_e low	all supergiants + rest

however: NLTE-effects also in cooler stars, e.g.. iron in sun

TE - LTE - NLTE : a summary

	TE	LTE	NLTE
velocity distribution of particles **Maxwellian (T_e=T_i)**	✓	✓	✓
excitation **Boltzmann**	✓	✓	no
ionization **Saha**	✓	✓	no
source function	$B_\nu(T)$	$B_\nu(T)$, except scattering component	only $S_\nu^{ff} = B_\nu(T)$
radiation field	$J_\nu = B_\nu(T)$	$J_\nu \neq B_\nu(T)$, equality only for $\tau_\nu \geq \left(\frac{1}{\varepsilon_\nu}\right)^{1/2}$	$J_\nu \neq B_\nu(T)$ dito

Kinetic equilibrium

NLTE – Kinetic equilibrium

- do NOT use Saha-Boltzmann, however calculate occupation numbers by assuming statistical equilibrium
- for stationarity ($\partial/\partial t = 0$) and as long as kinematic time-scale >> atomic transition time scales (usually valid)

$$\boxed{\sum_{j \neq i} n_i P_{ij} = \sum_{j \neq i} n_j P_{ji}} \quad \forall i$$

n_i: occupation number (atomic species, ionization stage, level)

P_{ij} transitionrate from level $i \to j$ ($\dim P_{ij} = s^{-1}$)

- in words: the number of all possible transitions from level i into other states j is balanced by the number of transitions from all other states j into level i.

$\Rightarrow$ linear equation system for n_i, has to be closed by abundance equation

$$\sum n_{ik} = n_k$$

if n_{ik} the occupation numbers of species k and n_k the total particle density of k

Transition rates

- collisional processes bb, ionization/rec.
- radiative processes bb, ionization/rec.

Radiative processes depend on radiation field
radiation field depends on opacities
opacities depend on occupation numbers

Iteration required!

... no so easy, however possible

Note: to obtain reliable results, order of

30 species
3-5 ionizationstages / species
20...1000 level / ion
100,000... some 10^6 transitions
to be considered in parallel

requires large data base of atomic quantities (energies, transitions, cross sections)
fast algorithm to calculate radiative transfer!

Solution of the rate equations – a simple example

HAD: for each atomic level, the sum of all populations must be equal to the sum of all depopulations (for stationary situations)

example: 3-niveau atom with continuum

assume: all rate coefficients are known (i.e., also the radiation field)

=> rate equations (equations of statistical equilibrium)

$$-n_1\left[R_{1k}+C_{1k}+R_{12}+C_{12}+R_{13}+C_{13}\right]+n_2(R_{21}+C_{21})+n_3(R_{31}+C_{31})+n_k(R_{k1}+C_{k1})=0$$

$$n_1(R_{12}+C_{12})-n_2\left[R_{2k}+C_{2k}+R_{21}+C_{21}+R_{23}+C_{23}\right]+n_3(R_{32}+C_{32})+n_k(R_{k2}+C_{k2})=0$$

$$n_1(R_{13}+C_{13})+n_2(R_{23}+C_{23})-n_3\left[R_{3k}+C_{3k}+R_{31}+C_{31}+R_{32}+C_{32}\right]+n_k(R_{k3}+C_{k3})=0$$

$$n_1(R_{1k}+C_{1k})+n_2(R_{2k}+C_{1k})+n_3(R_{3k}+C_{1k})-n_k\left[R_{k1}+C_{k1}+R_{k2}+C_{k2}+R_{k3}+C_{k3}\right]=0$$

with

R_{ij}, radiative bound-bound transitions (lines!)

R_{ik} radiative bound-free transitions (ionizations)

R_{ki} radiative free-bound transitions (recombinations)

C_{ij} collisional bound-bound transitions

C_{ik} collisional bound-free transitions

C_{ki} collisonal free-bound transitions

in matrix representation =>

$$P = \underbrace{\begin{pmatrix} -(R_{1k}+C_{1k}+R_{12}+C_{12}+R_{13}+C_{13}) & (R_{21}+C_{21}) & (R_{31}+C_{31}) & (R_{k1}+C_{k1}) \\ (R_{12}+C_{12}) & -(R_{2k}+C_{2k}+R_{21}+C_{21}+R_{23}+C_{23}) & (R_{32}+C_{32}) & (R_{k2}+C_{k2}) \\ (R_{13}+C_{13}) & (R_{23}+C_{23}) & -(R_{3k}+C_{3k}+R_{31}+C_{31}+R_{32}+C_{32}) & (R_{k3}+C_{k3}) \\ (R_{1k}+C_{1k}) & (R_{2k}+C_{2k}) & (R_{3k}+C_{3k}) & -(R_{k1}+C_{k1}+R_{k2}+C_{k2}+R_{k3}+C_{k3}) \end{pmatrix}}_{\text{rate matrix, diagonal elements sum of all depopulations}}$$

$$P * \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 (= n_k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Rate matrix is singular, since, e.g., last row linear combination of other rows (negative sum of all previous rows)

THUS: LEAVE OUT arbitrary line (mostly the last one, corresponding to ionization equilibrium) and REPLACE by inhomogeneous, linearly independent equation for all n_i, to obtain unique solution

particle number conservation for considered atom:

$$\sum_{i=1}^{N} n_i = \alpha_k N_{\mathrm{H}}, \text{ with } \alpha_{\mathrm{k}} \text{ the abundance of element k}$$

NOTE 1: numerically stable equation solver required, since typically hundreds of levels present, and (rate-) coefficients of highly different orders of magnitude

NOTE 2: occupation numbers n_i depend on radiation field (via radiative rates), and radiation field depends (non-linearly) on n_i (via opacities and emissivities)
=> Clever iteration scheme required!!!!

Example for extreme NLTE condition
Nebulium (= [OIII] 5007, 4959) in Planetary Nebulae

mechanism suggested by *I. Bowen* (1927):

- low-lying meta-stable levels of OIII(2.5 eV) **collisionally excited** by free electrons (resulting from photoionization of hydrogen via "hot", *diluted* radiation field from central star)
- Meta-stable levels become **strongly populated**
- **radiative decay** results in **very strong** [OIII] emission lines
- impossible to observe suggested process in laboratory, since *collisional deexitation* (no photon emitted)) *much stronger than radiative decay under terrestrial conditions*.
- Thus, after detection new element proposed , "nebulium"

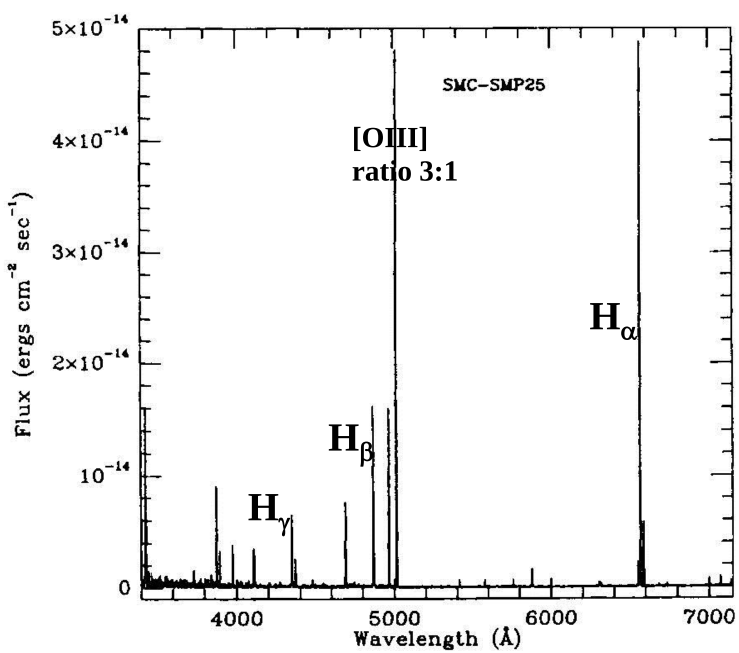


FIG. 1*a*

Condition for radiative decay

NOTE: $A_{ml} \le 10^{-2}$ (typical values are 10^{7})

$n_m A_{ml} \gg n_m n_e q_{ml}(T_e)$, with metastable level m

$\rightarrow n_e \ll n_e(\text{crit})$,

$$n_e(\text{crit}) = \frac{A_{ml}}{q_{ml}(T_e)}, \quad \mathrm{q}_{\mathrm{ml}} = 8.63 \cdot 10^{-6} \frac{\Omega(l,m)}{g_m \sqrt{T_e}}$$

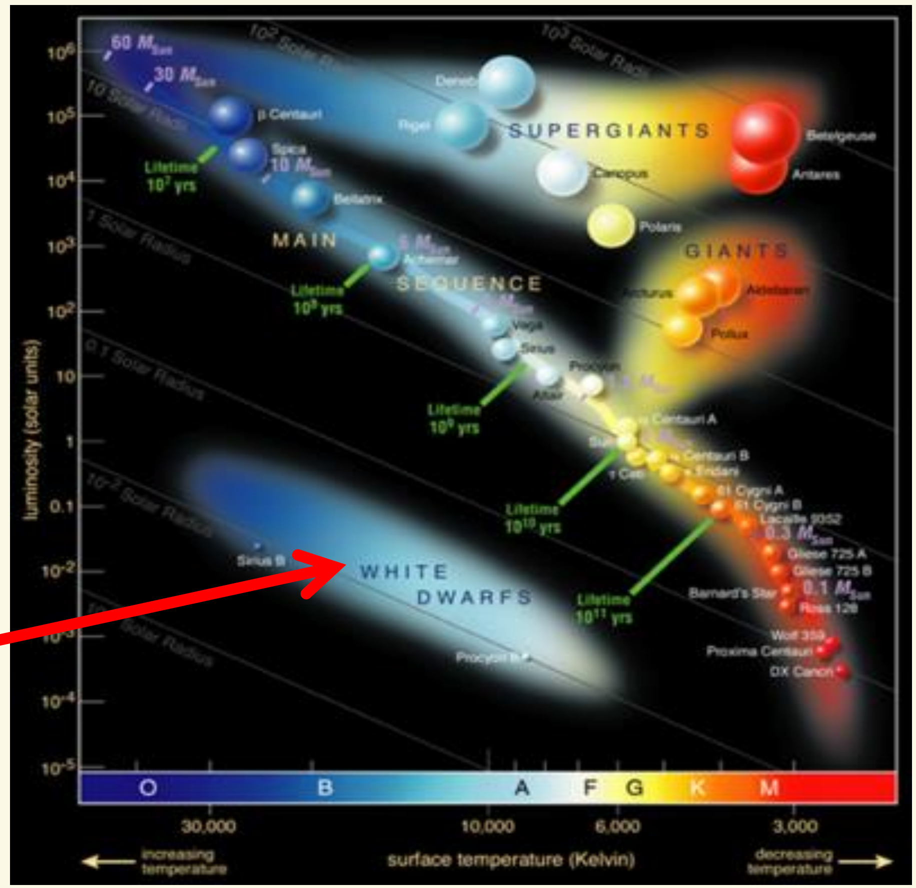
$\Omega(l,m)$ collisional strength, order unity

for typical temperatures $T_e \approx 10{,}000\text{K}$ and [OIII] 5007,

we have $n_e(\text{crit}) \approx 4.9 \cdot 10^5 \text{cm}^{-3}$,

much larger than typical nebula densities

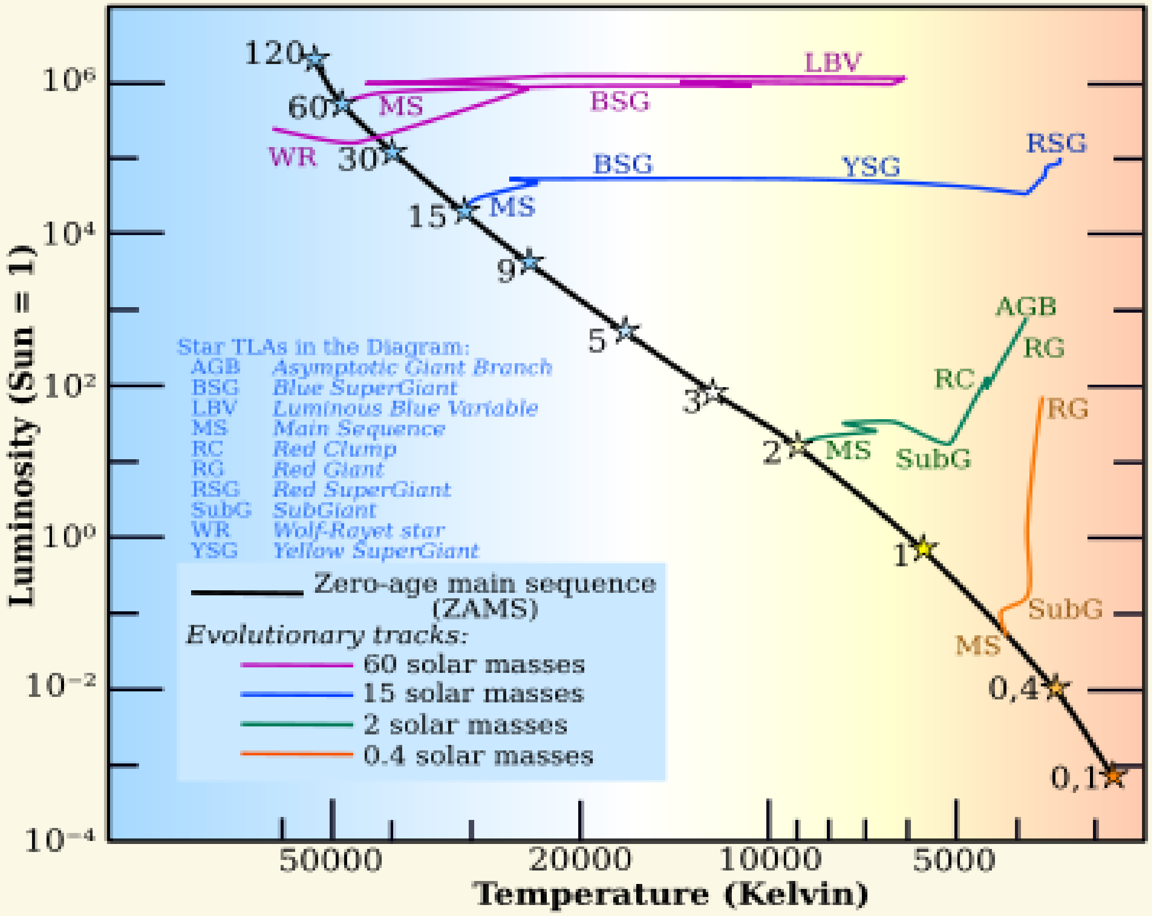
A tour de modeling and analysis of stellar atmospheres throughout the HRD



except for
white dwarfs

Some different types of stars…

Hot luminous stars:
Massive,
main-sequence (MS) or
evolved, ~10 Rsun.
Strong, fast stellar
winds

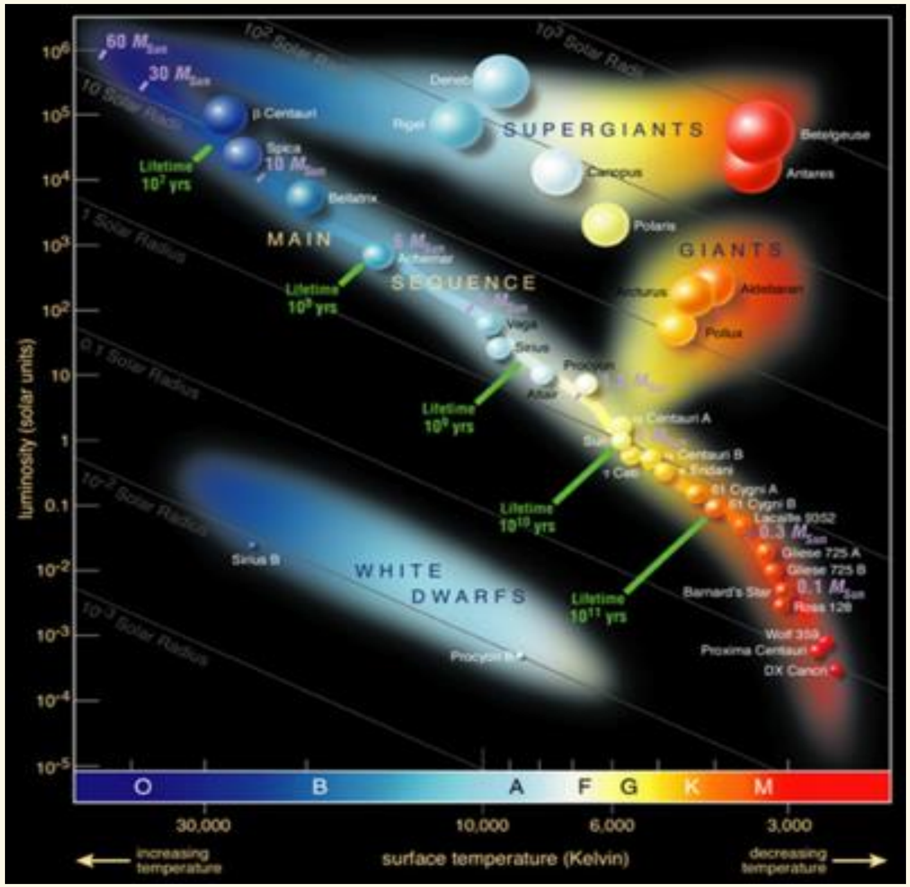


Cool, luminous stars
(RSG, AGB):
Massive or low/intermediate mass, evolved,
several 100 (!) Rsun.
Strong, slow stellar winds

Solar-type stars:
Low-mass, on or near MS,
hot surrounding coronae,
weak stellar wind
(e.g. solar wind)

A tour de modeling and analysis of stellar atmospheres throughout the HRD

Different regimes require different key input physics and assumptions

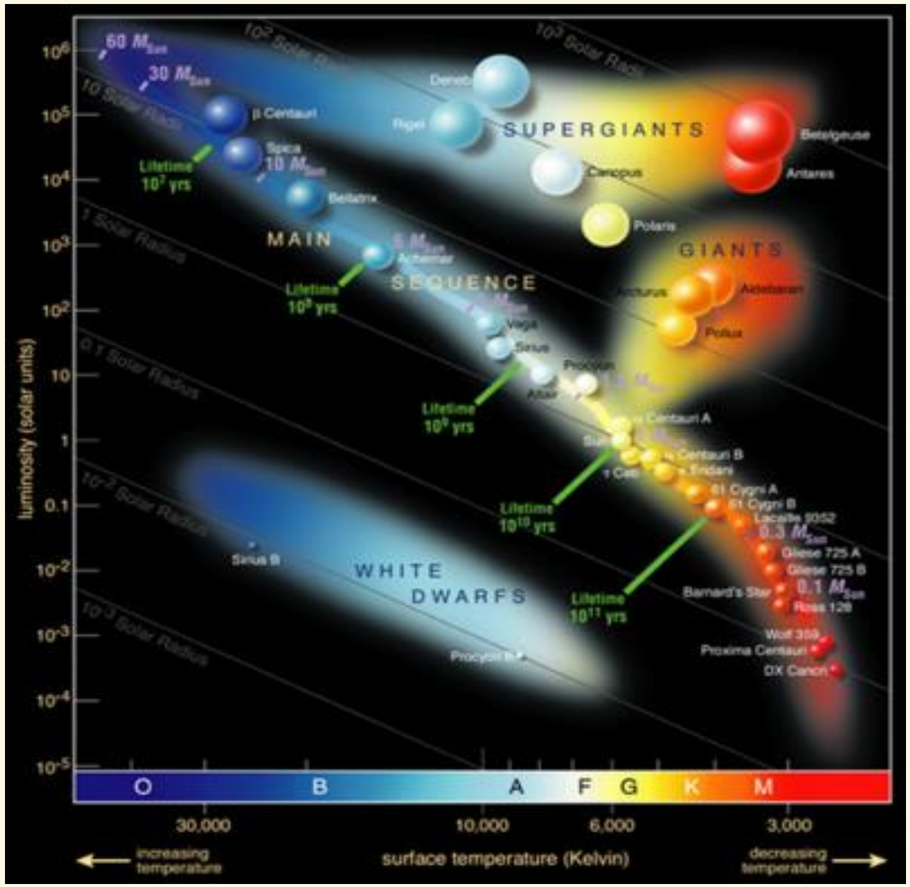


- LTE or NLTE
- Spectral line blocking/blanketing
- (sub-) Surface convection
- Geometry and dimensionality
- Velocity fields and outflows

Spectroscopy and Photometry

ALSO:
Analysis
of different
WAVELENGTH
BANDS
is different

(X-ray, UV, optical, infra-red…)



Depends on where in atmosphere light escapes from

Question: Why is this “formation depth” different for different wavebands and diagnostics?

Spectroscopy and Photometry (see Chap. 2)

...gives insight into and understanding of our cosmos

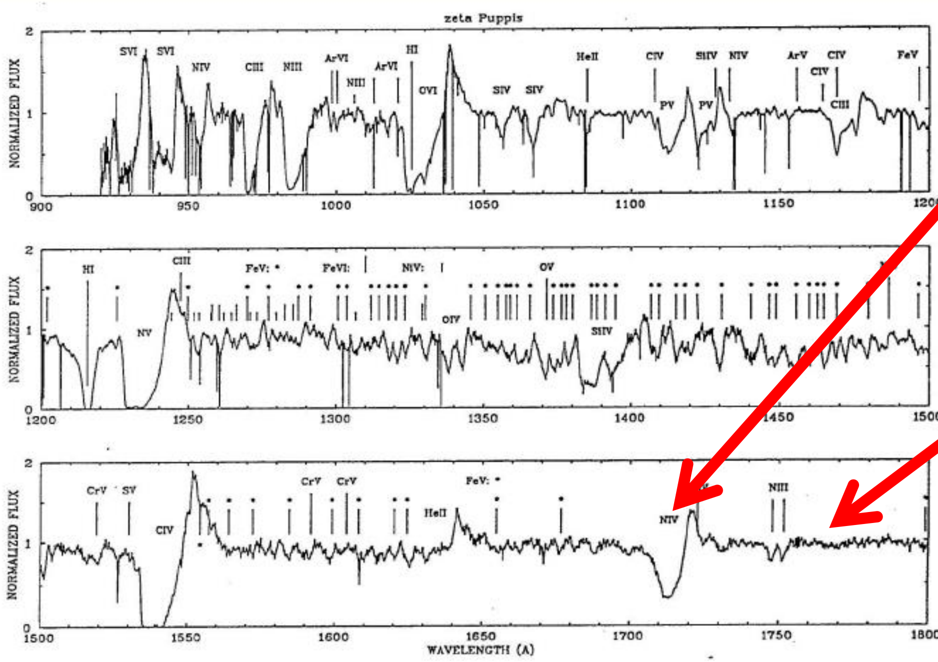
- requires
  - plasma physics, plasma is "normal" state of atmospheres and interstellar matter (plasma diagnostics, line broadening, influence of magnetic fields,...)
  - atomic physics/quantum mechanics, interaction light/matter (micro quantities)
  - radiative transfer, interaction light/matter (macroscopic description)
  - thermodynamics, thermodynamic equilibria: TE, LTE (local), NLTE (non-local)
  - hydrodynamics, atmospheric structure, velocity fields, shockwaves,...

- provides
  - stellar properties, mass, radius, luminosity, energy production, chemical composition, properties of outflows
  - properties of (inter) stellar plasmas, temperature, density, excitation, chemical comp., magnetic fields

- INPUT for stellar, galactic and cosmologic evolution and for stellar and galactic structure

Spectroscopy (see Chap. 2)

UV spectrum of the O4I(f) supergiant ζ Pup



montage of **Copernicus** (**λ < 1500 Å, high res. mode, Δλ ≈ 0.05 Å, Morton & Underhill 1977**) **and IUE (Δλ ≈ 0.1 Å) observations**

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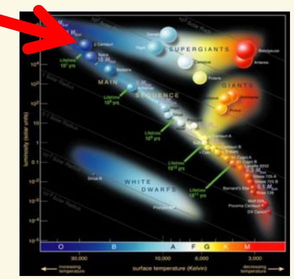
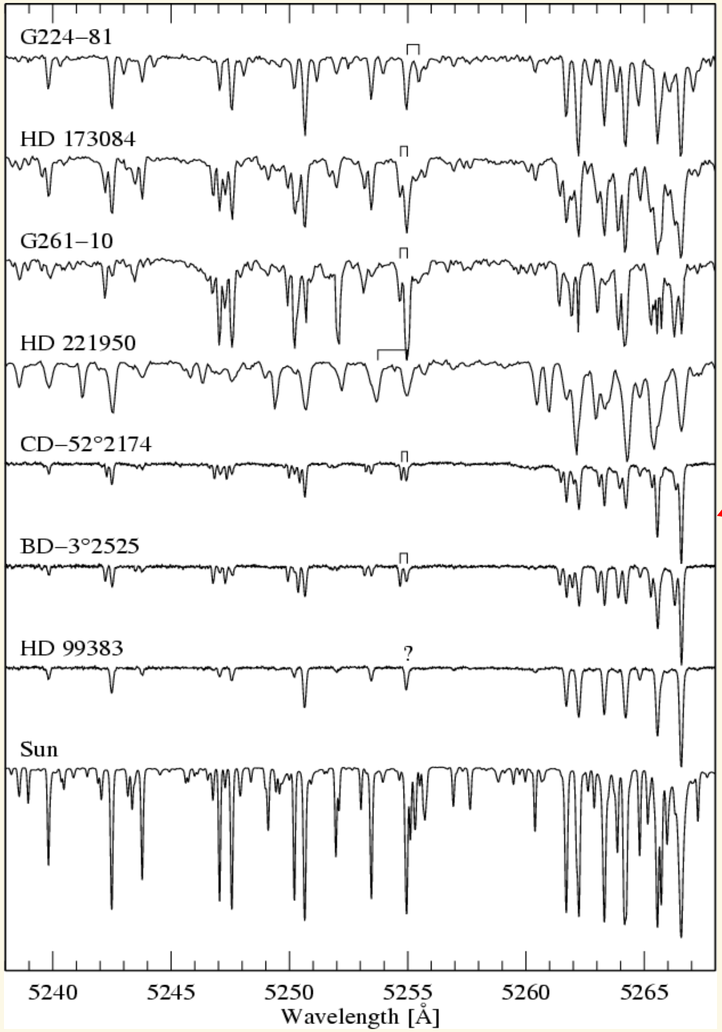
UV "P-Cygni" lines formed in rapidly accelerating, hot stellar winds

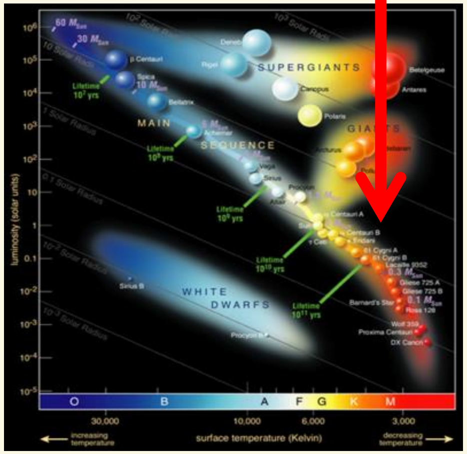
(quasi-) Continuum formed in (quasi-) hydrostatic photosphere



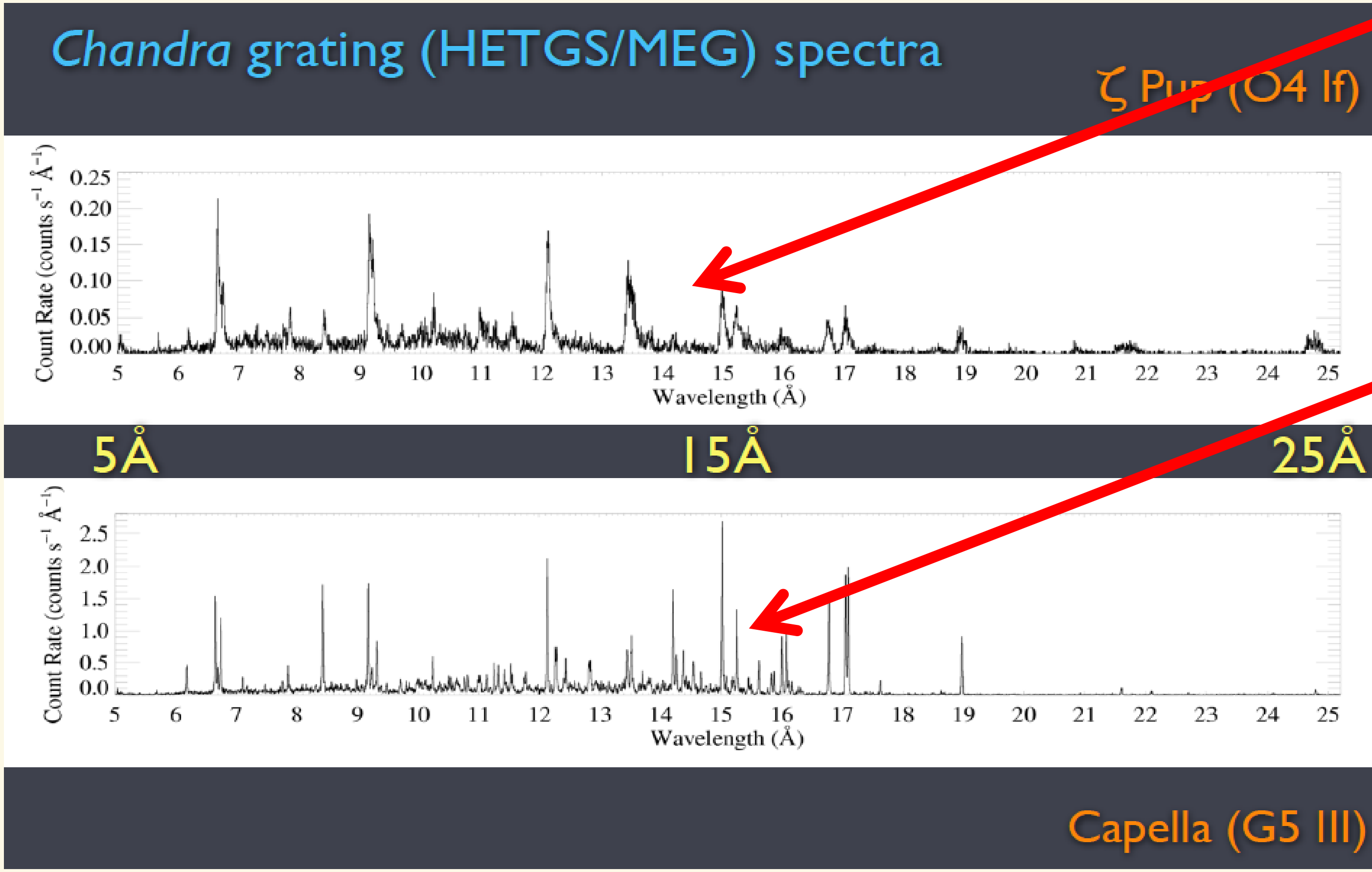
Spectroscopy



Lines and continuum in the optical around 5200 Å, in cool solar-type stars, formed in the photosphere



Spectroscopy



X-rays from hot stars, formed in shocks in stellar wind

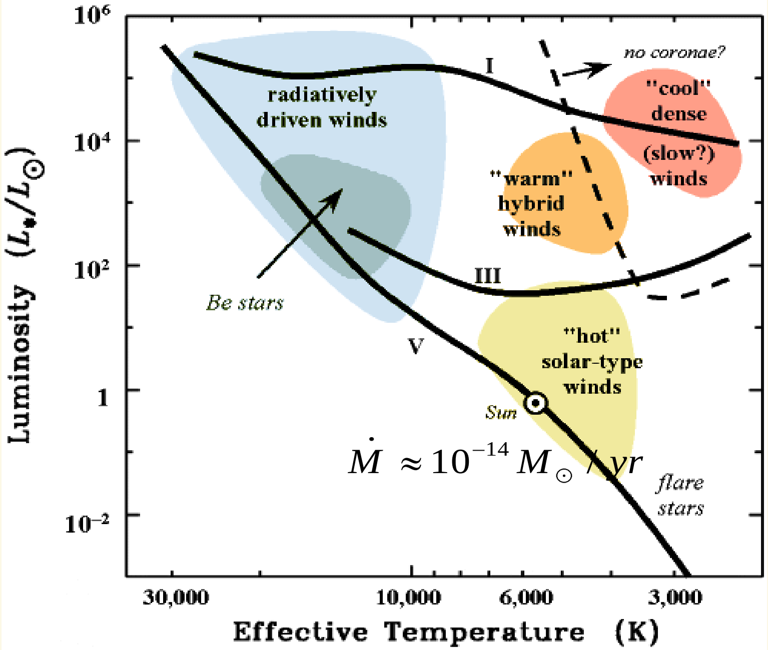
X-rays from cool stars, formed in hot corona

A tour de modeling and analysis of stellar atmospheres throughout the HRD

Stellar Winds (see Chap. 8)

KEY QUESTION: What provides the force able to overcome gravity?

$$\dot{M} \approx 10^{-4} \ldots 10^{-8} M_\odot / yr$$



- LTE or NLTE
- Spectral line blocking/blanketing
- (sub-) Surface convection
- Geometry and dimensionality
- Velocity fields and outflows

A tour de modeling and analysis of stellar atmospheres throughout the HRD

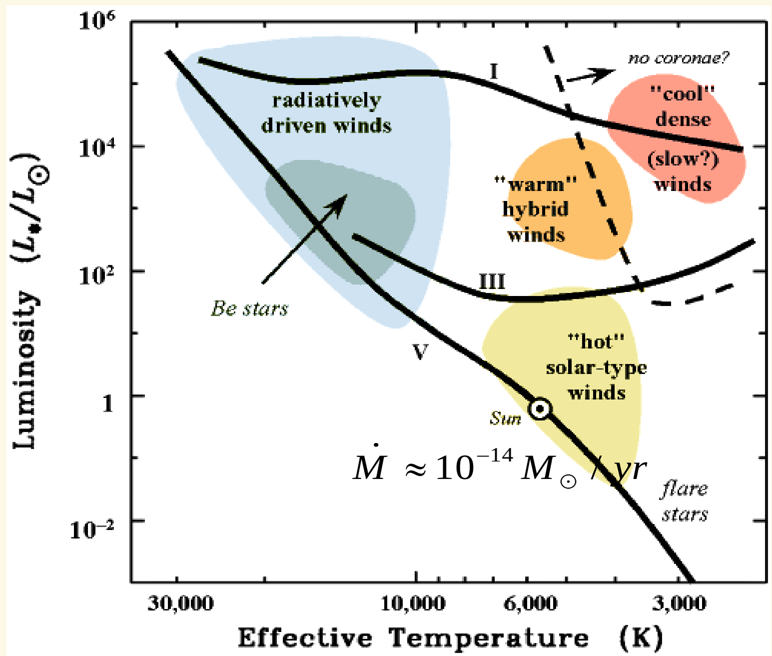
KEY QUESTION: What provides the force able to overcome gravity?

Pressure gradient
in hot coronae of solar-type stars

Radiation force:
Dust scattering
(in pulsation-levitated material)
in cool AGB stars
(Höffner and colleagues)

Same mechanism In cool RSGs?

$$\dot{M} \approx 10^{-4} ... 10^{-8} M_{\odot} / yr$$



- LTE or NLTE
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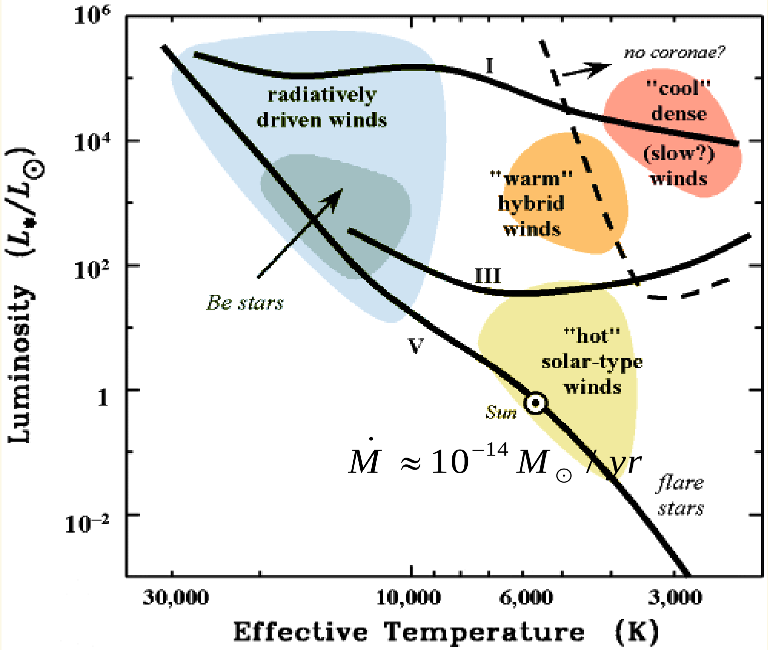
- Velocity fields and outflows

A tour de modeling and analysis of stellar atmospheres throughout the HRD

KEY QUESTION: What provides the force able to overcome gravity?

Radiation force:
line scattering in hot, luminous stars
→ done at USM, more to follow in Chap. 8

$$\dot{M} \approx 10^{-4} \ldots 10^{-8} M_\odot / yr$$



- LTE or NLTE
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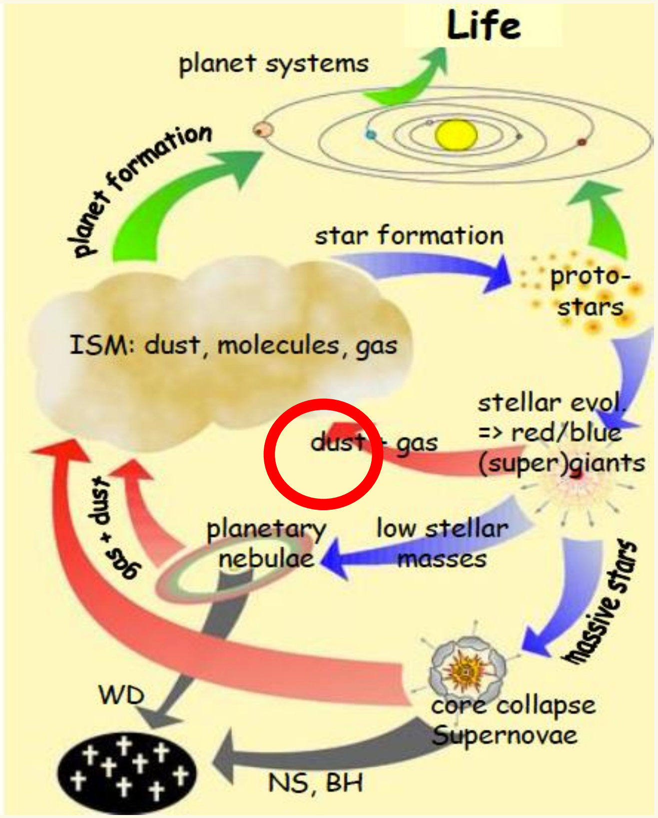
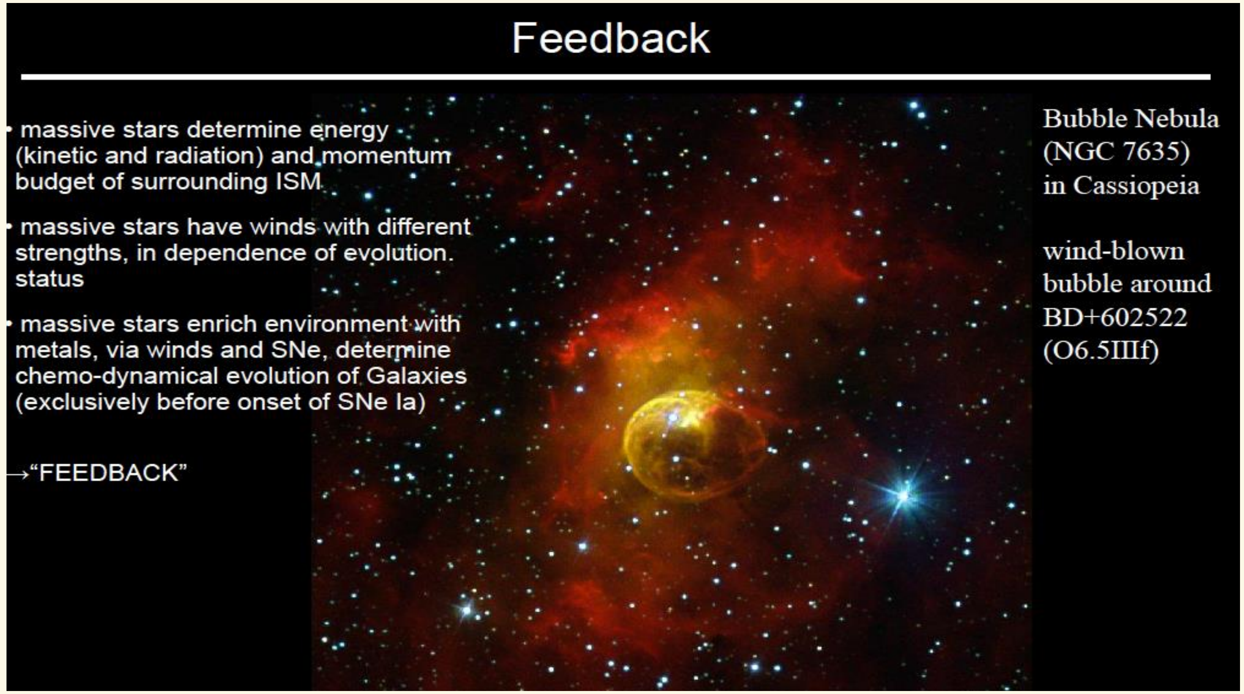
Question: How do you think the high mass loss of stars with high luminosities affects the evolution of the star and its surroundings?

Stellar Atmospheres in practice

from introductory slides …

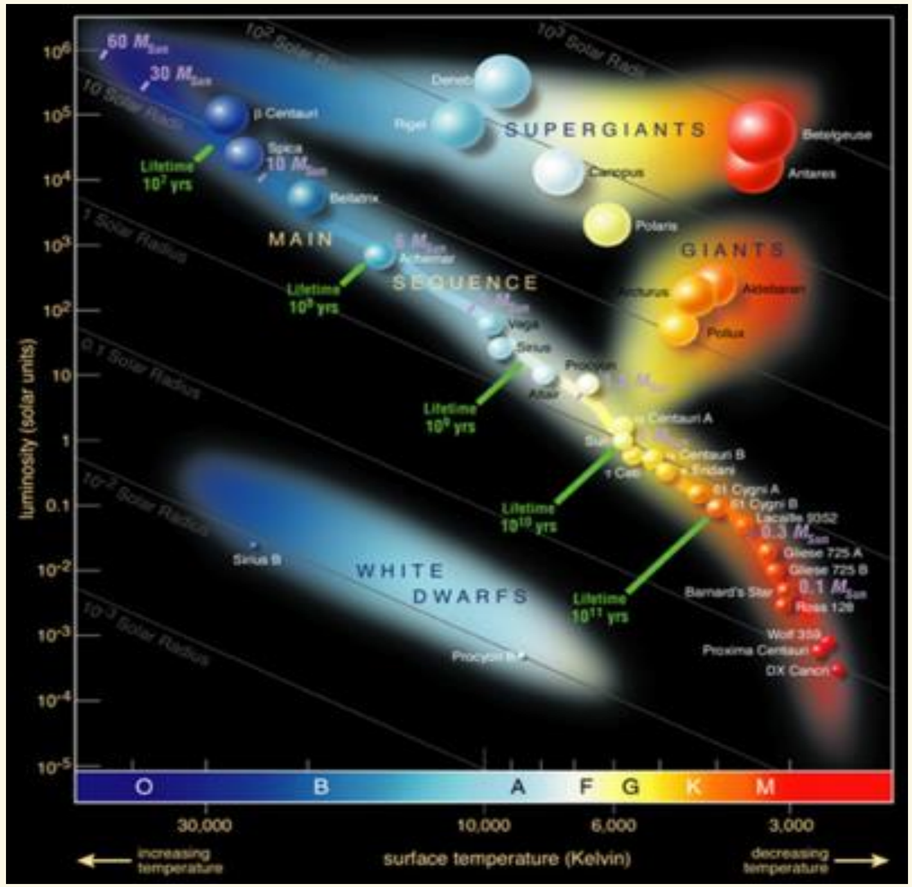




Stellar Winds from evolved hot and cool stars control late evolution, and feed the ISM with nuclear processed material

A tour de modeling and analysis of stellar atmospheres throughout the HRD

In the following, we focus on stellar photospheres



From Chap. 6

Summary: stellar atmospheres - the solution principle

THUS problem of stellar atmospheres solved (in principle, without convection, p-p geometry, static)

Given log g_*, T_{eff}, abundances

(A) hydrostatic equilibrium

$$\frac{dp_{gas}}{dz} = -\rho(g_* - g_{rad}) \; ; \quad g_{rad} = \frac{4\pi}{c\rho}\int_0^\infty \chi_\nu H_\nu d\nu = \frac{4\pi}{c\rho}\left(\sigma^{TH} H(z) + \int_0^\infty \chi_\nu^{rest} H_\nu d\nu\right)$$

$$\Rightarrow \frac{dp_{gas}}{dz} = -\rho g_* + \sigma^{TH}\frac{\sigma_B T_{eff}^4}{c} + \frac{4\pi}{c}\int_0^\infty \chi_\nu^{rest} H_\nu d\nu \qquad H = \frac{1}{4\pi}\sigma_B T_{eff}^4 \; \left(= \frac{1}{4\pi} F\right)$$

(B) equation of rad. transfer

$$\mu \frac{dI_\nu}{dz} = \chi_\nu(S_\nu - I_\nu) \quad \forall \nu, \mu \quad \Rightarrow J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu) d\mu \; ; \quad H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\mu d\mu$$

(C) a) radiative equilibrium

scattering terms cancel, since conservative

$$\int_0^\infty (\eta_\nu - \kappa_\nu J_\nu) d\nu = \int_0^\infty \left\{(\sigma^{TH} J_\nu + \chi_\nu^{rest} S_\nu^{rest}) - (\sigma^{TH} + \chi_\nu^{rest}) J_\nu\right\} d\nu = \int_0^\infty \chi_\nu^{rest}(S_\nu^{rest} - J_\nu) d\nu \stackrel{?}{=} 0$$

b) flux-conservation: $4\pi \int_0^\infty H_\nu(z) d\nu = 4\pi H(z) \stackrel{?}{=} \sigma_B T_{eff}^4$ $\Rightarrow \Delta T(z)$ $\rightarrow \Delta \chi_\nu(z)$ etc

(D) equation of state $p_{gas}(z) = \frac{k_B}{\mu m_H}\rho(z) T(z)$

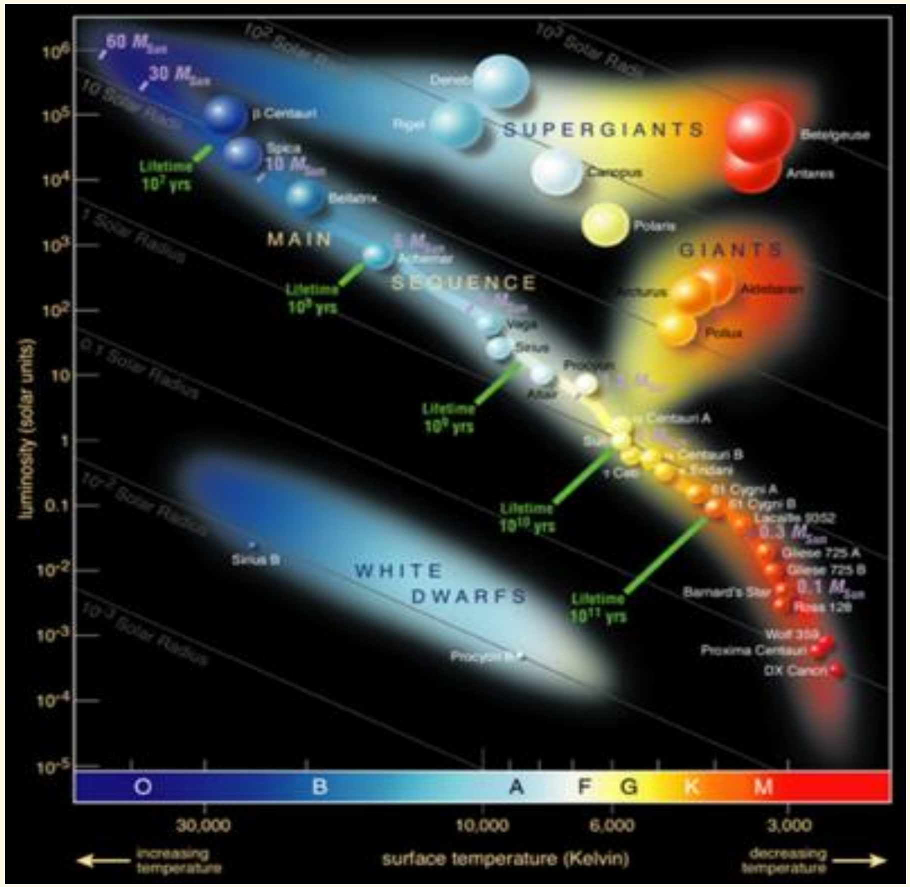
solution by iteration!

Solution of differential equations A and B by discretization
differential operators => finite differences
all quantities have to be evaluated on suitable grid

Eq. of radiative transfer (B) usually solved by the so-called Feautrier and/or Rybicki scheme

- OBSER- VATIONS!!!

A tour de modeling and analysis of stellar atmospheres throughout the HRD



- LTE or NLTE
- Spectral line blocking/blanketing
- (sub-) Surface convection
- Geometry and dimensionality
- Velocity fields and outflows

LTE or NLTE? (see Chap. 7)

When is LTE valid???

roughly: electron collisions $\propto n_e T^{1/2}$ >> photoabsorption rates $\propto I_\nu(T) \propto T^x, \; x \geq 1$

LTE: T low, n_e high — dwarfs (giants), late B and cooler
NLTE: T high, n_e low — all supergiants + rest

however: NLTE-effects also in cooler stars, e.g.. iron in sun

HOT STARS:

Complete model atmosphere and synthetic spectrum must be calculated in NLTE

NLTE calculations for various applications (including Supernovae remnants) within the expertise of USM

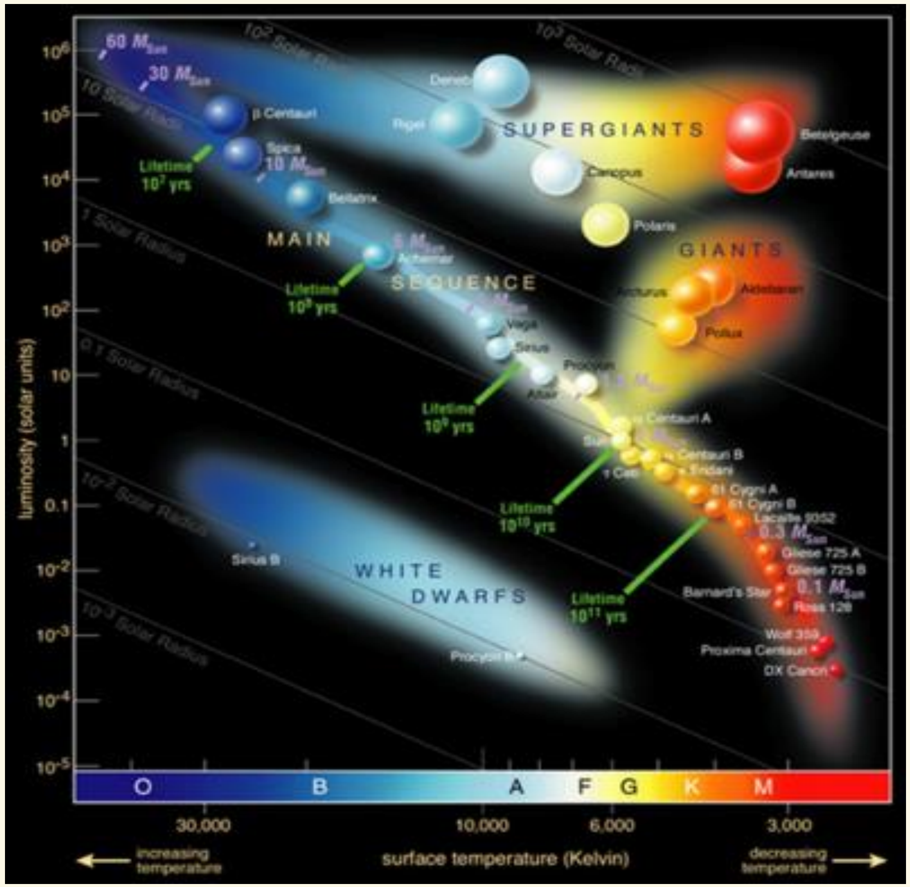
COOL STARS:

Standard to neglect NLTE-effects on atmospheric structure, might be included when calculating line spectra for individual "trace" elements (typically used for chemical abundance determinations)

BUT: See work by Phoenix-team (Hauschildt et al.)
ALSO: RSGs still somewhat open question

Stellar Atmospheres in practice

A tour de modeling and analysis of stellar atmospheres throughout the HRD



- LTE or NLTE
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Spectral line blocking/blanketing

- Effects of numerous -- literally millions -- of (primarily metal) spectral lines upon the atmospheric structure and flux distribution
- Q: Why is this tricky business?

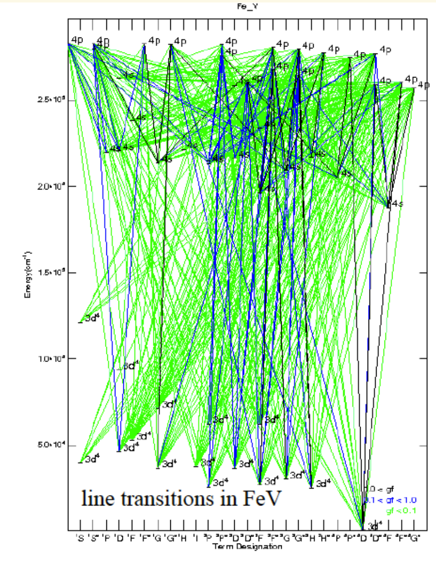
Spectral line blocking/blanketing

- Effects of numerous -- literally millions -- of (primarily metal) spectral lines upon the atmospheric structure and flux distribution
- Q: Why is this tricky business?

- Lots of atomic data required (thus atomic physics and/or experiments)
- LTE or NLTE?
- What lines are relevant?
  (i.e., what ionization stages? Are there molecules present?)

Techniques:

Opacity Distribution Functions
Opacity-Sampling
Direct line by line calculations



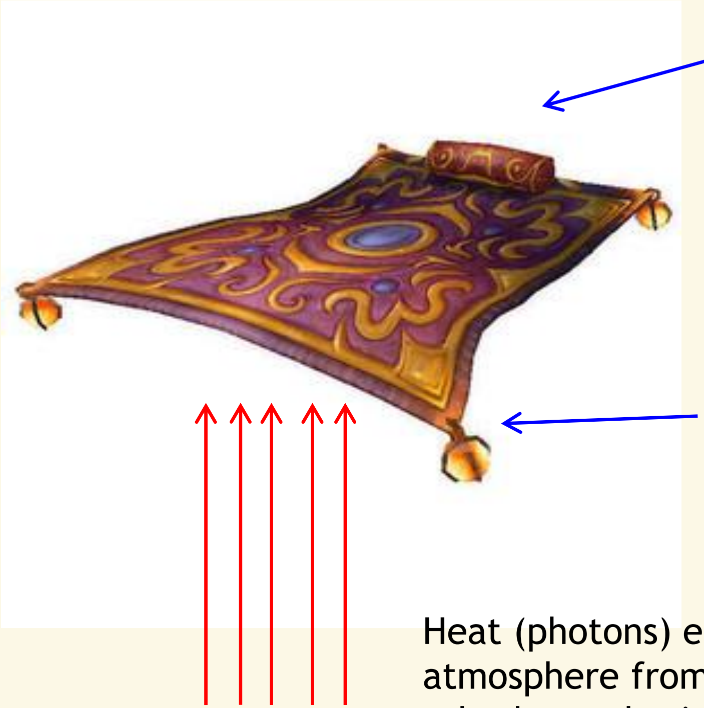
Spectral line blocking/blanketing

Back-warming (and surface-cooling)

Numerous absorption lines
- "block" (E)UV radiation flux

Total flux conservation
- demands these photons be emitted elsewhere → redistributed to optical/infra-red

Lines act as "blanket",
- whereby back-scattered line photons are (partly) thermalized and thus heat up deeper layers



Spectral line blocking/blanketing

Back-warming and flux redistribution

...occur in stars of all spectral types

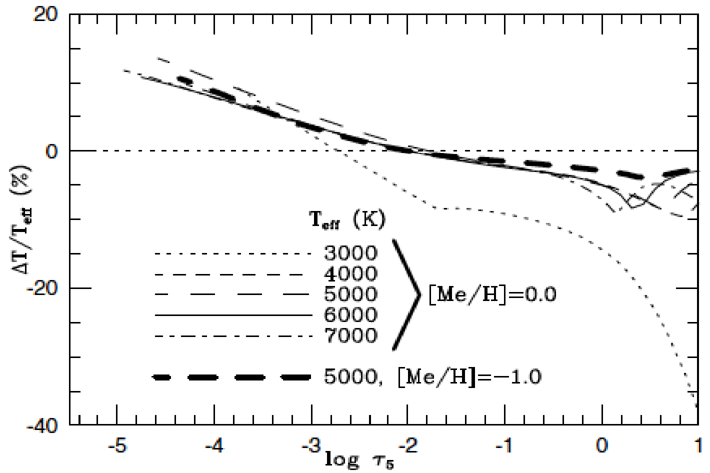


Fig. 4. The effects of switching off line absorption on the temperature structure of a sequence of models with $\log g = 3.0$ and solar metallicity. Note that $\Delta T \equiv T(\mathrm{nolines}) - T(\mathrm{lines})$. It is seen that the blanketing effects are fairly independent of effective temperature for models with $T_{\mathrm{eff}} \geq 4000$.

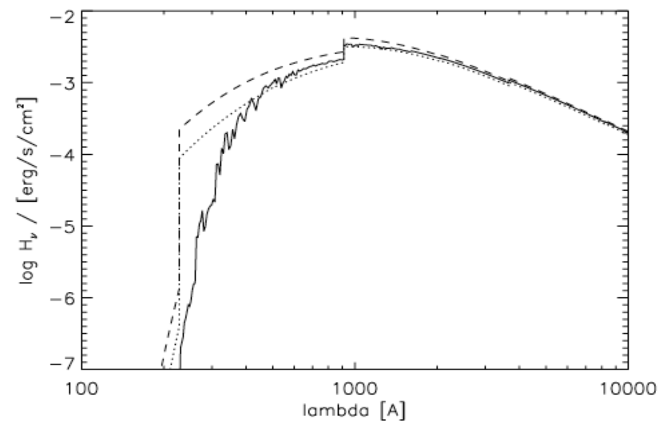


Fig. 10. Emergent Eddington flux H_ν as function of wavelength. Solid line: Current model of HD 15629 (O5V((f)) with parameters from Table 1 ($T_{\mathrm{eff}} = 40\,500$ K, $\log g = 3.7$, "model 1"). Dotted: Pure H/He model without line-blocking/blanketing and negligible wind, at same T_{eff} and $\log g$ ("model 2"). Dashed: Pure H/He model, but with $T_{\mathrm{eff}} = 45\,000$ K and $\log g = 3.9$ ("model 3").

Back warming in cool stars
(from Gustafsson et al. 2008)

UV to optical flux redistribution in hot stars
(from Repolust, Puls & Hererro 2004)

Spectral line blocking/blanketing

Back-warming and flux redistribution

…occur in stars of all spectral types

From Puls et al. 2008

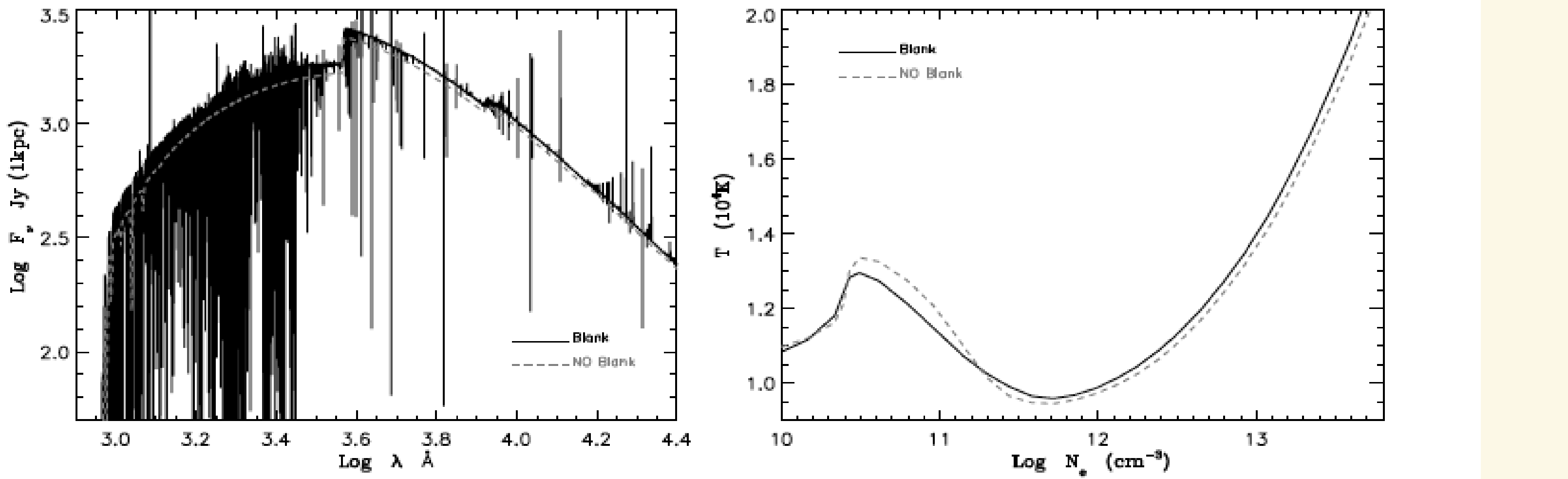


Fig. 9 Effects of line blanketing (solid) vs. unblanketed models (dashed) on the flux distribution ($\log F_\nu$ (Jansky) vs. $\log \lambda$ (Å), left panel) and temperature structure ($T\,(10^4$ K) vs. $\log n_e$, right panel) in the atmosphere of a late B-hypergiant. Blanketing blocks flux in the UV, redistributes it towards longer wavelengths and causes back-warming.

Spectral line blocking/blanketing

Back-warming – effect on effective temperature

RECALL: T_{eff} -- or total flux (plane-parallel) -- fundamental input parameter in model atmosphere!

$$F = \sigma_B T_{eff}^4$$

T_{eff} in cool stars derived, e.g., by optical photometry

From Gustafsson et al. 2008: Estimate effect by assuming a blanketed model with T_{eff} such that the deeper layers correspond to an unblanketed model with effective temperature $T'_{eff} > T_{eff}$

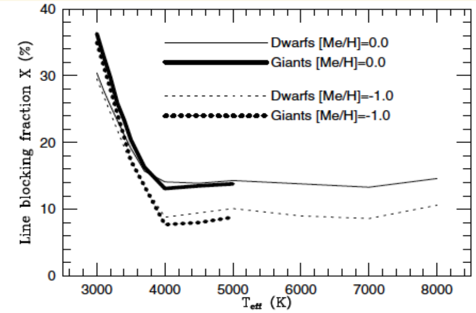


Fig. 3. The blocking fraction X in percent for models in the grid with two different metallicities. The dwarf models all have $\log g = 4.5$ while the giant models have $\log g$ values increasing with temperature, from $\log g = 0.0$ at $T_{eff} = 3000$ K to $\log g = 3.0$ at $T_{eff} = 5000$ K.

$$T'_{eff} = (1 - X)^{-\frac{1}{4}} \cdot T_{eff}, \tag{35}$$

where X is the fraction of the integrated continuous flux blocked out by spectral lines,

$$X = \frac{\int_0^\infty (F_{cont} - F_\lambda) d\lambda}{\int_0^\infty F_{cont} d\lambda}. \tag{36}$$

Question: Why does the line blocking fraction increase for very cool stars?

Spectral line blocking/blanketing

Back-warming – effect on effective temperature

RECALL: T_{eff} -- or total flux (plane-parallel) -- fundamental input parameter in model atmosphere!

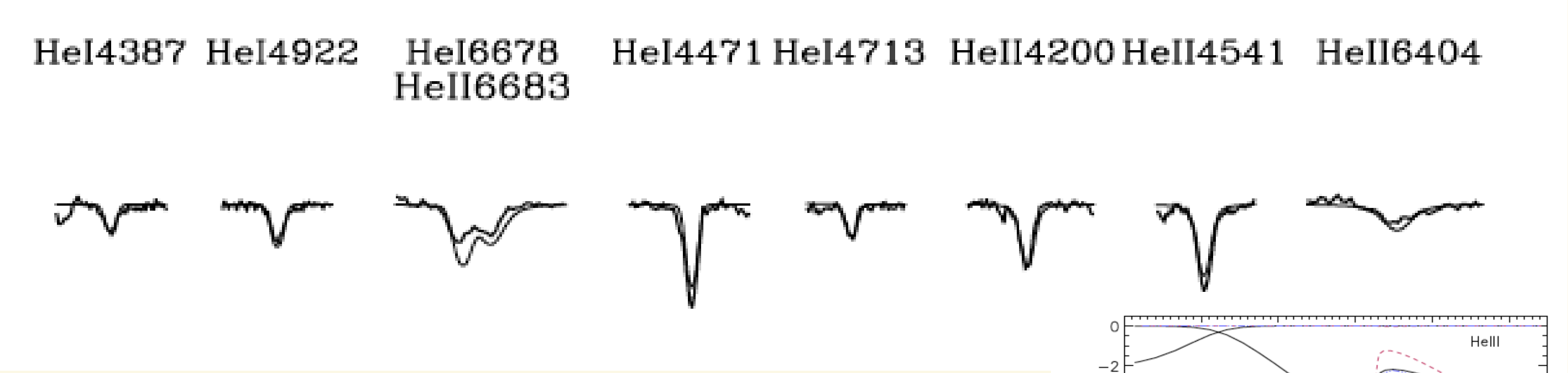
Previous slide were LTE models. In hot stars, everything has to be done in NLTE...

$$F = \sigma_B T_{\text{eff}}^4$$

Question: Why is optical photometry generally NOT well suited to derive Teff in hot stars?

Spectral line blocking/blanketing

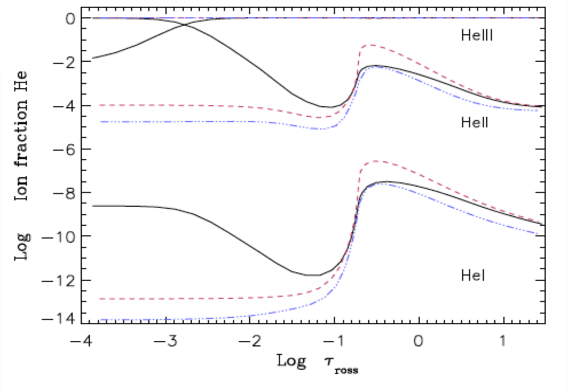
Instead, He ionization-balance is typically used
(or N for the very hottest stars, or, e.g., Si for B-stars)



Simultaneous fits to observed HeI and HeII lines
-- from Repolust, Puls, Hererro (2004)

- Back-warming shifts ionization balance toward more completely ionized Helium in blanketed models

→ thus fitting the same observed spectrum requires lower T_{eff} than in unblanketed models

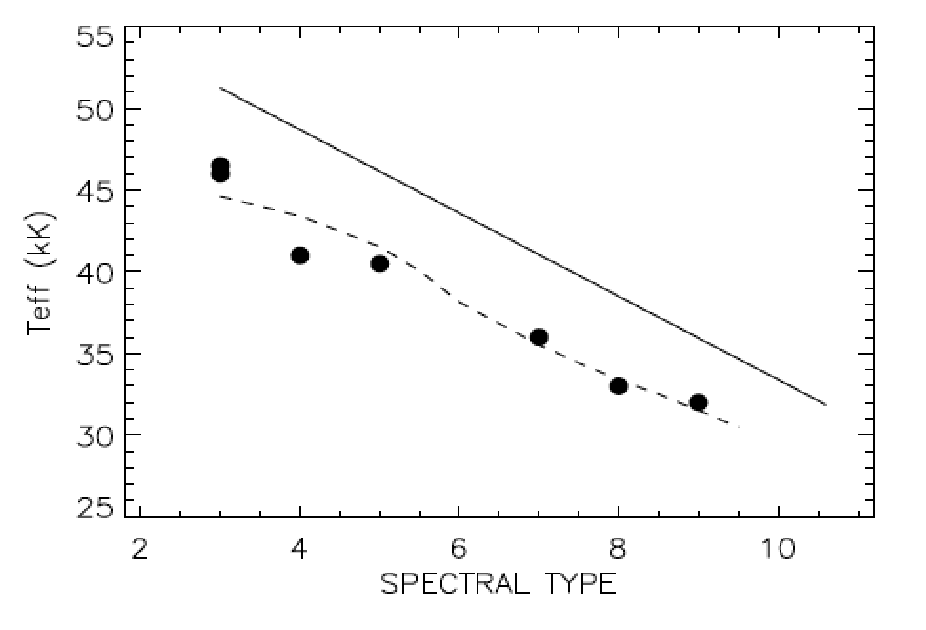


- black – blanketed Teff=45 kK
- red – unblanketed Teff=45 kK
- blue – unblanketed Teff= 50 kK

Spectral line blocking/blanketing

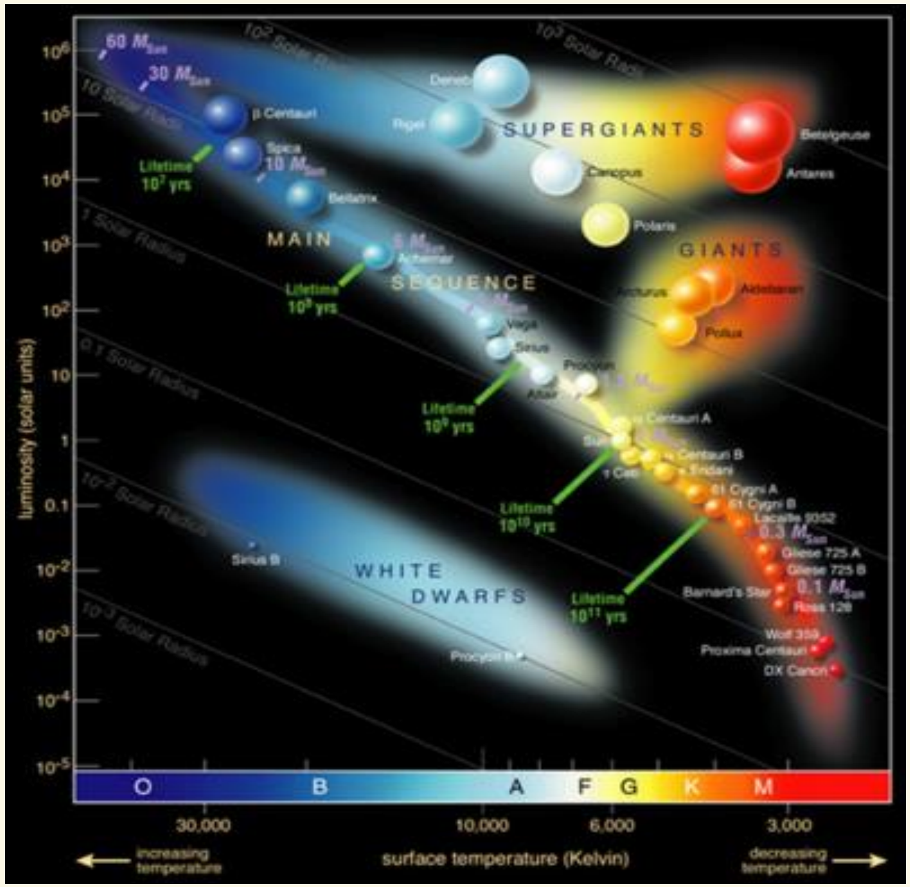
Instead, He ionization-balance is typically used
(or N for the very hottest stars, or, e.g., Si for B-stars)

Result: In hot O-stars with Teff~40,000 K, back-warming can lower the derived T_{eff} as compared to unblanketed models by several thousand degrees! (~ 10 %)



New T_{eff} scale for O-dwarf stars. Solid line – unblanketed models. Dashed – blanketed calibration, dots – observed blanketed values (from Puls et al. 2008)

A tour de modeling and analysis of stellar atmospheres throughout the HRD



- LTE or NLTE
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Surface Convection

from Chap. 6

Convection

energy transport not only by radiation, however also by

- waves
- heat conduction
- convection

$\Big\}$ not efficient in typical stellar atmospheres, but ... coronae, chromospheres, white dwarfs

Thus

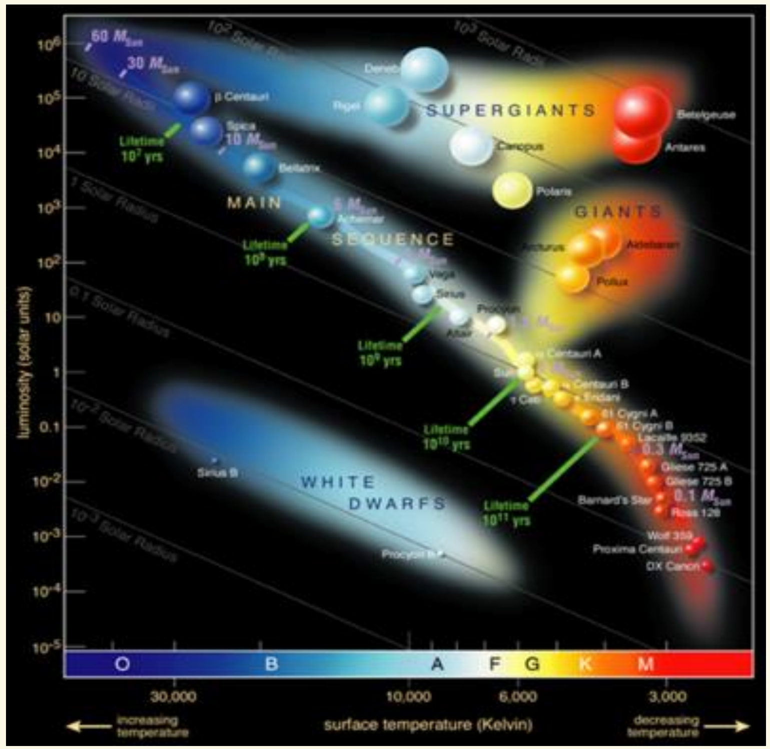
total flux = const

$$\nabla \cdot \left(\underline{F}^{\mathrm{rad}} + \underline{F}^{\mathrm{conv}} \right) = 0$$

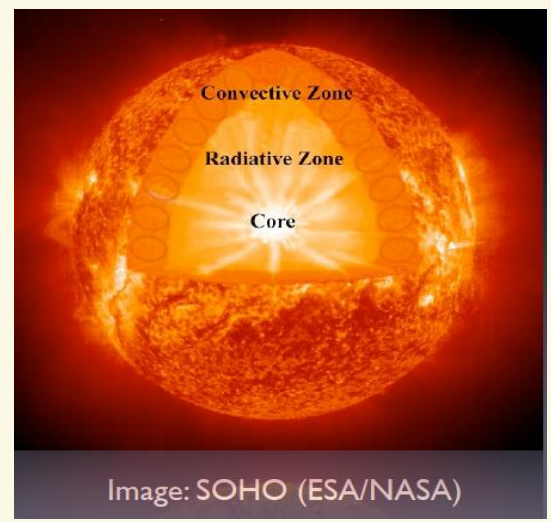
(in quasi-hydrostatic atmospheres)

Surface Convection

OBSERVATIONS:
"Sub-surface" convection in layers T~160,000 K (due to iron-opacity peak) currently discussed also in hot stars



- H/He recombines in atmospheres of cool stars
  - → Provides MUCH opacity
  - → Convective Energy transport

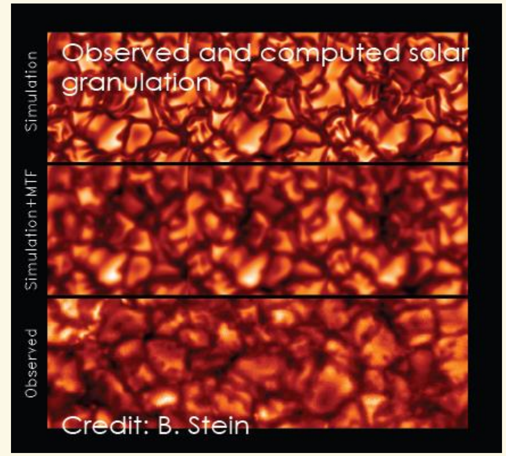


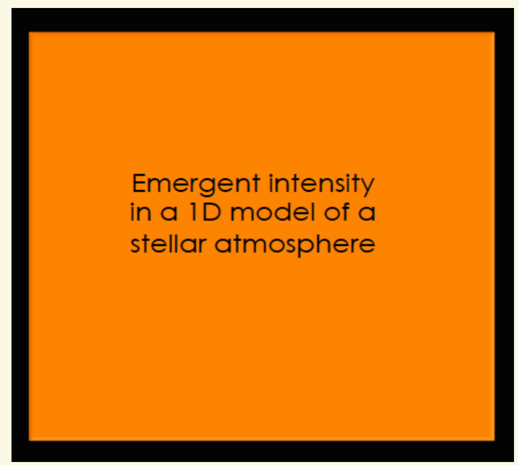
Surface Convection

Traditionally accounted for by rudimentary "mixing-length theory" (see Chap. 6) in 1-D atmosphere codes

BUT:
- Solar observations show very dynamic structure
- Granulation and lateral inhomogeneity

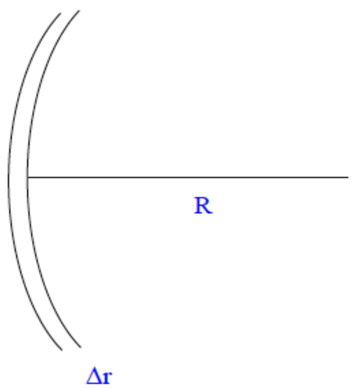
→ Need for full 3-D radiation-hydrodynamics simulations in which convective motions occur spontaneously if required conditions fulfilled (all physics of convection 'naturally' included)





Surface Convection

Solar-type stars:
Photospheric extent << stellar radius
Small granulation patterns



example: **the sun**

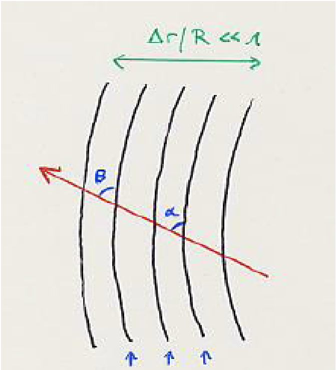
$R_{sun} \approx 700{,}000$ km
Δr (photo) ≈ 300 km

=> $\Delta r / R \approx 4\ 10^{-4}$

BUT corona
$\Delta r / R$ (corona) ≈ 3

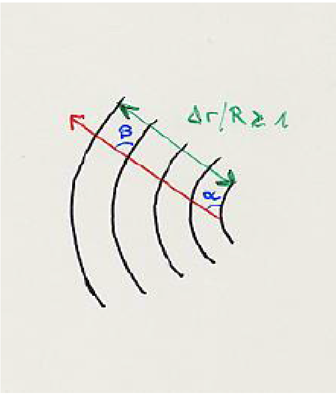
as long as $\Delta r / R << 1$ => plane-parallel symmetry

light ray through atmosphere



lines of constant temperature and density (isocontours)

curvature of atmosphere insignificant for photons' path : $\alpha = \beta$



significant curvature : $\alpha \neq \beta$, spherical symmetry

examples

solar photosphere / cromosphere	solar corona
atmospheres of	atmospheres of
main sequence stars	supergiants
white dwarfs	expanding envelopes (stellar winds)
giants (partly)	of OBA stars, M-giants and supergiants

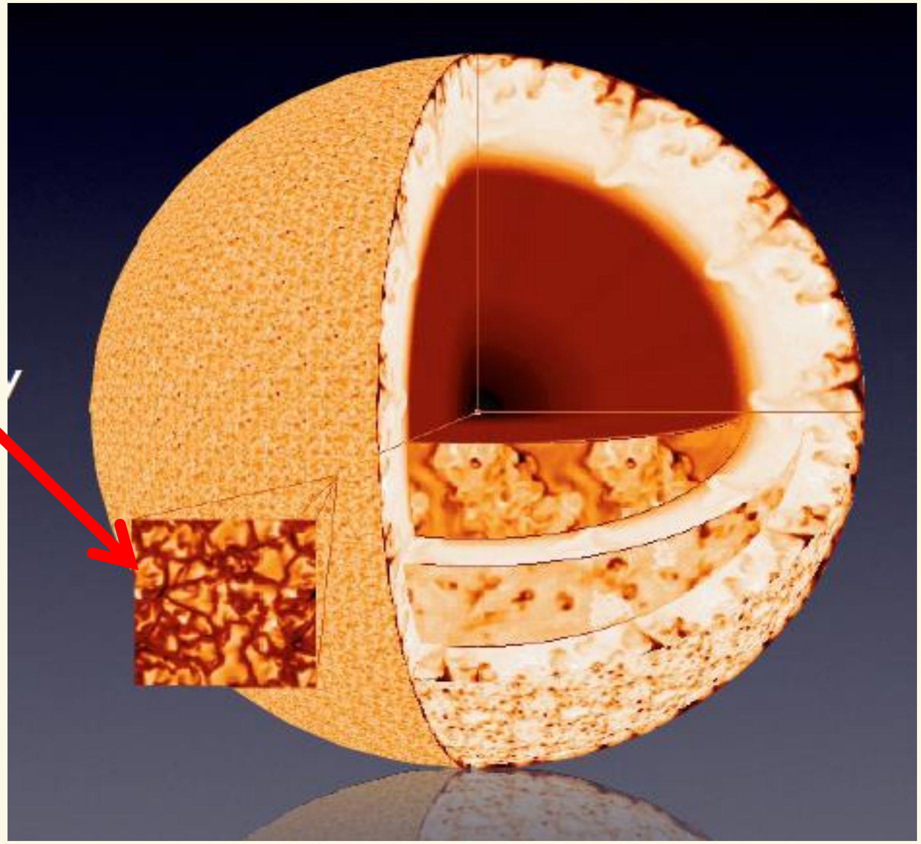
from Chap. 3

Surface Convection

Solar-type stars:
Atmospheric extent << stellar radius
Small granulation patterns

→
Box-in-a-star Simulations

(cmp. plane-parallel approximation)



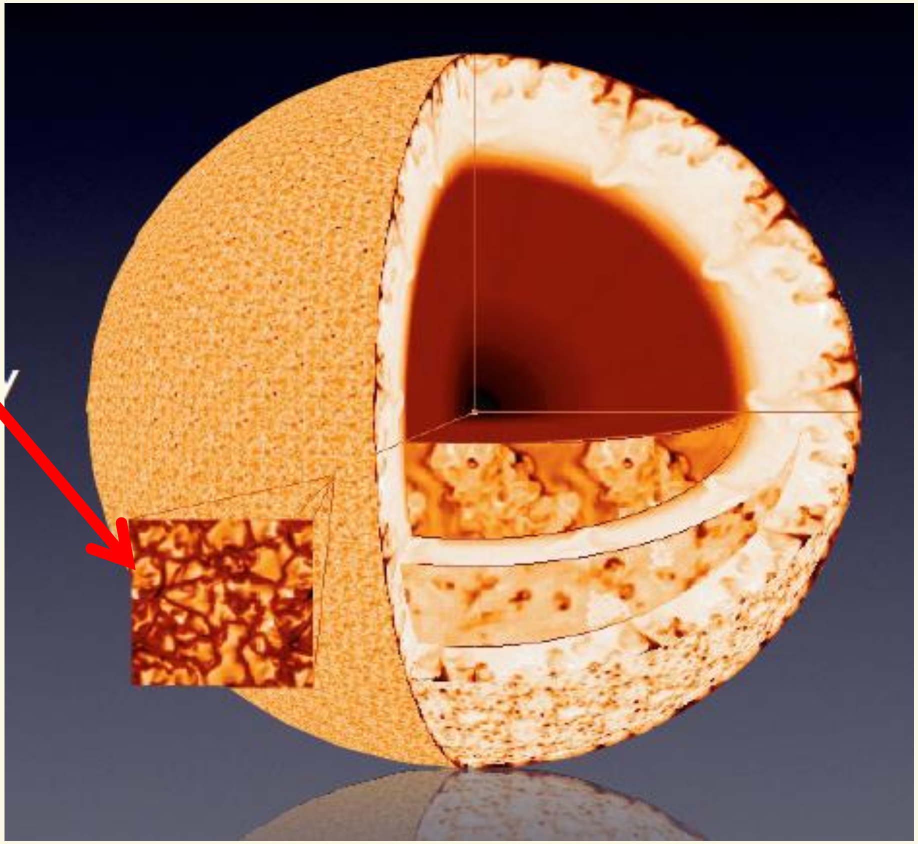
From Wolfgang Hayek

Surface Convection

Approach
(teams by Nordlund, Steffen):

Solve radiation-hydrodynamical conservation equations of mass, momentum, and energy (closed by equation of state).

3-D radiative transfer included to calculate net radiative heating/cooling q_{rad} in energy equation, typically assuming LTE and a very simplified treatment of line-blanketing

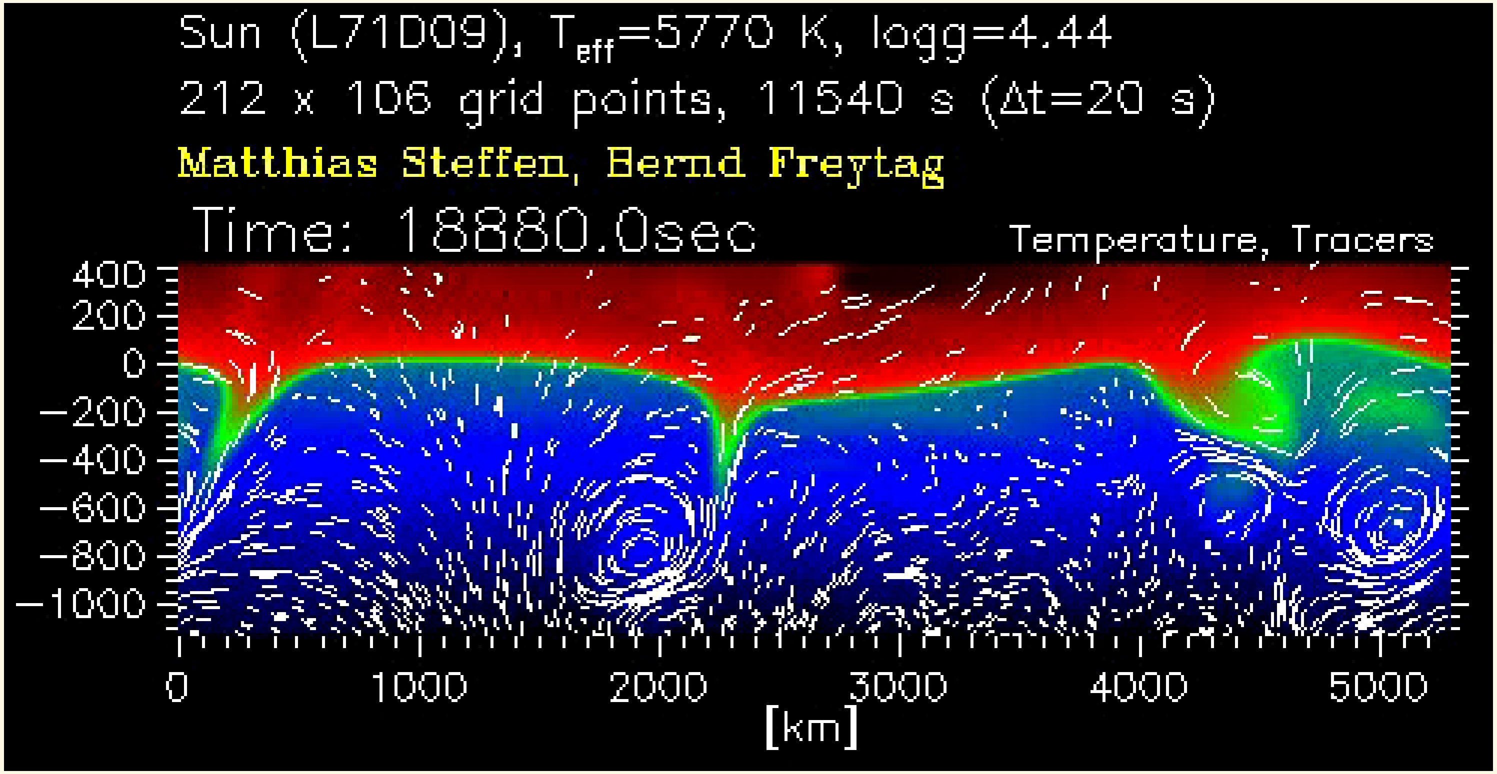


From Wolfgang Hayek

$$q_{\mathrm{rad}} = 4\pi\rho \int_\lambda \kappa_\lambda (J_\lambda - S_\lambda)\, d\lambda,$$

(= 0 in case of radiative equilibrium)

Surface Convection



From Berndt Freytag's homepage: http://www.astro.uu.se/~bf/

Surface Convection

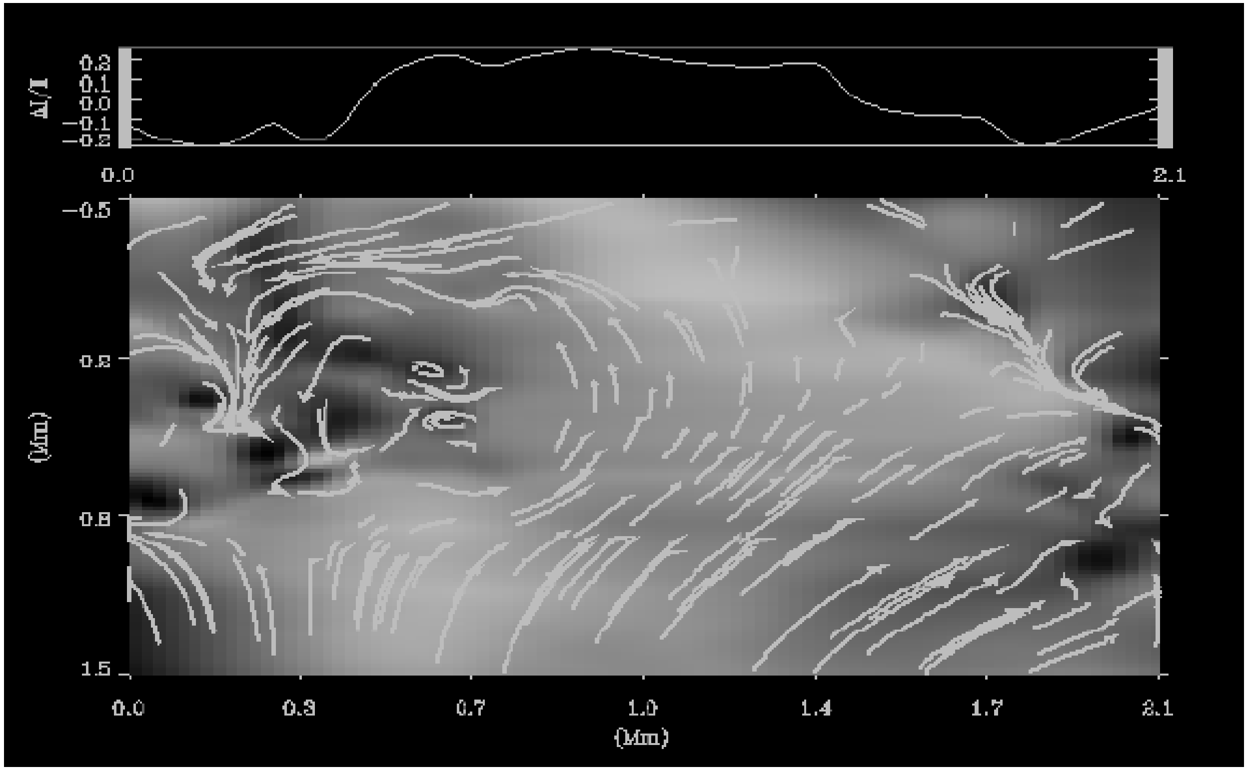


FIG. 4.—Pressure fluctuations about the mean hydrostatic equilibrium and the velocity field in an xz slice through a granule. The pressure is high above the centers of granules, which decelerates the warm upflowing fluid and diverts it horizontally. High pressure also occurs in the intergranular lanes where the horizontal motions are halted and gravity pulls the now cool, dense fluid down into the intergranular lanes. Horizontal rolls of high vorticity occur at the edges of the intergranular lanes. The emergent intensity profile across the slice is shown at the top.

From Stein & Nordlund (1998)

Surface Convection

Some key features:

Slow, broad upward motions, and faster, thinner downward motions

Non-thermal velocity fields

Overshooting from zone where convection is efficient according to stability criteria (see Chap. 6)

Energy balance in upper layers not only controlled by radiative heating/cooling, but also by cooling from adiabatic expansion

See Stein & Nordlund (1998); Collet et al. (2006), etc

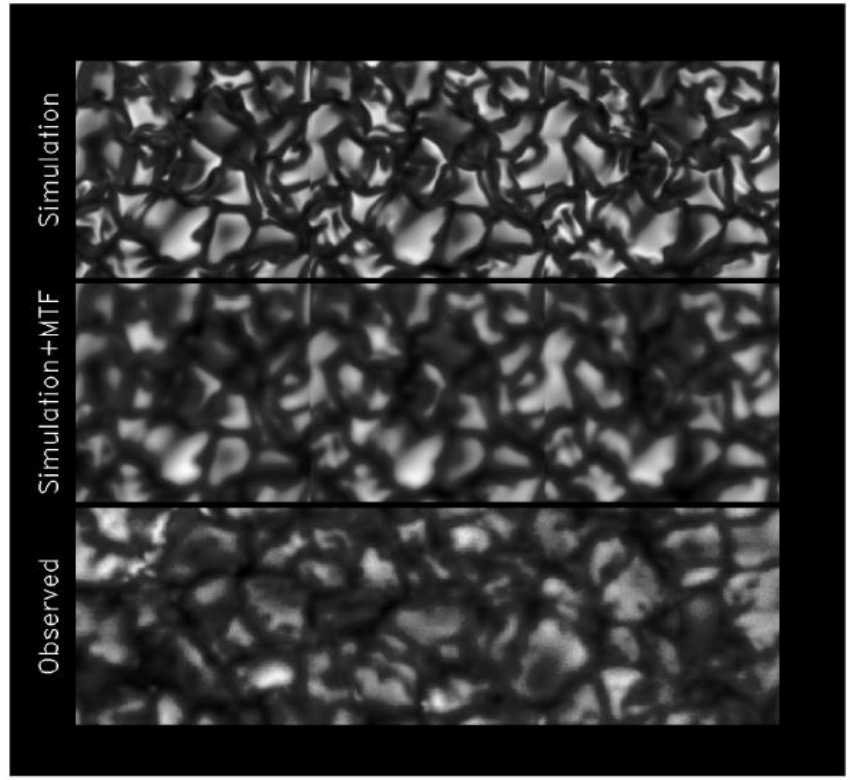


Fig. 19.—Comparison of granulation as seen in the emergent intensity from the simulations and as observed by the Swedish Vacuum Solar Telescope on La Palma. The top row shows three simulation images at 1 minute intervals, which together make a composite image 18 × 6 Mm in extent. The middle row shows this image smoothed by an Airy plus exponential point-spread function. The bottom row shows an 18 × 6 Mm white-light image from La Palma. Note the similar appearance of the smoothed simulation image and the observed granulation. The common edge brightening in the simulation is reduced when smoothed. Images by (Title 1996, private communication) taken in the CH G-band have much more contrast than white light and clearly reveal the edge brightening of granules.

Question: This does not look much like the traditional 1-D models we've discussed during the previous lecture! – Do you think we should throw them in the garbage?

Surface Convection

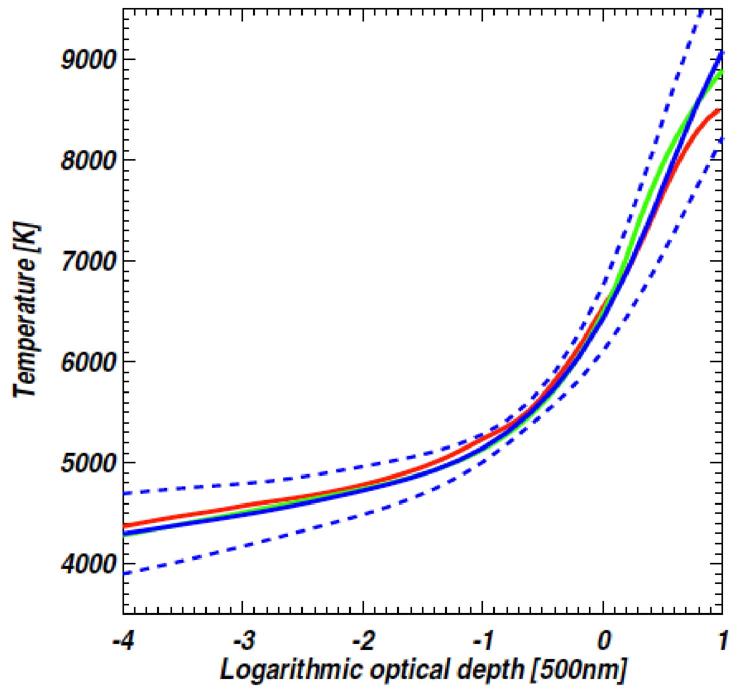


Figure 1: The mean temperature structure of the 3D hydrodynamical model of Trampedach et al. (2009) is shown as a function of optical depth at 500 nm (blue solid line). The blue dashed lines correspond to the spatial and temporal rms variations of the 3D model, while the red and green curves denote the 1D semi-empirical Holweger & Müller (1974) and the 1D theoretical MARCS (Gustafsson et al. 2008) model atmospheres, respectively.

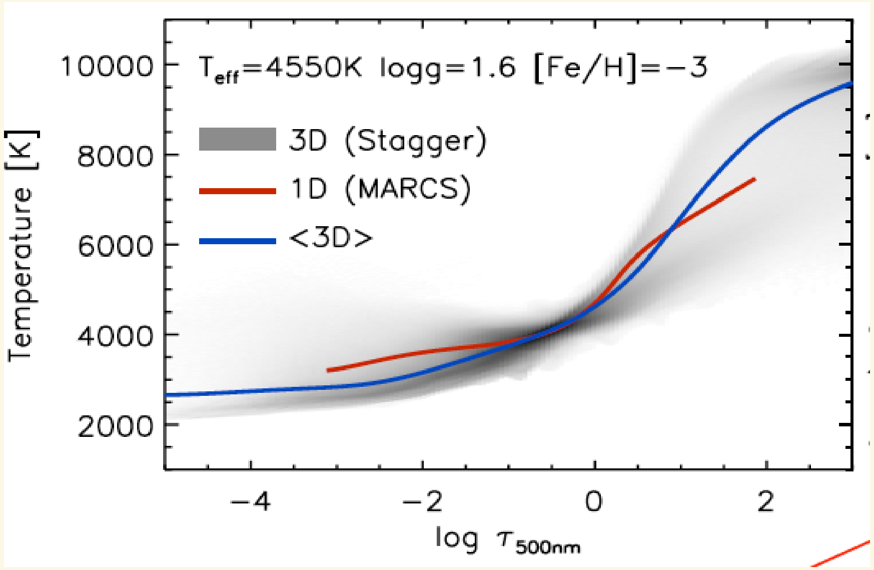
In many (though not all) cases, AVERAGE properties still quite OK:

Convection in energy balance approximated by
“mixing-length theory”
Non-thermal velocity fields due to convective motions
included by means of so-called “micro-” and
“macro-turbulence”

BUT quantitatively we always need to ask:
To what extent can average properties be modeled by traditional 1-D codes?

Unfortunately, a general answer very difficult to give, need to be considered case by case

Surface Convection



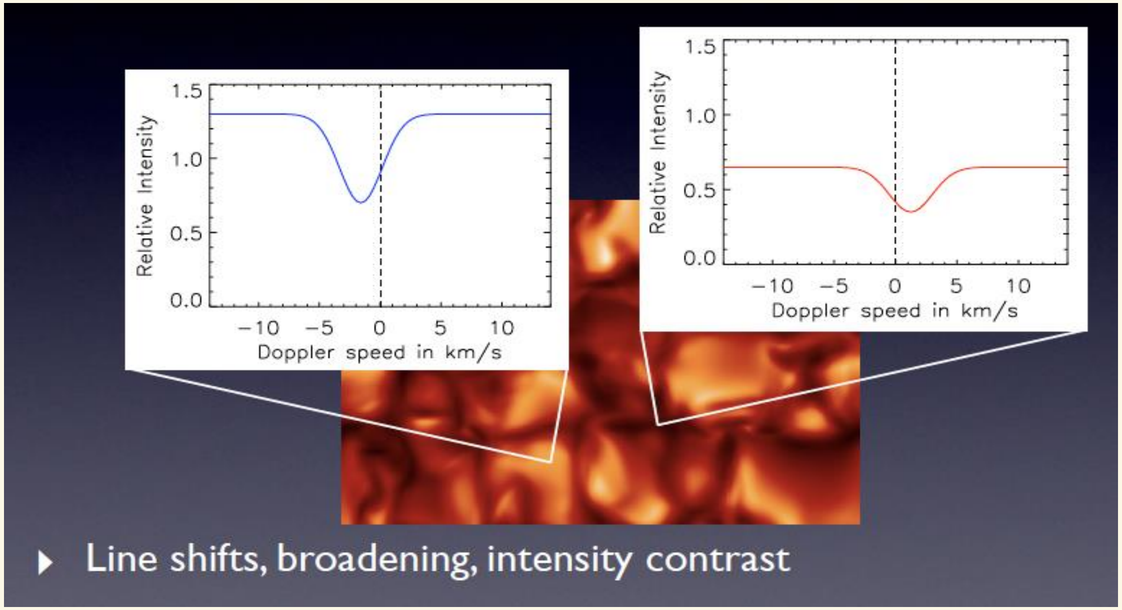
Metal-poor red giant, simulation by Remo Collet, figure from talk by M. Bergemann

For example:

In metal-poor cool stars spectral lines are scarce (Question: Why?), and energy balance in upper photosphere controlled to a higher degree by adiabatic expansion of convectively overshot material.

In classical 1-D models though, these layers are convectively stable, and energy balance controlled only by radiation (radiative equilibrium, see Chap. 4).

Surface Convection



From talk by Hayek

3-D radiation-hydro models successful in reproducing many solar features (see overview in Asplund et al. 2009), e.g:

Center-to-limb intensity variation

Line profiles and their shifts and variations (without micro/macroturbulence)

Observed granulation patterns

Surface Convection

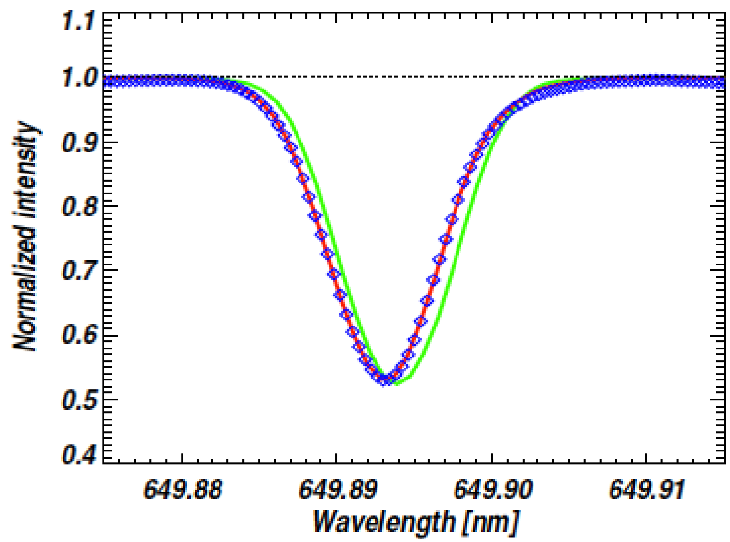


Figure 3: The predicted spectral line profile of a typical Fe I line from the 3D hydrodynamical solar model (red solid line) compared with the observations (blue rhombs). The agreement is clearly very satisfactory, which is the result of the Doppler shifts arising from the self-consistently computed convective motions that broaden, shift and skew the theoretical profile. For comparison purposes also the predicted profile from a 1D model atmosphere (here Holweger & Müller 1974) is shown; the 1D profile has been computed with a microturbulence of 1 km s^{-1} and a tuned macroturbulence to obtain the right overall linewidth. Note that even with these two free parameters the 1D profile can neither predict the shift nor the asymmetry of the line.

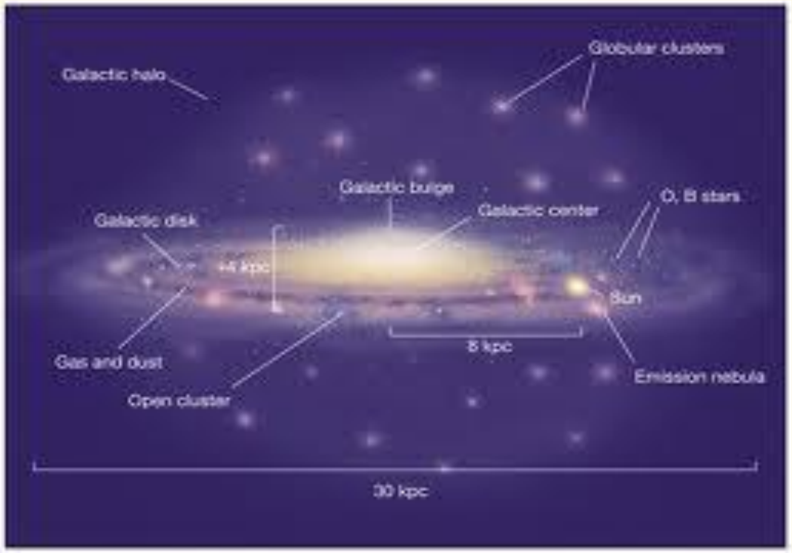
affects chemical abundance (determined by means of line profile fitting to observations)

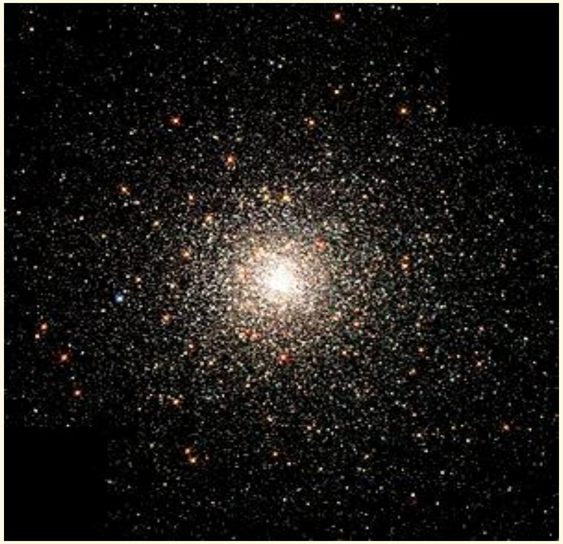
One MAJOR result:
Effects on line formation has led to a downward revision of the CNO solar abundances and the solar metallicity, and thus to a revision of the *standard cosmic chemical abundance scale*

Fig. from Asplund et al. (2009) – "The Chemical Composition of the Sun"

Surface Convection

Also potentially critical for Galactic archeology…

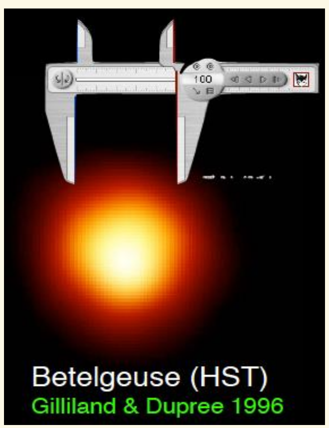


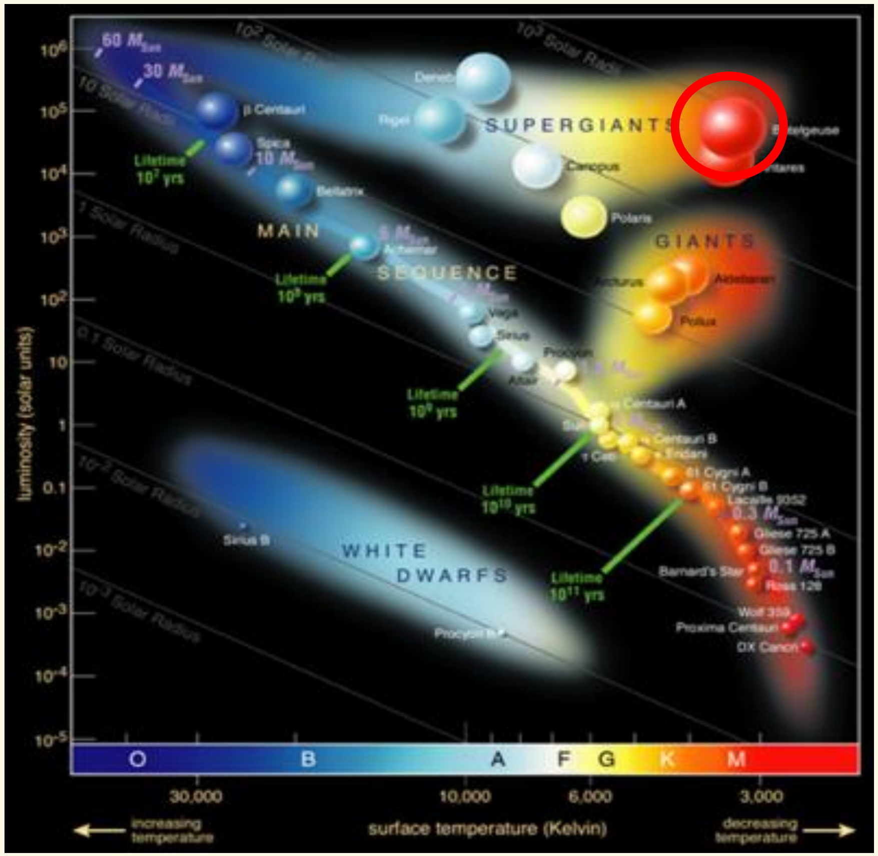


…which traces the chemical evolution of the Universe by analyzing VERY old, metal-poor Globular Cluster stars –- relics from the early epochs (e.g. Anna Frebel and collaborators)

Surface Convection

Out to Jupiter…





- Giant Convection Cells in the low-gravity, extended atmospheres of Red Supergiants
- Question: Why extended?

$H = a^2 / g$ (with a the isothermal speed of sound)

$$a^2_{RSG} / a^2_{sun} \approx T_{RSG} / T_{sun} = 0.5...0.6$$

$$g_{RSG} / g_{sun} \approx 10^{-4}!$$

(see Chap. 6)

Surface Convection

Supergiants (or models including a stellar wind):
Atmospheric extent > stellar radius:

Box-in-a-star → Star-in-a-box

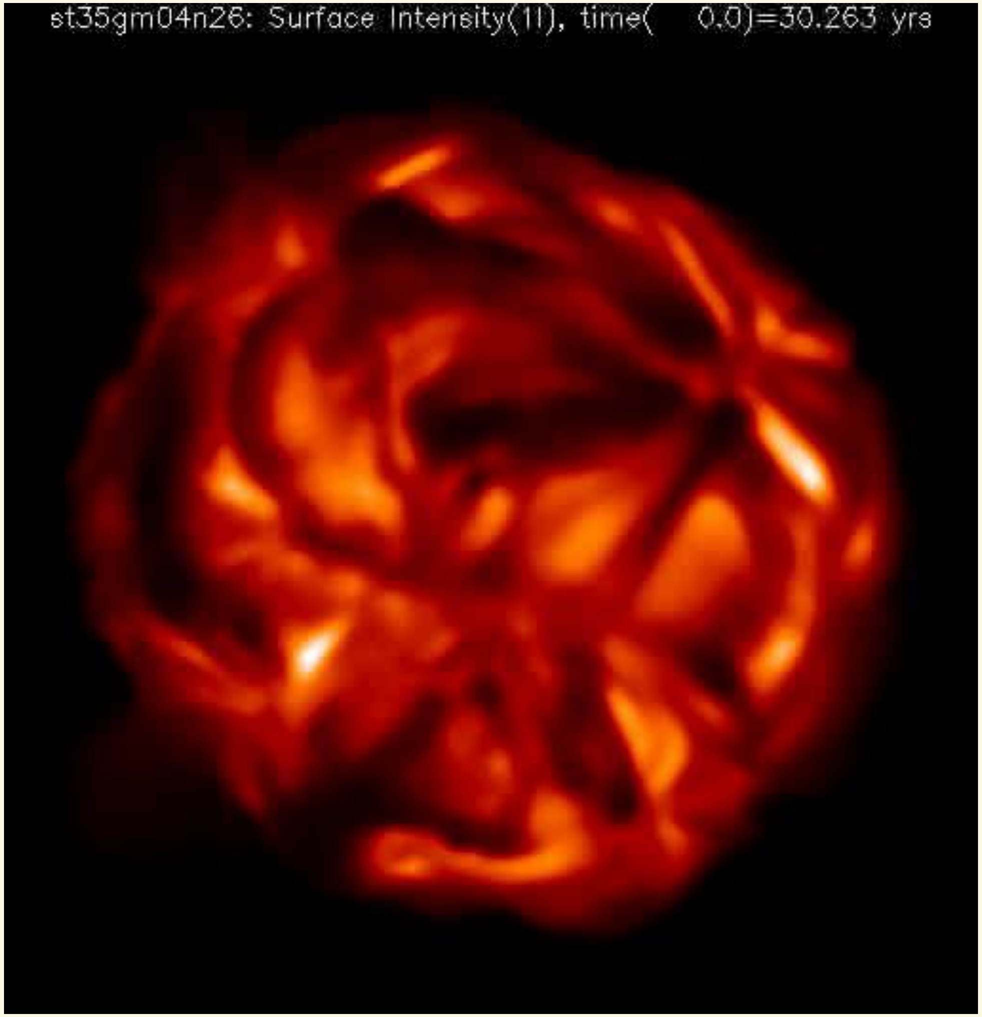
(1D: Plane-parallel → Spherical symmetry, see Chap. 3)

Star to model: Betelgeuse
Mass: 5 solar masses
Radius: 600 Rsun
Luminosity: 41400 Lsun
Grid: Cartesian cubical grid with 171^3 points
Edge length of box 1674 solar radii

Model by Berndt Freytag, note the HUGE convective cells visible in the emergent intensity map!!

Surface Convection

Star to model: Betelgeuse
Mass: 5 solar masses
Radius: 600 Rsun
Luminosity: 41400 Lsun
Grid: Cartesian cubical grid with 171^3 points
Edge length of box 1674 solar radii
Movie time span: 7.5 years



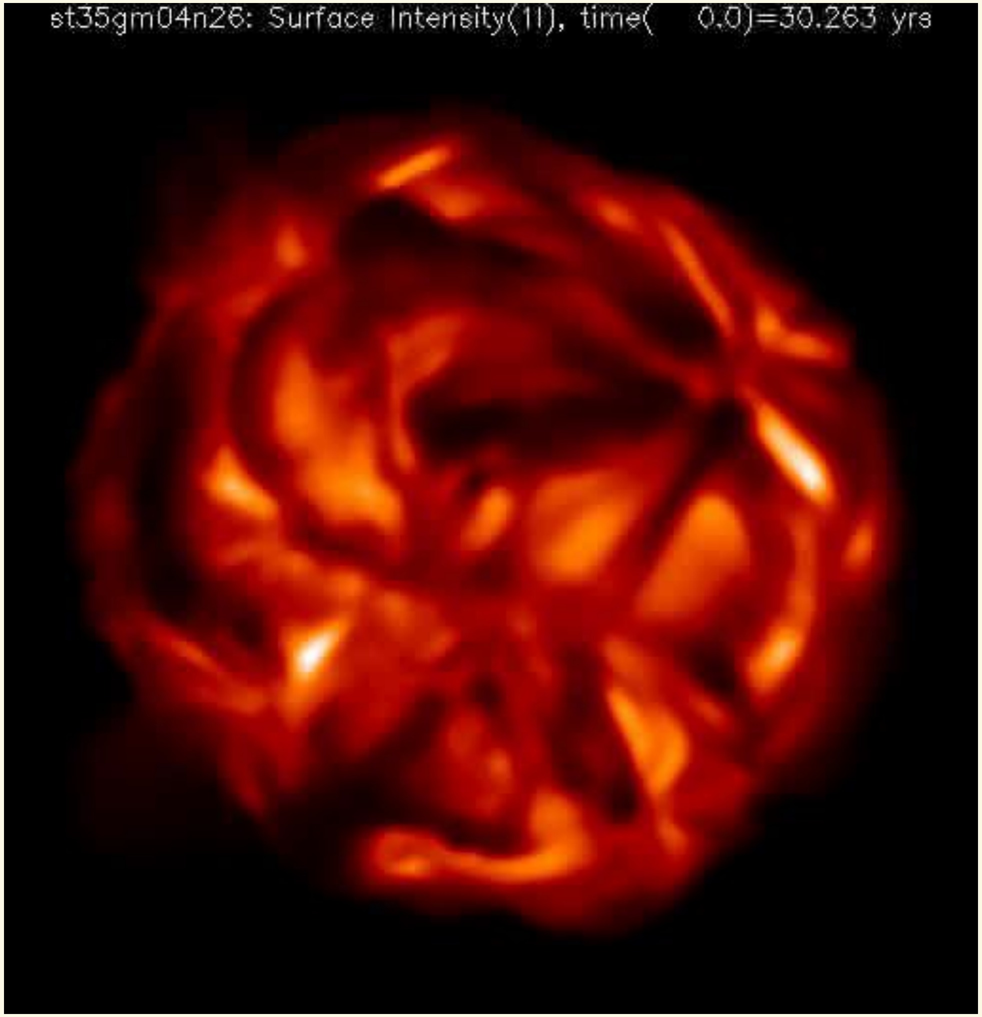
http://www.astro.uu.se/~bf/movie/dst35gm04n26/movie.html

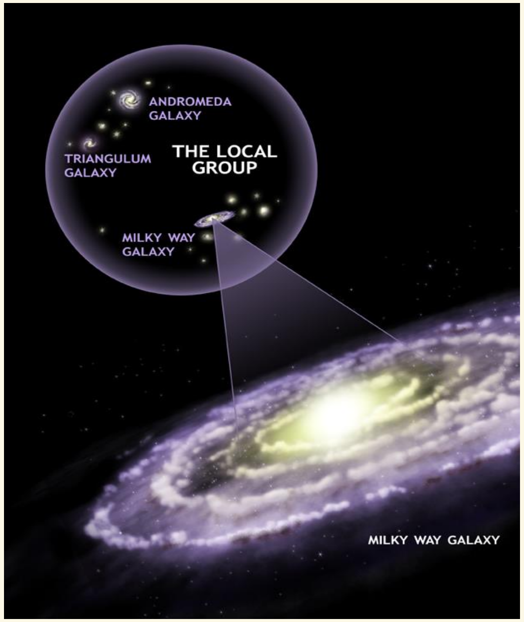
Surface Convection

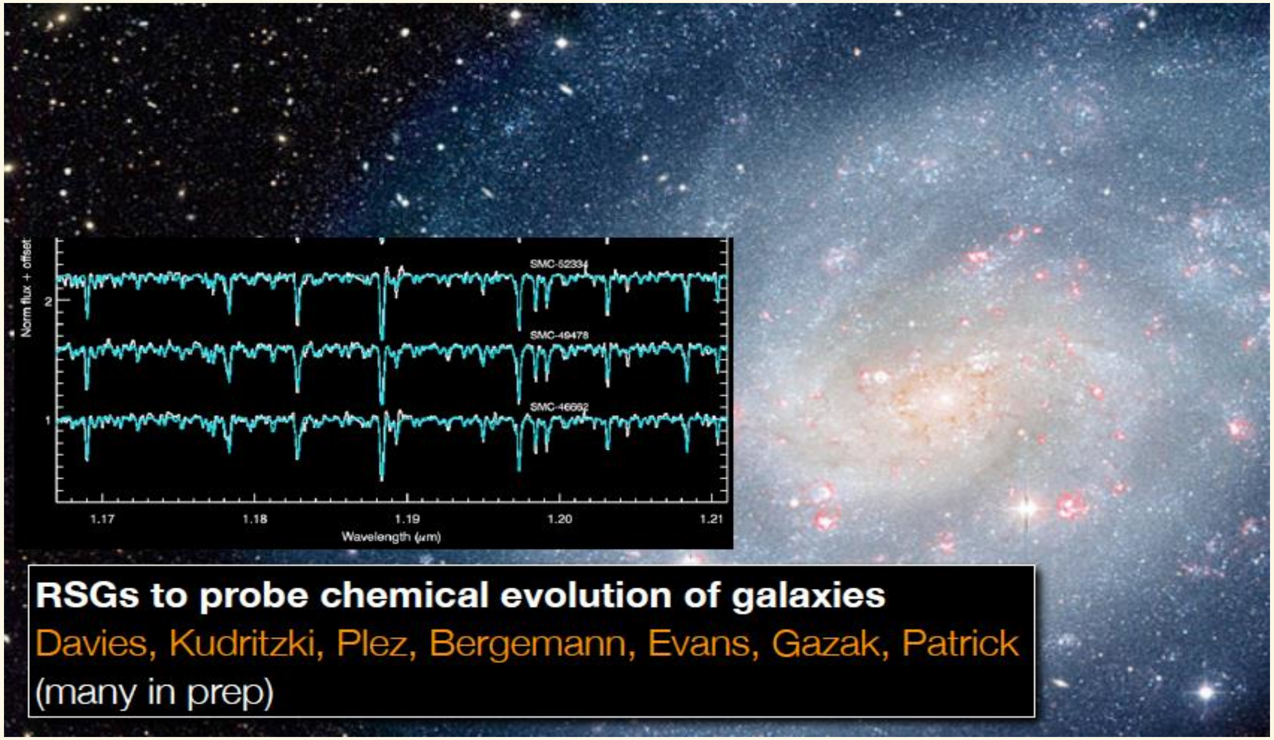
Extremely challenging,
models still in their infancies.
LOTS of exciting physics to explore, like

Pulsations
Convection
Numerical radiation-hydrodynamics
Role of magnetic fields
Stellar wind mechanisms

Also, to what extent can main effects
be captured by 1-D models?
For quantitative applications like....







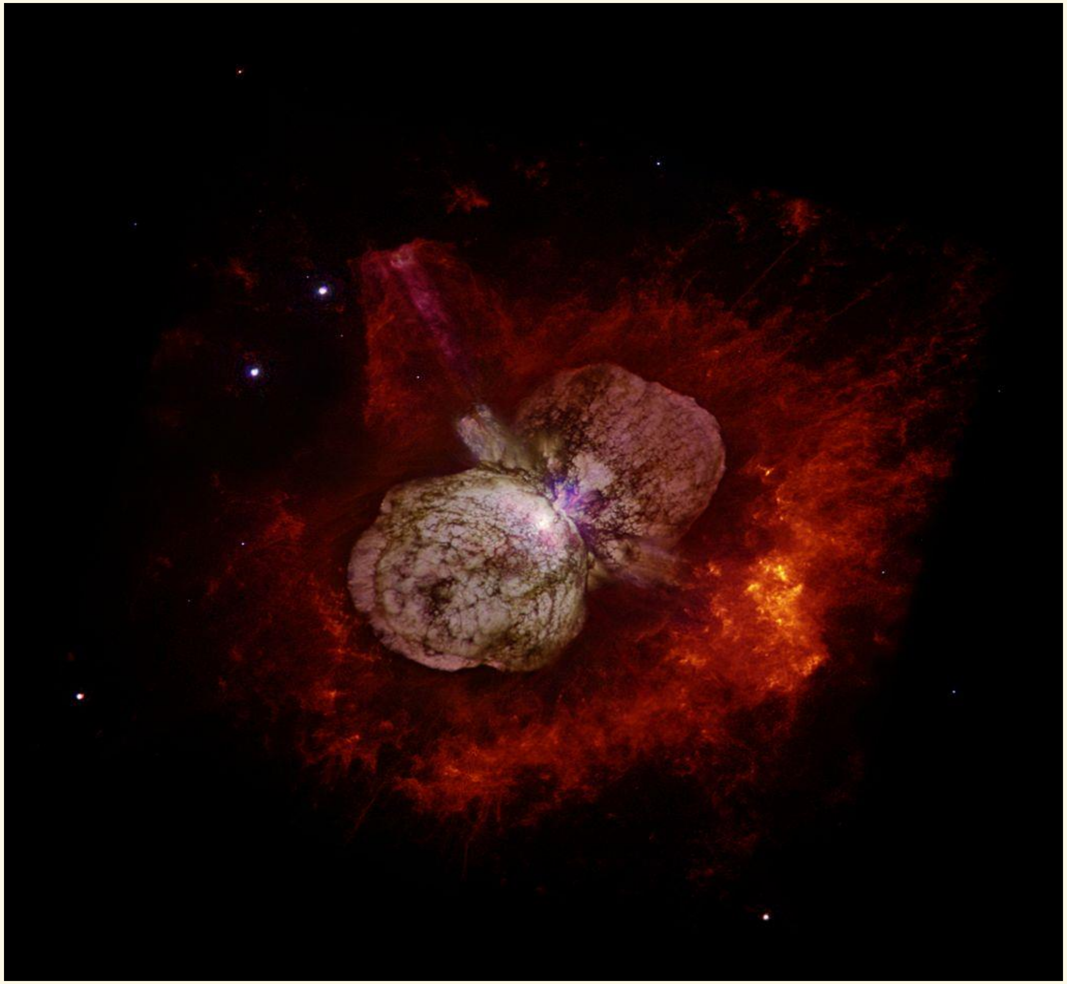
Question: Why are RSGs ideal for
extragalactic observational stellar astrophysics
using new generations of extremely large infra-red telescopes?

important codes (not complete) and their features ....

Codes	FASTWIND CMFGEN PoWR	WM-basic	TLUSTY Detail/Surface	Phoenix	MARCS Atlas	CO5BOLD STAGGER
geometry	1-D spherical	1-D spherical	1-D plane-parallel	1-D/3-D spherical/ plane-parallel	1-D plane-parallel (MARCS also spherical)	3-D Cartesian
LTE/NLTE	NLTE	NLTE	NLTE	NLTE/LTE	LTE	LTE simplified
dynamics	quasi-static photosphere + prescribed supersonic outflow	time-independent hydrodynamics	hydrostatic	hydrostatic or allowing for supersonic outflows	hydrostatic	hydrodynamic
stellar wind	yes	yes	no	yes	no	no
major application	hot stars with winds	hot stars with dense winds, ion. fluxes, SNRs	hot stars with negligible winds	cool stars, brown dwarfs, SNRs	cool stars	cool stars
comments	CMFGEN also for SNRs; FASTWIND using approx. line-blocking	line-transfer in Sobolev approx. (see part 2)	Detail/Surface with LTE-blanketing	convection via mixing-length theory	convection via mixing-length theory	very long execution times, but model grids start to emerge

Stellar Atmospheres in practice

And then there are, e.g.,



- Luminous Blue Variables (LBVs) like Eta Carina,
- Wolf-Rayet Stars (WRs)
- Planetary Nebulae (and their Central Stars)
- Be-stars with disks
- Brown Dwarfs
- Pre main-sequence T-Tauri and Herbig stars

…and many other interesting objects

Stellar astronomy alive and kicking! Very rich in both

Physics
Observational applications

A first application – The D4000 break in early type galaxies

- spectroscopic study of region around 4000 Å: useful tool to investigate stellar populations in composite stellar systems

$$D_{4000} = \frac{(\lambda_2^- - \lambda_1^-)}{(\lambda_2^+ - \lambda_1^+)} \frac{\int_{\lambda_1^+}^{\lambda_2^+} F_\nu \, d\lambda}{\int_{\lambda_1^-}^{\lambda_2^-} F_\nu \, d\lambda},$$

where $(\lambda_1^-, \lambda_2^-, \lambda_1^+, \lambda_2^+) = (3750, 3950, 4050, 4250)$ Å.

definition by Bruzual (1983)

D4000 pseudo color
(combination of λ and ν,
not logarithmically defined

- D4000 break in early type galaxies
  - only low signal to noise required
  - only weakly contaminated by reddening
  - no absolute fluxes required
  - same def. for red-shifted objects, only int. range has to be modified
  - BUT: many lines contribute to break, complex behavior
- easy detection of young populations => star formation history

Spectral energy distribution of A-K stars

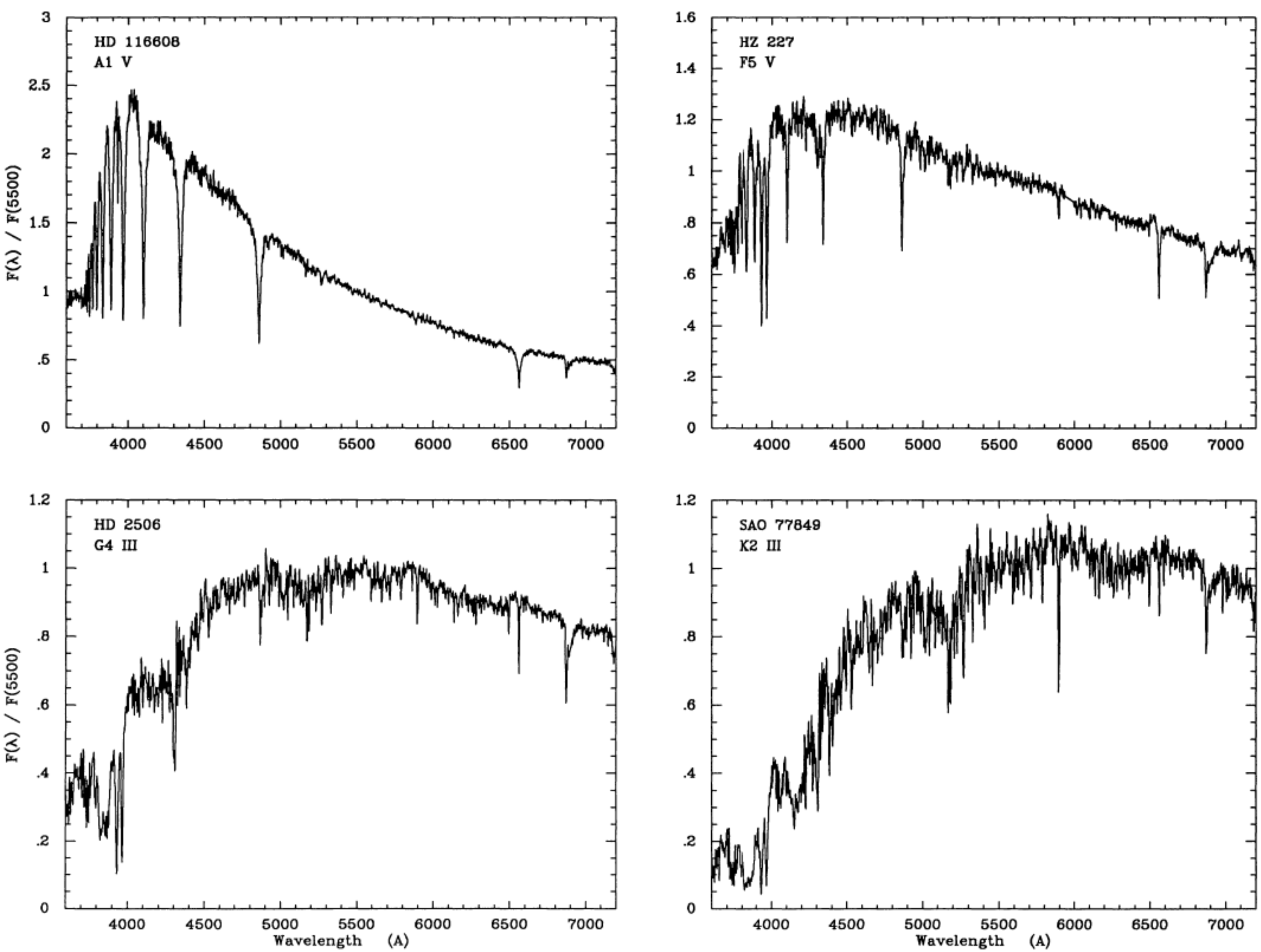


FIG. 4.—Spectra of four Galactic stars, taken from the spectral library of Jacoby et al. (1984). These spectra can be used to identify some of the major stellar absorption features in the galaxy spectra.

Note the change in the energy distribution as a function of spectral type (Teff)

Spectral energy distribution of elliptical galaxies

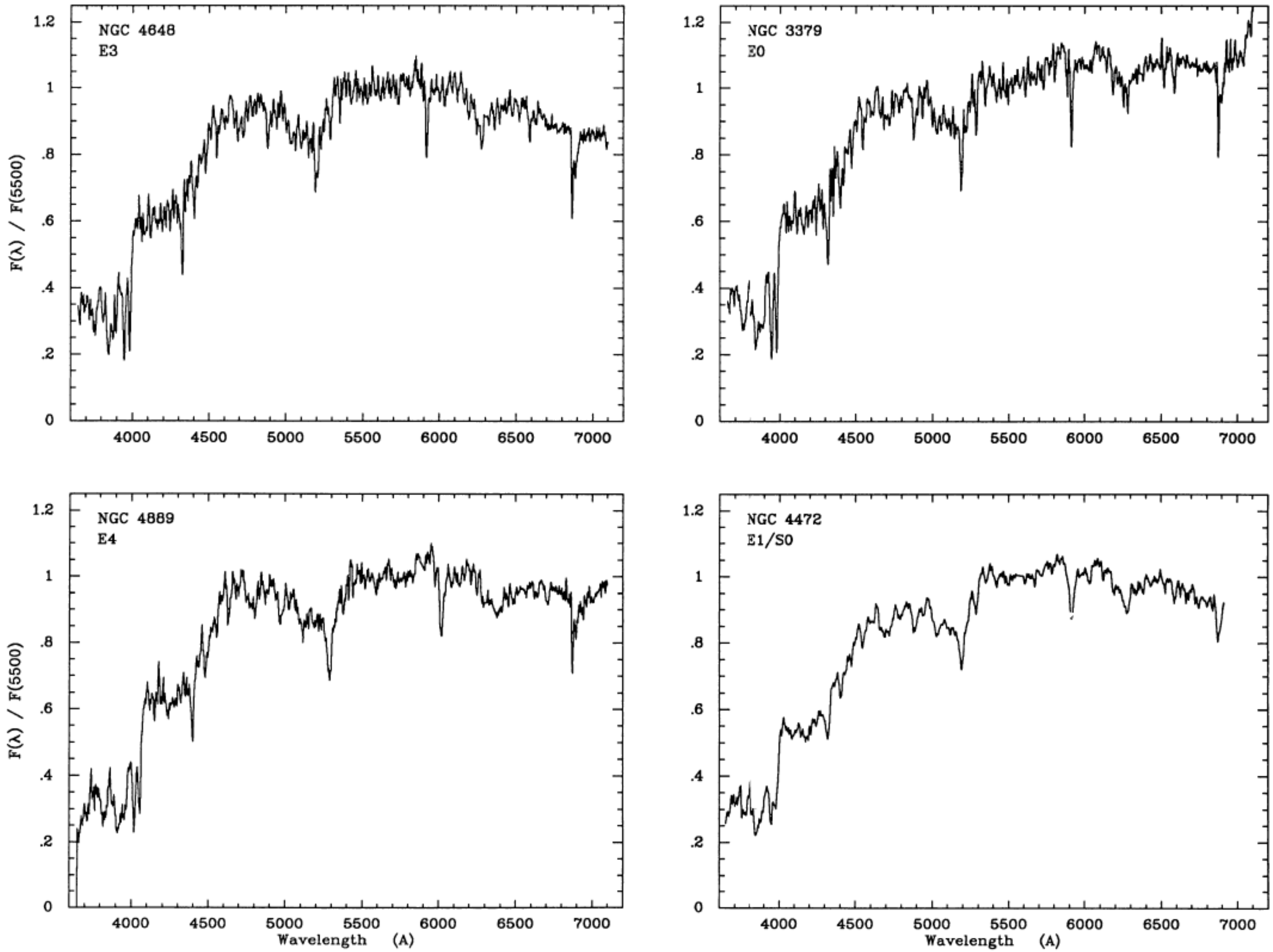


FIG. 5.—Integrated spectra of four elliptical galaxies. The spectrum of NGC 4472 (*lower right*) was obtained at lower resolution with the IRS scanner.

D4000 break clearly visible, Mg/MgH complex strong
⇒dominated by G/K-giants

from Kennicutt, 1992, ApJS 79

Spectral energy distribution of spiral galaxies

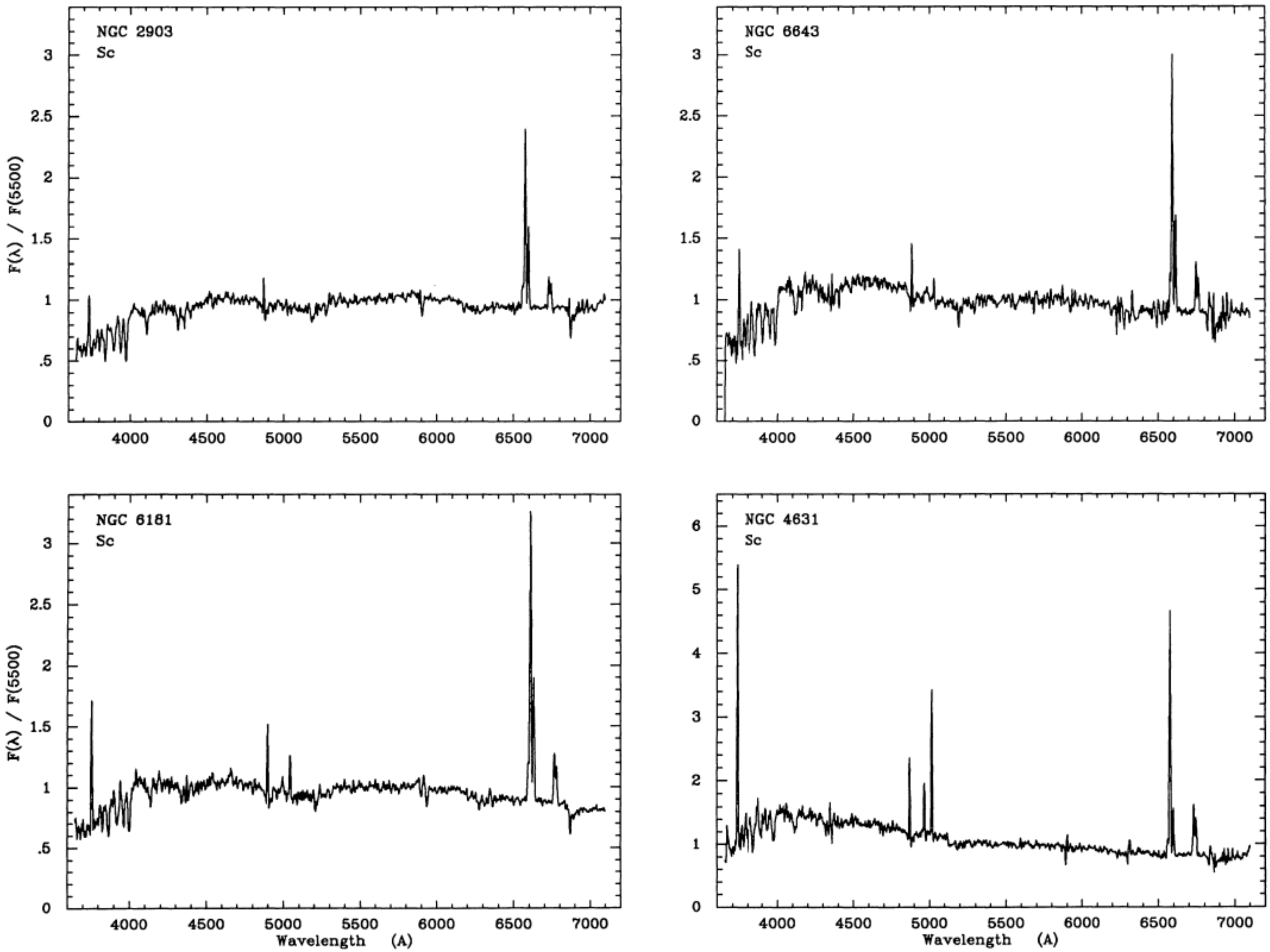
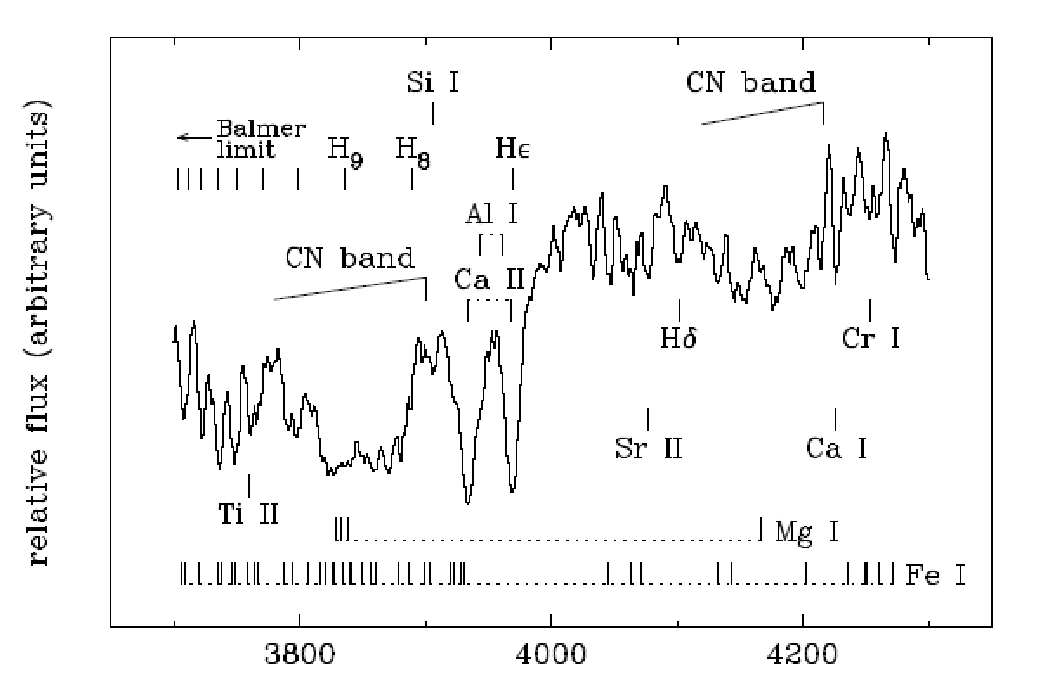


FIG. 10.—Integrated spectra of four Sbc–Sc galaxies, selected to illustrate the range in excitation in the emission-line spectra. See Fig. 9 for other examples.

no break,
Balmer decrement,
nebular emission lines (Halpha, Hbeta, [OIII],…)

⇒presence of early type stars plus HII-regions

The 4000 Å region: a closer inspection



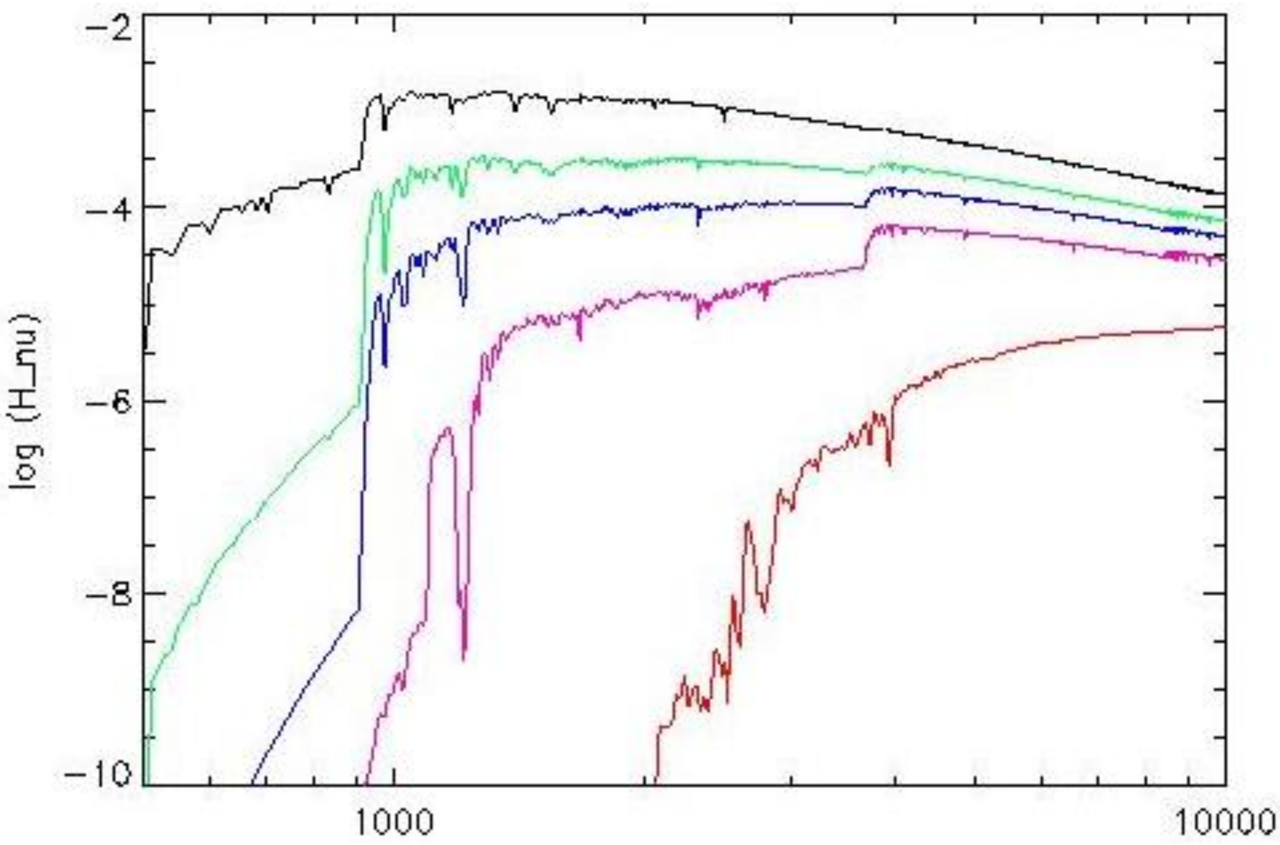
From Gorgas et al., 1999, A&A

spectrum of HD72324(G9 III)

- very strong CaII H/K lines
  - major ion, resonance lines (almost all Ca-atoms are in ground-state of CaII)
    => very strong lines
- weaker Balmer lines (almost all hydrogen in ground-state)
- multitude of FeI and MgI lines
- + CN band lines

=> Strong D4000 break

Theoretical energy distributions of supergiants

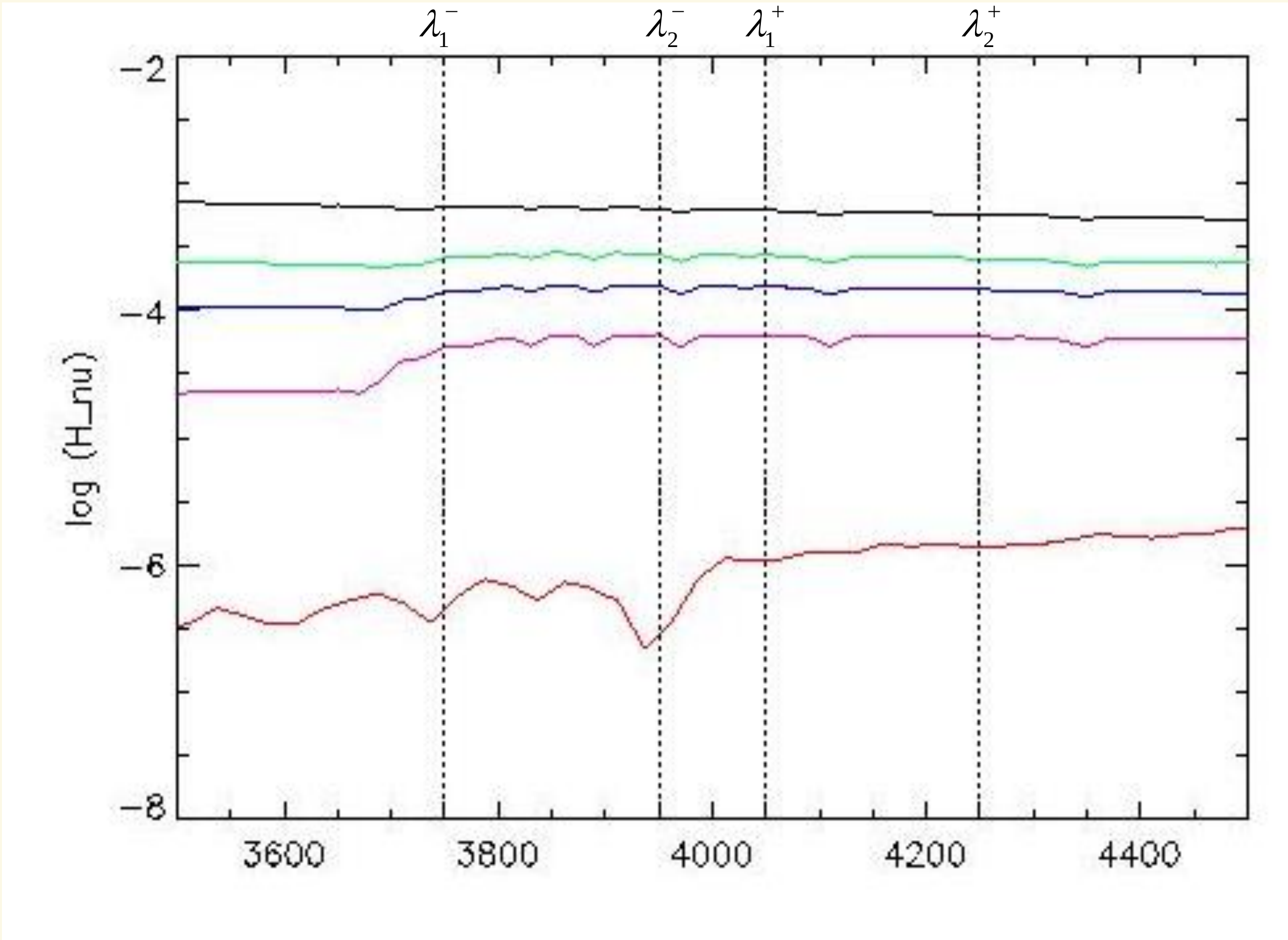


Teff [K] =

30000
20000
15000
10000
5000

calculated by means of 'Atlas' (Kurucz) model atmospheres (LTE)

Theoretical energy distributions of supergiants: zoom into the 4000 Å region

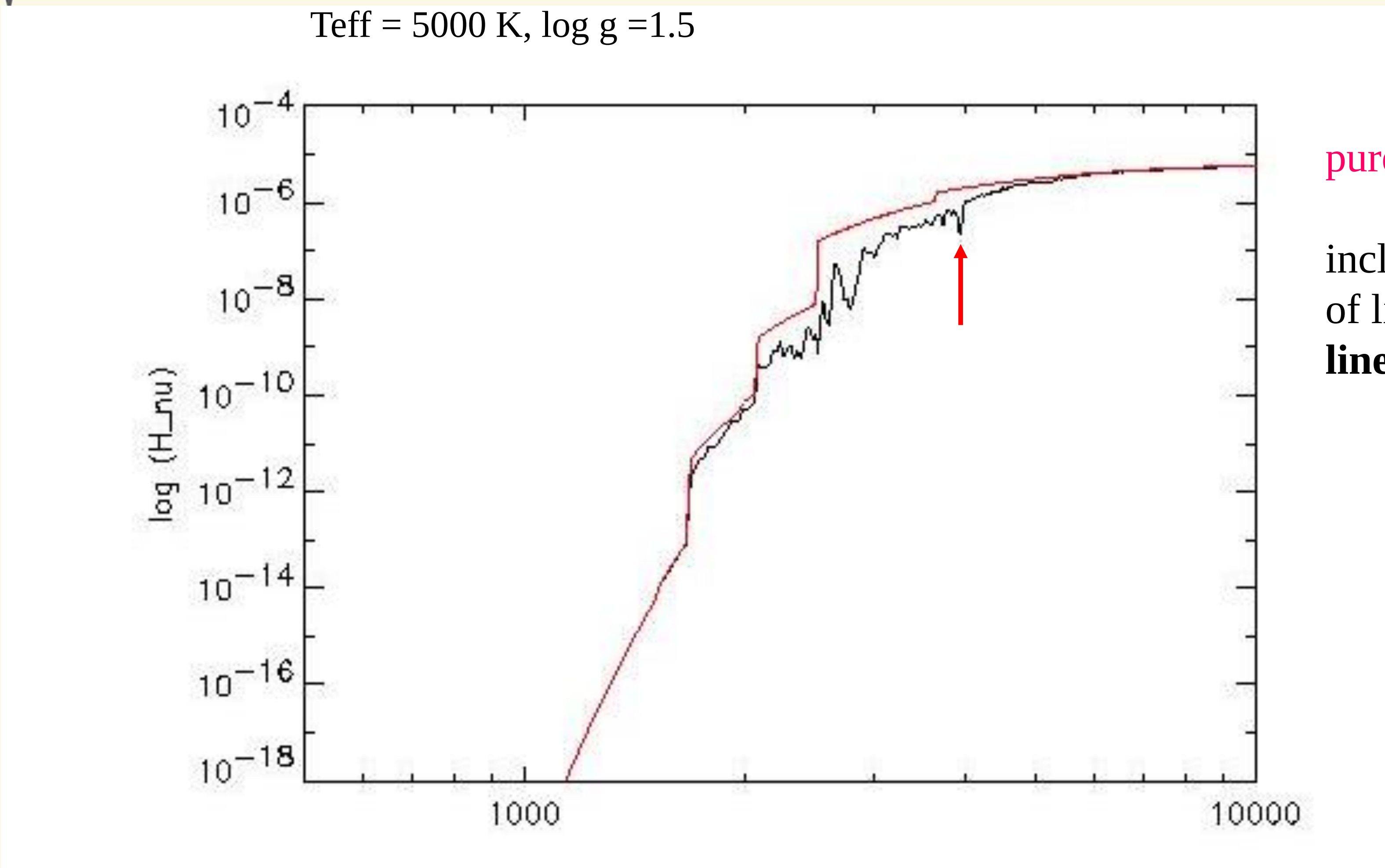


Teff [K] =

30000
20000
15000
10000
5000

calculated by means of 'Atlas' (Kurucz) model atmospheres (LTE)

The D4000 break: consequence of line-blocking



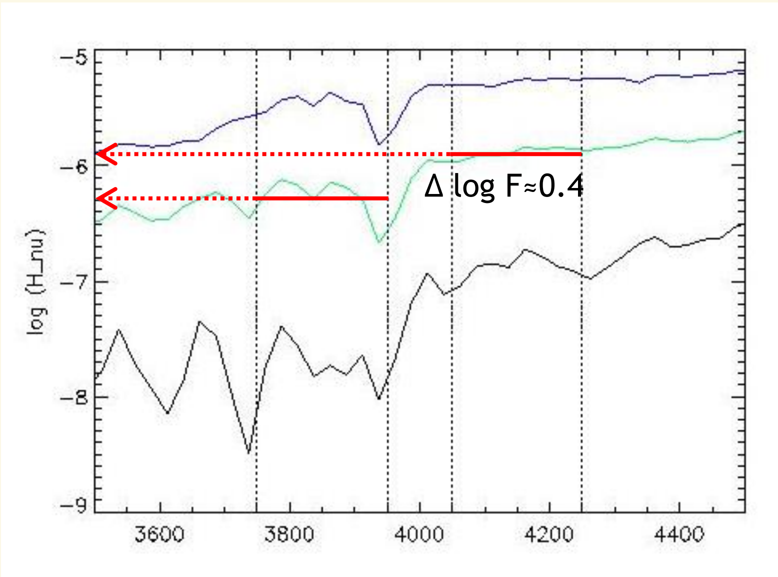
pure continuum

including multitude of lines:
line-blocking

The D4000 break: dependence on parameters

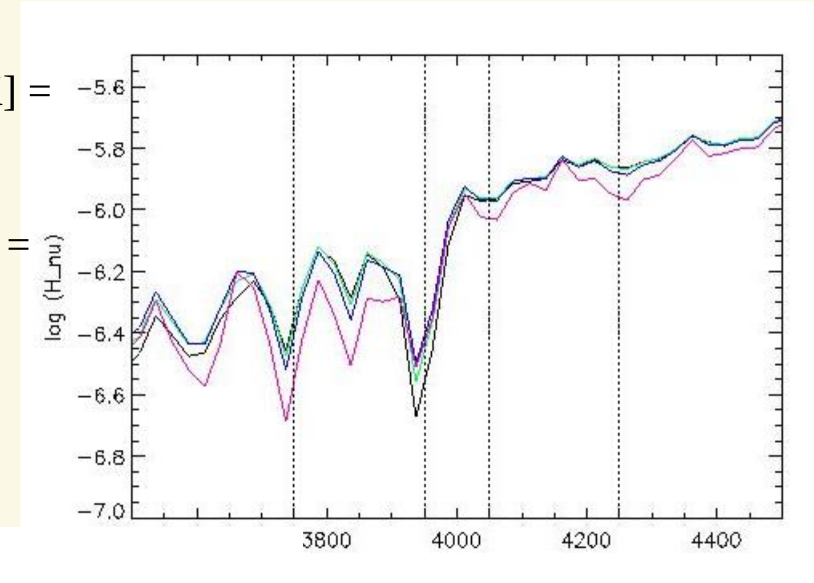
supergiant
Teff [K] =

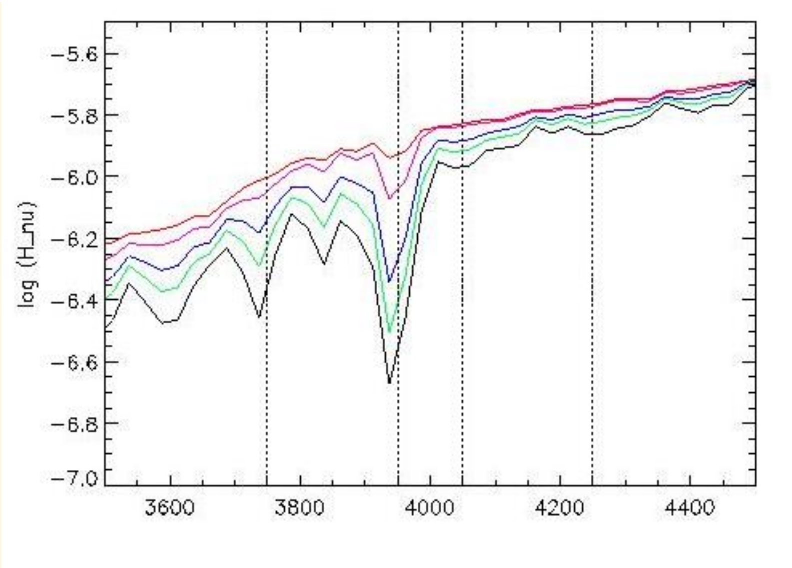
6000
5000
4000



Teff [K] =
5000

log g =
1.50
2.25
3.00
4.50





Teff [K] = 5000
log g = 1.50

metallicity (log Z/Z_{sun})
0.00
-0.50
-1.00
-2.00
-3.00

The D4000 break: empirical calibration

factor 2.5
corresponds t[o]
Δ log F=0.4

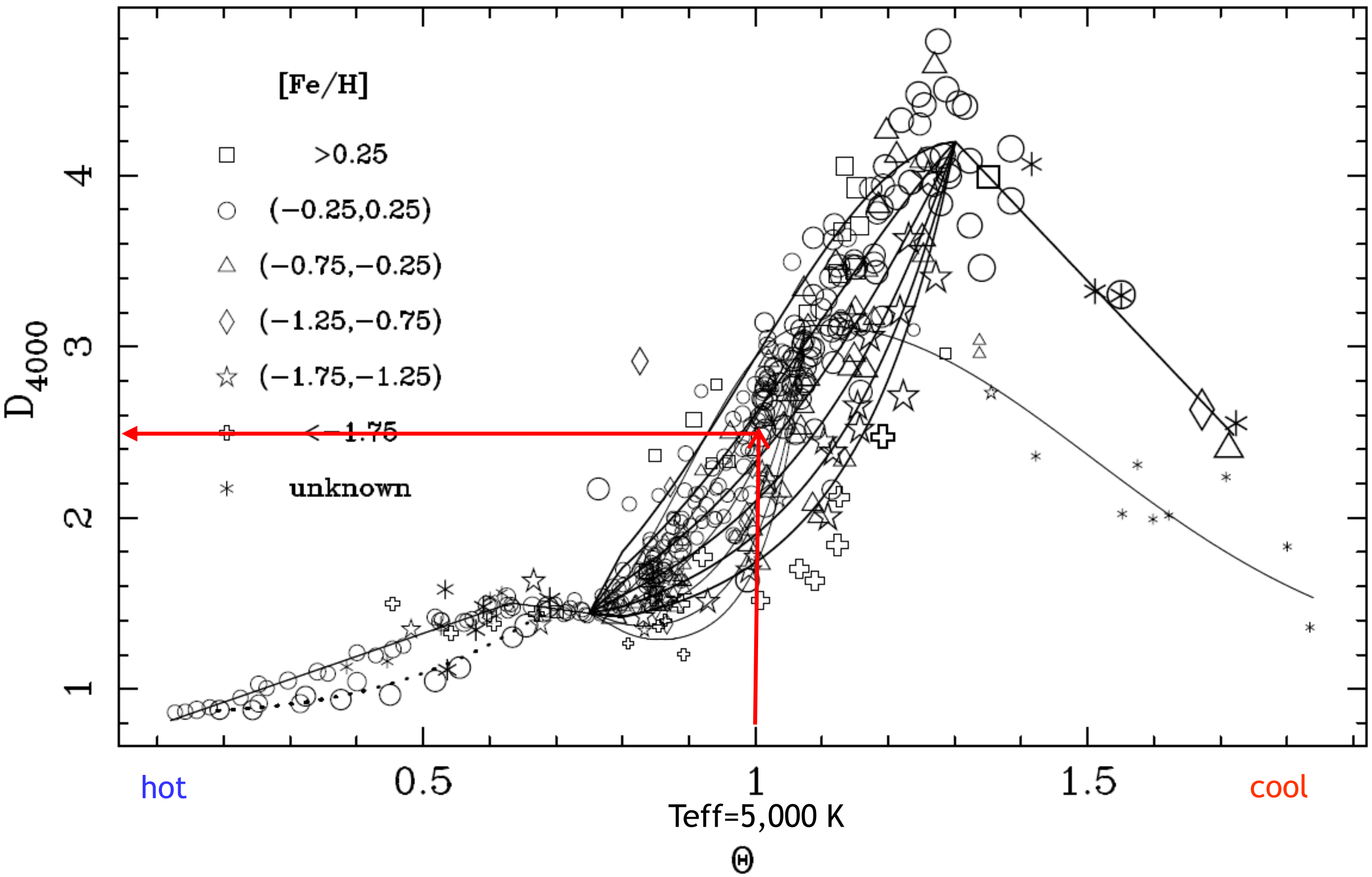
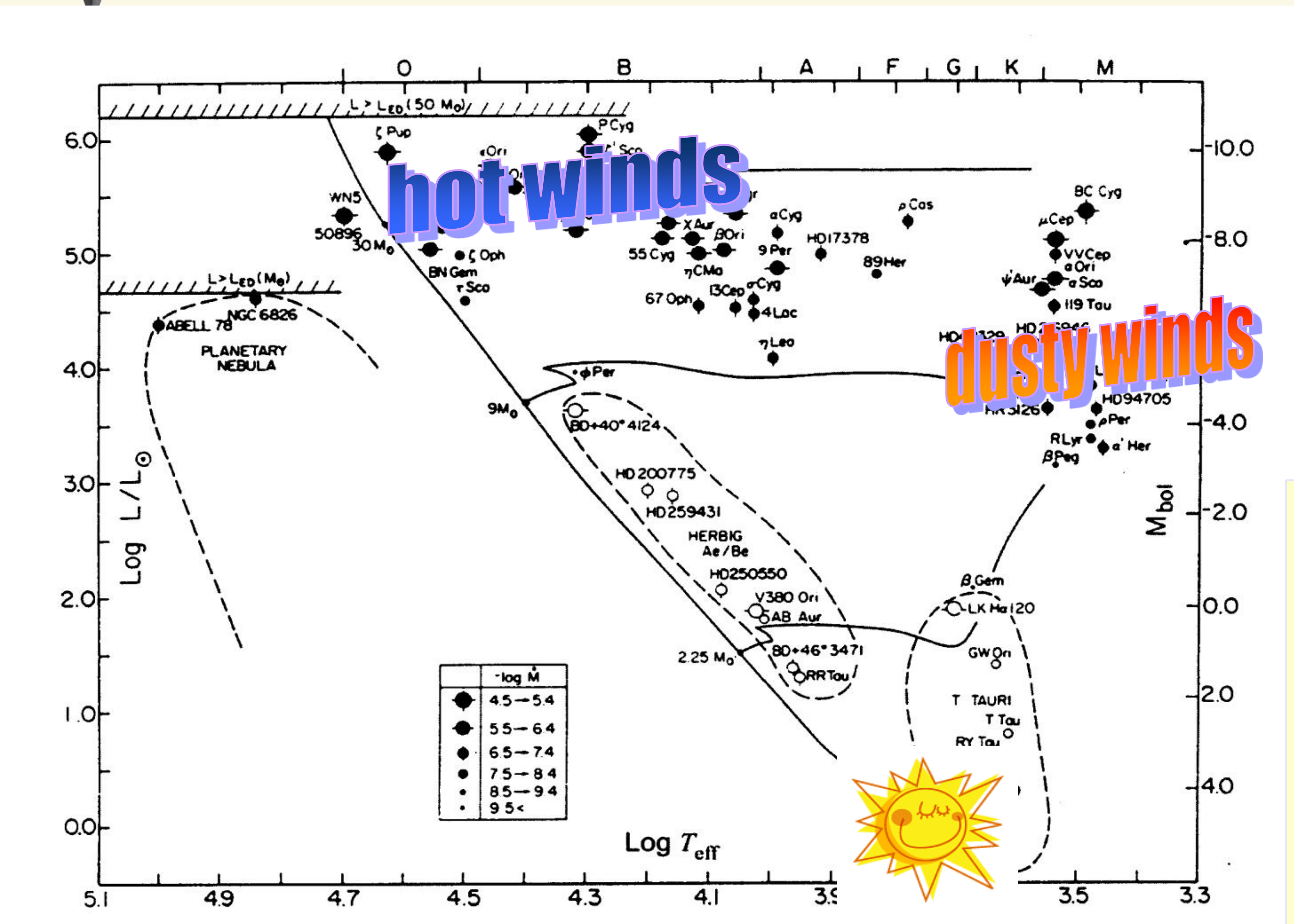


Fig. 5. D_{4000} as a function of $\theta \equiv 5040/T_{\rm eff}$ for the sample, together with the derived fitting functions. Stars of different metallicities are shown with different symbol types, with sizes giving an indication of the surface gravity (in the sense that low-gravity stars, i.e. giants, are plotted with larger symbols). Concerning the fitting functions, in the low θ range, the solid line corresponds to dwarf and giant stars, whereas the dashed line is used for supergiants. For lower temperatures, thick and thin lines refer to giant and dwarf stars respectively. For each of these groups in the mid-temperature range, the different lines represent the metallicities [Fe/H] $= +0.5, 0, -0.5, -1, -1.5, -2$, from top to bottom.



ubiquitous phenomenon

- solar type stars (incl. the sun)
- red supergiants/AGB-stars ("normal" + Mira Variables)
- hot stars (OBA supergiants, Luminous Blue Variables, OB-dwarfs, Central Stars of PN, sdO, sdB, Wolf-Rayet stars)
- T-Tauri stars
- and many more

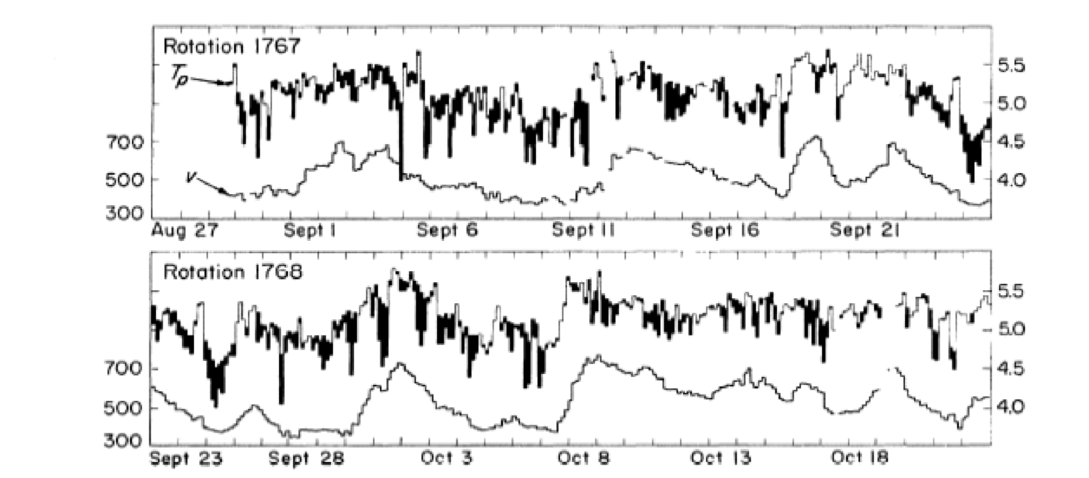
The solar wind - a suspicion

comet Halley, with „kink“ in tail

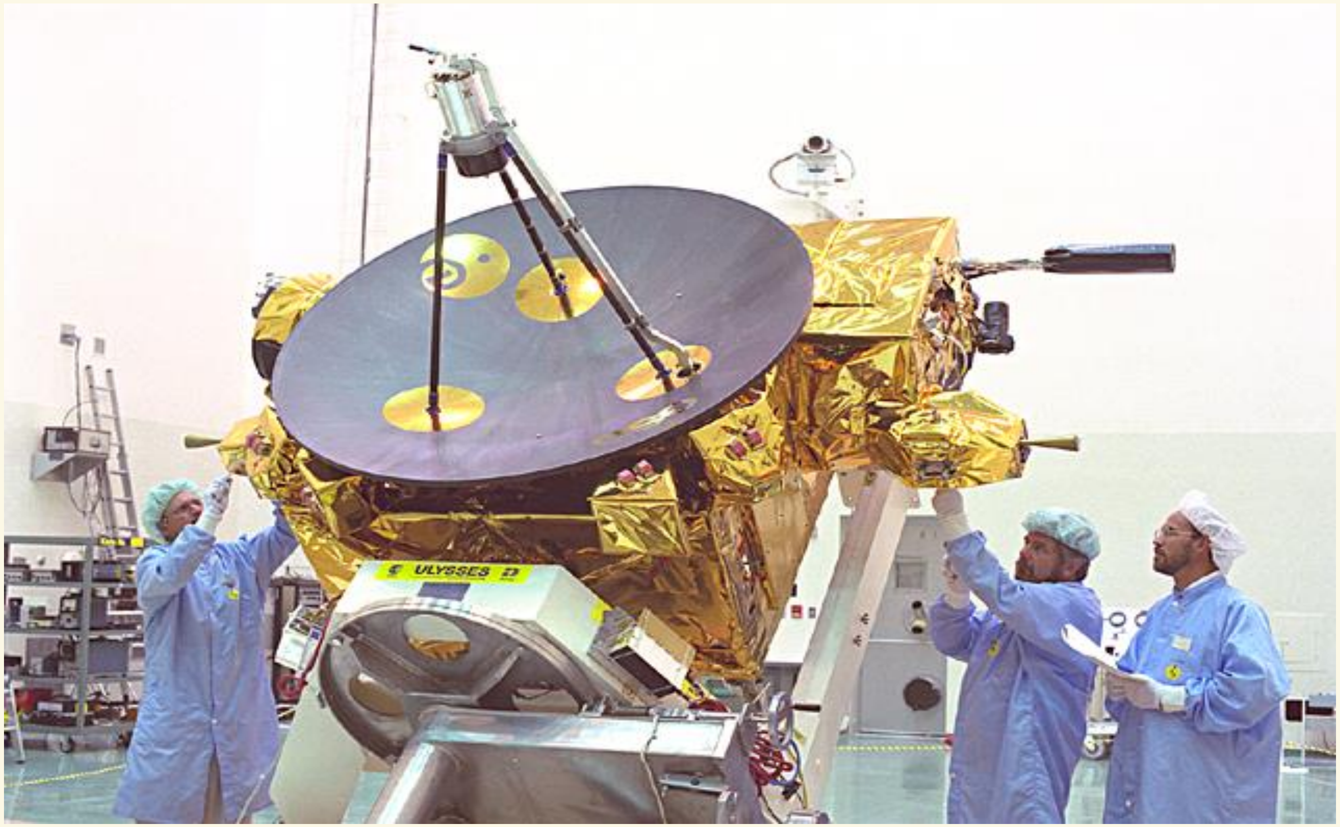
- comet tails *directed away from the sun*
- **Kepler:** influence of solar radiation pressure (-> radiation driven winds)
- *Ionic tail*: emits own radiation, sometimes different direction
- **Hoffmeister** (1943, subsequently Biermann): *solar particle radiation* different direction, since **v** (particle) comparable to **v** (comet)

The solar wind – the discovery

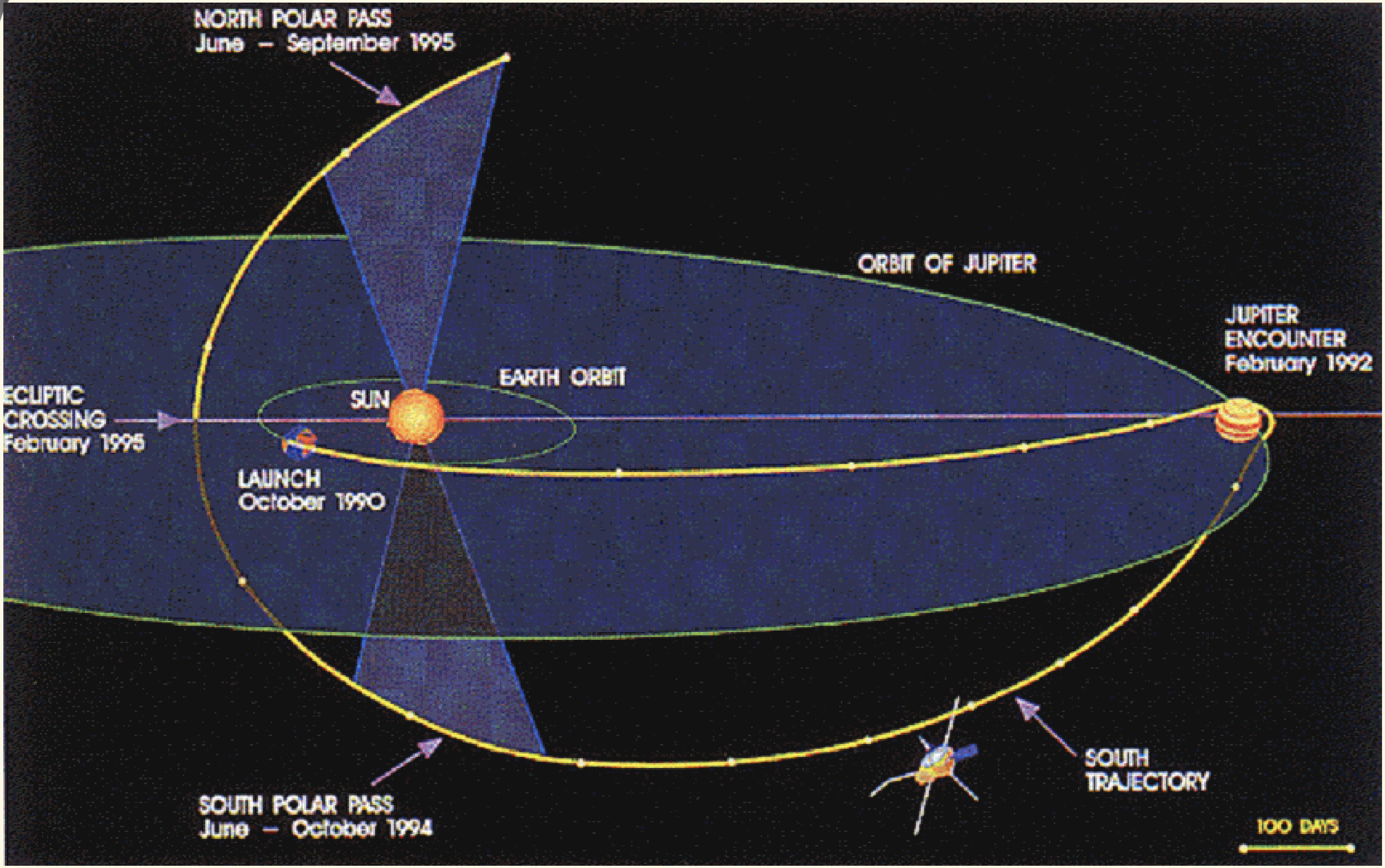
- **Eugene Parker (1958):** theoretical(!) investigation of coronal equilibrium: high temperature leads to (solar) wind (more detailed later on)
- **confirmed by**
  - Soviet measurements (Lunik2/3) with “ion-traps” (1959)
  - Explorer 10 (1961)
  - Mariner II (1962): measurement of fast and slow flows (27 day cycle -> co-rotating, related “coronal holes” and sun spots)

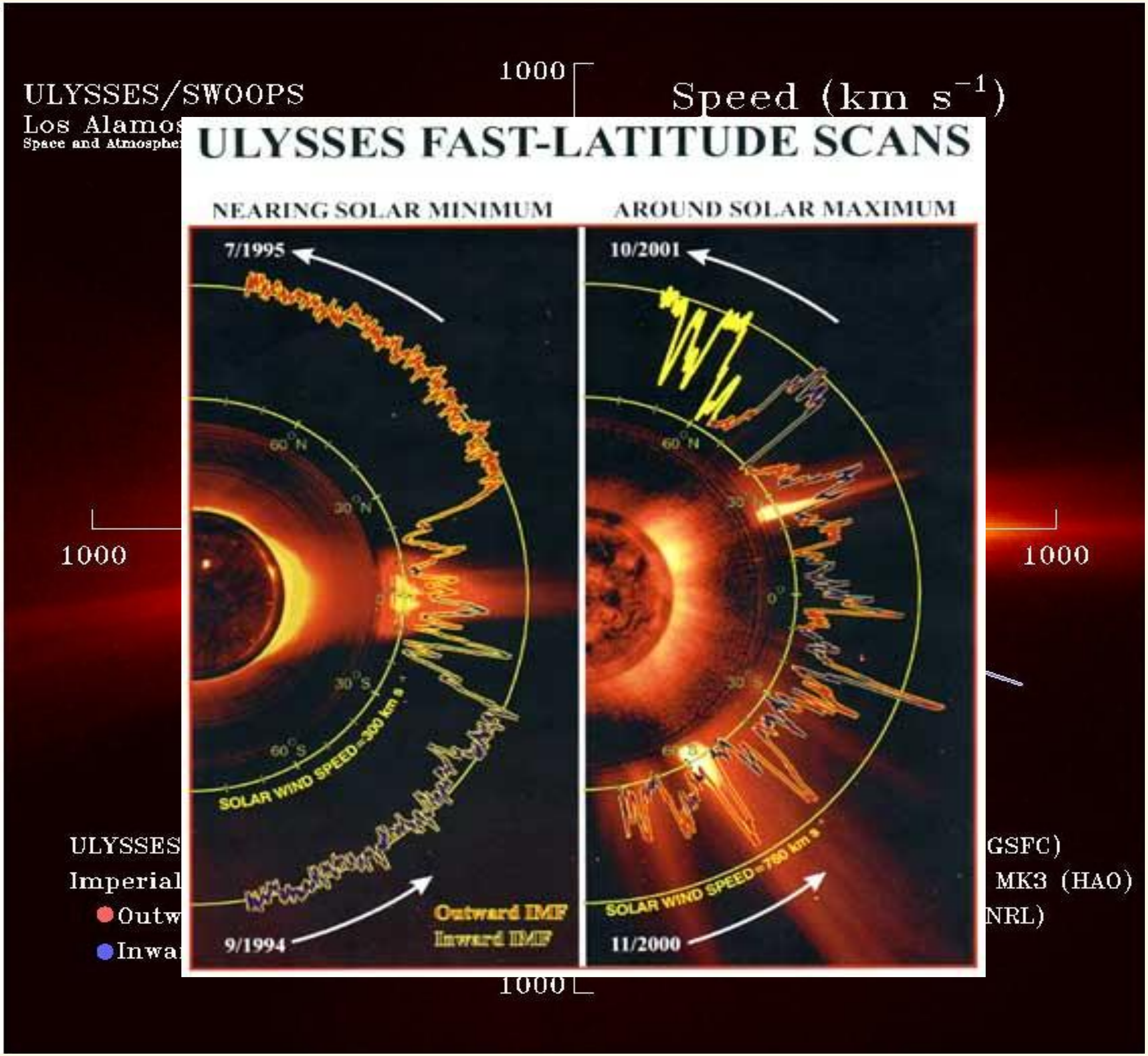


The solar wind – Ulysses ...



… surveying the polar regions

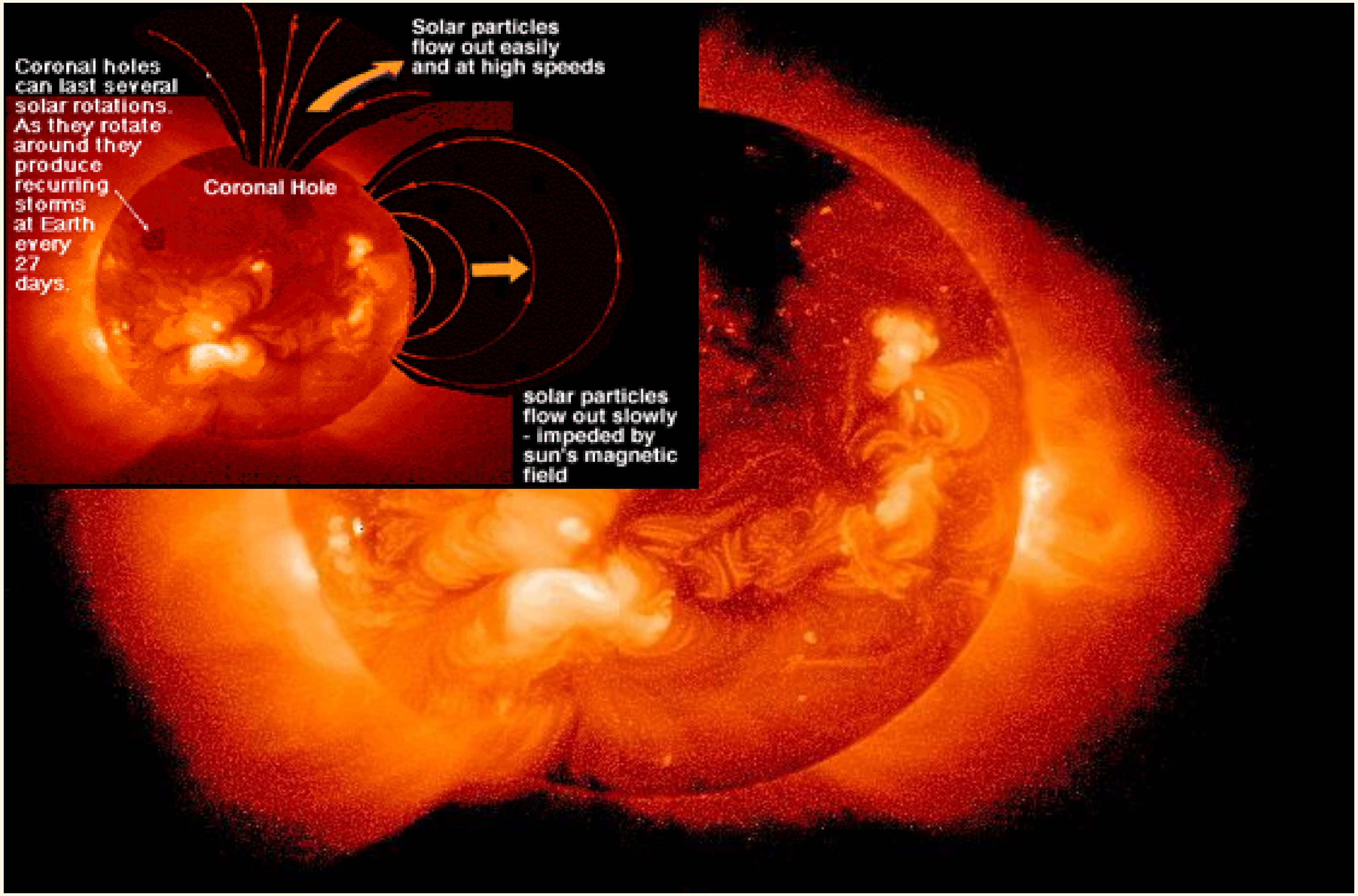




polar wind:
fast and thin

equatorial wind:
slow and dense

The solar wind - coronal holes



fast wind:
over coronal holes (dark corona, "open" field lines, e.g., in polar regions)

coronal X-ray emission

$\Rightarrow$

very high temperatures

(Yohkoh Mission)

The sun and its wind: mean properties

The sun

radius = 695,990 km = 109 terrestrial radii
mass = 1.989 10^{30} kg = 333,000 terrestrial masses
luminosity = 3.85 10^{33} erg/s = 3.85 10^{20} MW $\approx 10^{18}$ nuclear power plants
effective temperature = 5770 °K
central temperature = 15,600,000 °K
life time approx. 10 10^9 years
age = 4.57 10^9 years
distance sun earth approx. 150 10^6 km $\approx$ 400 times earth-moon

The solar wind

temperature when leaving the corona: approx.1 10^6 K
average speed approx. 400-500 km/s (travel time sun-earth approx. 4 days)
particle density close to earth: approx. 6 cm^{-3}
temperature close to earth: $\lesssim 10^5$ K
mass-loss rate: approx 10^{12} g/s (1 Megaton/s) $\approx 10^{-14}$ solar masses/year
$\approx$ one Great-Salt-Lake-mass/day $\approx$ one Baltic-sea-mass/year
$\Rightarrow$ no consequence for solar evolution, since only 0.01% of total mass lost over total life time

Stellar winds – hydrodynamic description

Need mechanism which accelerates material beyond escape velocity:

- **pressure** driven winds
- **radiation** driven winds

Note: red giant winds still not understood, only scaling relations available ("Reimers-formula")

remember equation of motion (conservation of momentum + stationarity, cf. Chap. 6, page 87)

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dp}{dr} + g^{ext} \quad \text{(in spherical symmetry)}$$

$\Rightarrow$ With mass-loss rate $\dot{M}$, radius r, density ρ and velocity v

$$\dot{M} = 4\pi r^2 \rho v,$$

equation of continuity: conservation of mass

and with isothermal sound-speed a

$$\left(1-\frac{a^2}{v^2}\right)v\frac{dv}{dr} = -\frac{GM}{r^2} + g_{rad} + \frac{2a^2}{r} - \frac{da^2}{dr}$$

equation of motion: from conservation of momentum

vel. field	grav. accel.	radiative accel.	(part of) accel. by pressure gradient
positive for v > a **negative** for v < a	inwards	outwards	outwards

Pressure driven winds

$$\left(1-\frac{a^2}{v^2}\right)v\frac{dv}{dr} = -\frac{GM}{r^2} + \cancel{g_{rad}} + \frac{2a^2}{r} - \cancel{\frac{da^2}{dr}}$$

vel. field | grav. accel. | radiative accel. | "pressure"

The solar wind as a proto-type for pressure driven winds

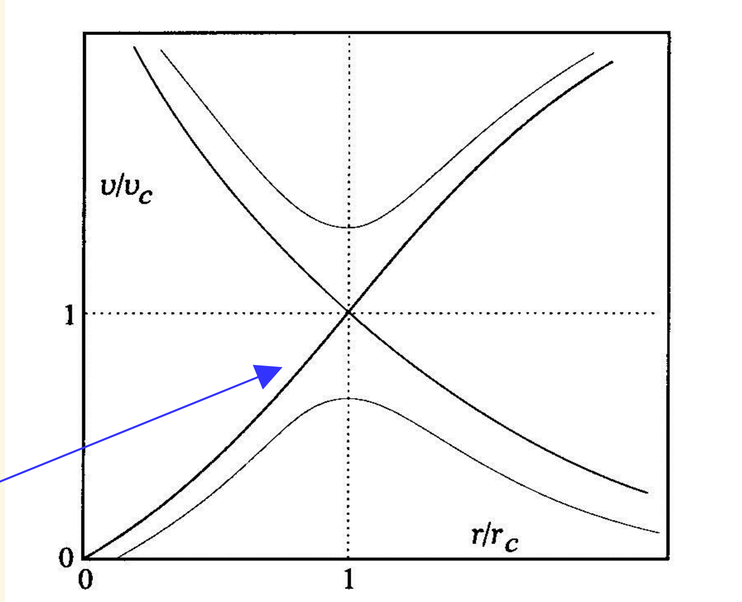
- present in stars which have an (extremely) hot corona ($T \approx 10^6$ K)
- with $g_{rad} \approx 0$ and $T \approx$ const, the rhs of the equation of motion changes sign at

$$r_c = \frac{GM}{2a^2}; \quad \text{with a } (T{=}1.5\cdot 10^6\text{ K}) \approx 160\text{ km/s},$$

we find for the sun $r_c \approx 3.9 R_{\rm sun}$

and obtain four possible solutions for v/v_c ("c" = critical point)

- only one (the "transonic") solution compatible with observations
- pressure driven winds as described here rely on the presence of a hot corona (large value of a!)
- Mass-loss rate $\dot{M} \approx 10^{-14}$ $M_{\rm sun}$ / yr, terminal velocity $v_\infty \approx 500$ km/s
- has to be heated (dissipation of acoustic and magneto-hydrodynamic waves)
- not completely understood so far



Radiation driven winds

accelerated by radiation pressure:

$$\left(1-\frac{a^2}{v^2}\right)v\frac{dv}{dr} = -\frac{GM}{r^2} + g_{rad} + \underbrace{\frac{2a^2}{r}-\frac{da^2}{dr}}_{\text{important only in lowermost wind}}$$

pressure terms only of secondary order
(a ≈ 20 km/s for hot stars,
≈ 3 km/s for cool stars)

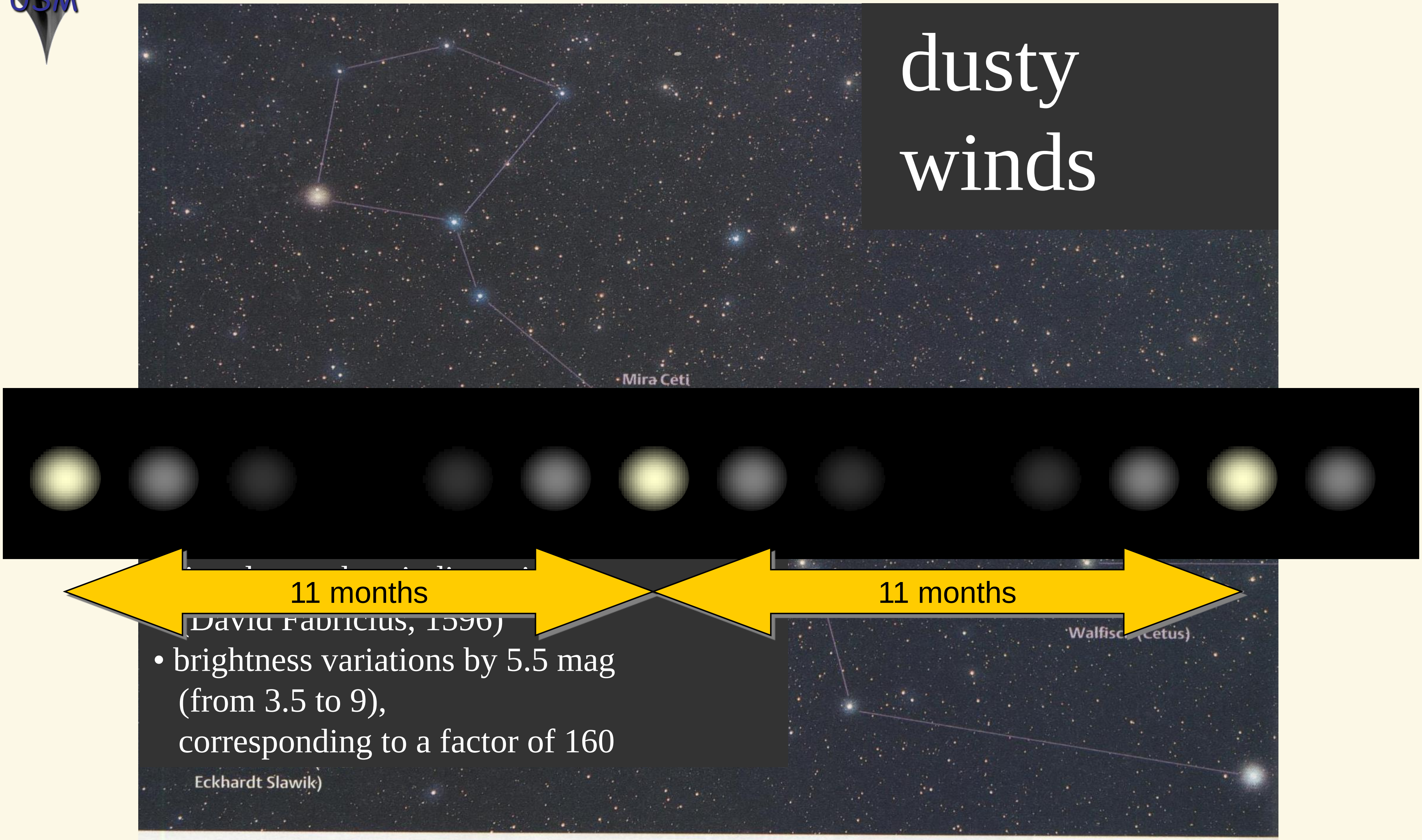
- cool stars (AGB): major contribution from **dust** absorption;
  coupling to "gas" by viscous drag force (gas - grain collisions)

  $\dot{M} \approx 10^{-6} \mathrm{M_{sun}}/\mathrm{yr},\ \mathrm{v}_\infty \approx 20\ \mathrm{km/s}$

- hot stars: major contribution from **metal line** absorption;
  coupling to bulk matter (H/He) by Coulomb collisions

  $\dot{M} \approx 10^{-6} ... 10^{-5}\ \mathrm{M_{sun}}/\mathrm{yr},\ \mathrm{v}_\infty \approx 2{,}000\ \mathrm{km/s}$

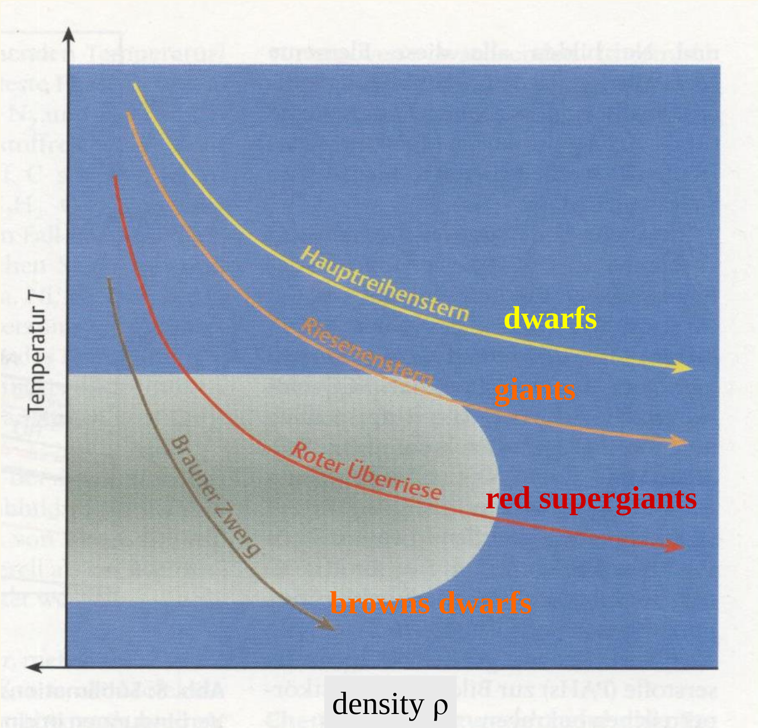
dusty winds



(David Fabricius, 1596)

- brightness variations by 5.5 mag (from 3.5 to 9), corresponding to a factor of 160

Cool supergiants: The dust-factories of our Universe



Material on this and following pages from Chr. Helling, *Sterne und Weltraum*, Feb/March 2002

dust: approx. 1% of ISM, 70% of this fraction formed in the winds of AGB-stars (cool, low-mass supergiants)

Red supergiants are located in dust-forming "window"

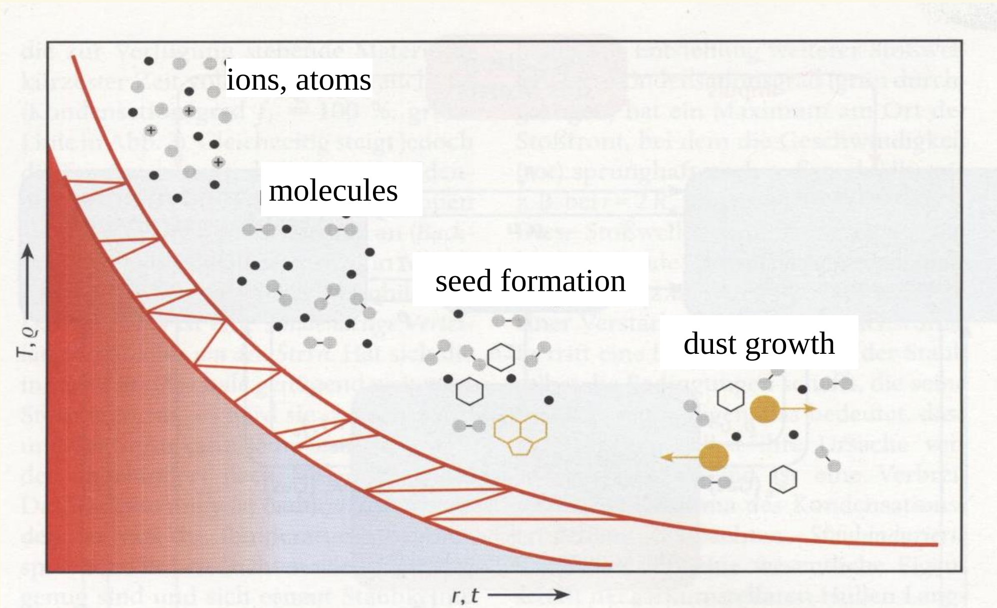
transition from gaseous phase to solid state possible only in **narrow range of temperature and density:**

gas density must be high enough and temperature low enough to allow for the chemical reactions:

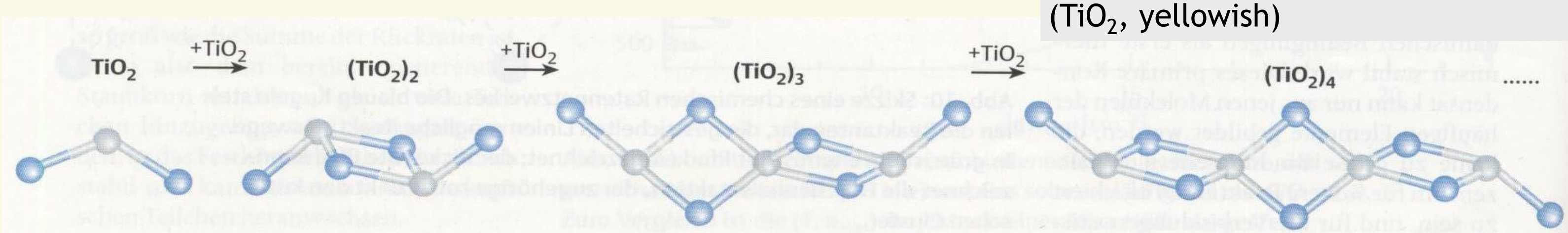
- sufficient number of dust forming molecules required
- the dust particles formed have to be thermally stable

Growth of dust in matter outflow

- decrease of density and temperature
- more and more complex structures are forming
- dust: macroscopic, solid state body, approx. 10^{-7} m (1000 Angstrom), 10^{9} atoms

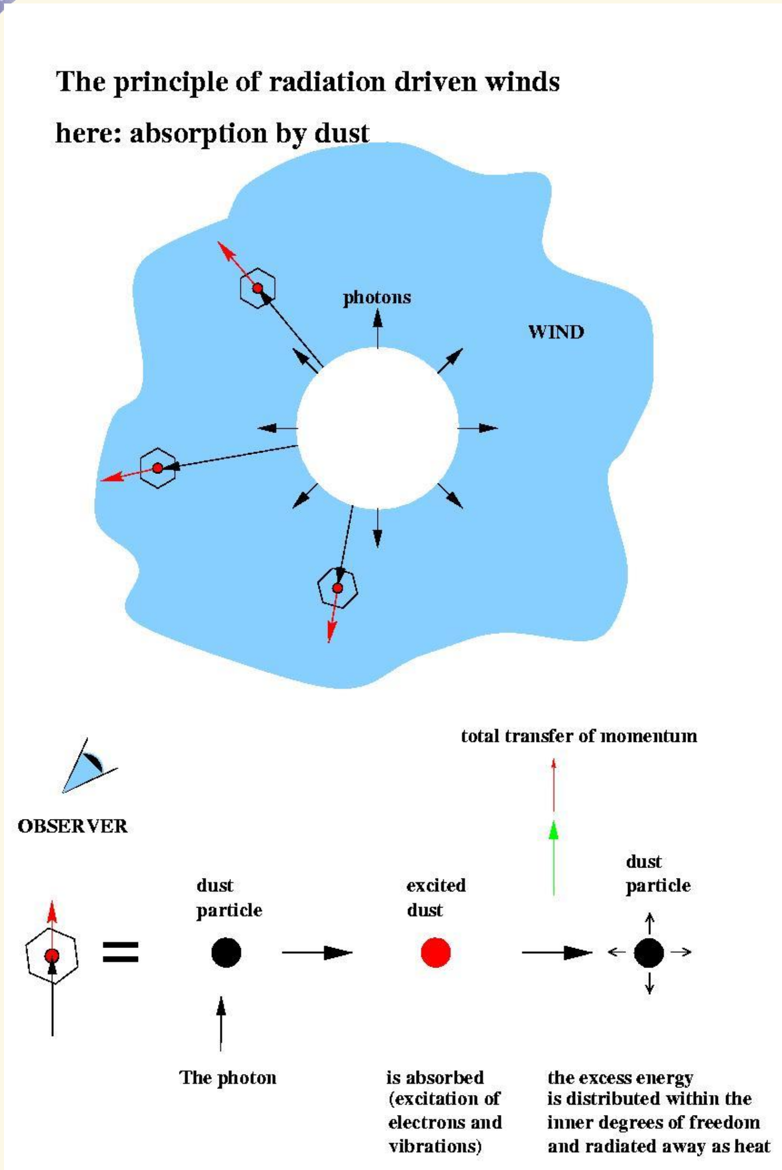


terrestrial, macroscopic rutile crystal (TiO_2, yellowish)



first steps of a linear reaction chain, forming the seed of $(TiO_2)_N$

Dust-driven winds: the principle



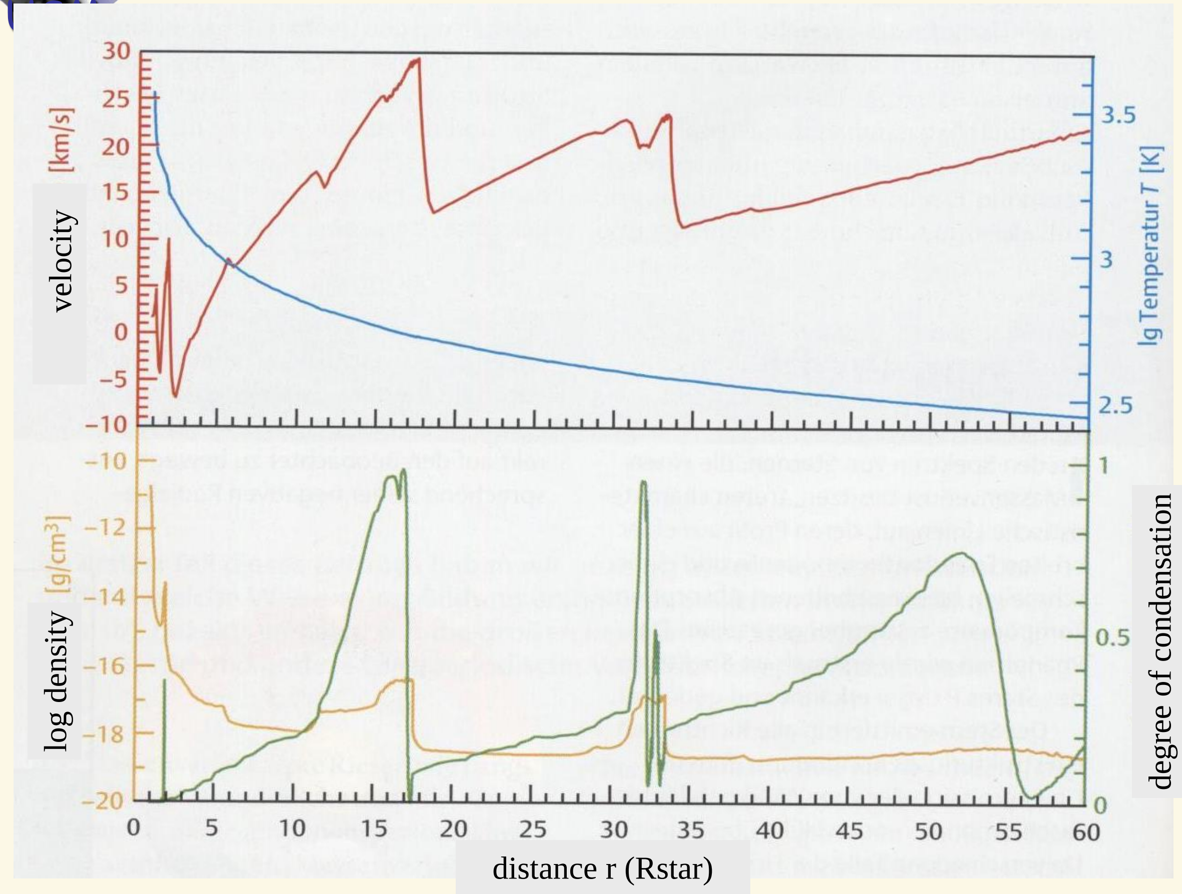
- star emits photons
- photons absorbed by dust
- momentum transfer accelerates dust
- gas accelerated by viscous drag force
  due to gas-dust collisions

acceleration
proportional to number of photons, i.e.,
proportional to *stellar luminosity* ***L***

$\Rightarrow$ mass-loss rate $\propto$ ***L***

dust driven winds at tip of AGB responsible
for ejection of envelope
$\Rightarrow$ Planetary Nebulae

winds from massive red supergiants still
not explained, but probably similar mechanism

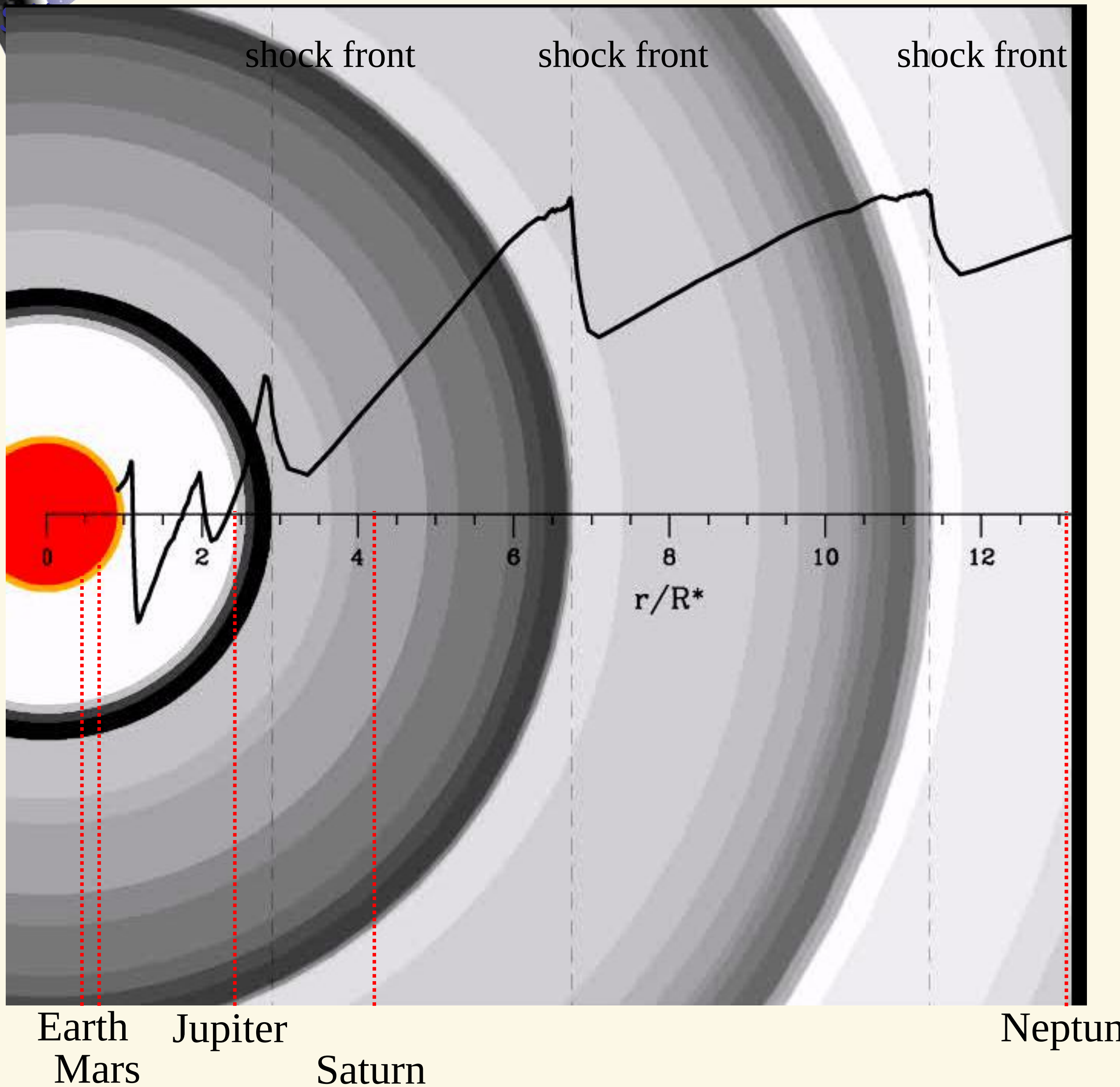


snapshot of a time-dependent hydro-simulation of a carbon-rich circumstellar envelope of an AGB-star. Model parameters similar to next slide.

- star ("surface") pulsates,
- sound waves are created,
- steepen into shocks;
- matter is compressed,
- dust is formed
- and accelerated by radiation pressure

dust shells are blown away, following the pulsational cycle

$\Rightarrow$ periodic darkening of stellar disc
$\Rightarrow$ **brightness variations**



dark colors: dust shells

velocity

simulation of a dust-driven wind (*working group E. Sedlmayr, TU Berlin*)

$T = 2600$ K, $L = 10^4 L_{sun}$,
$M = 1 M_{sun}$, $\Delta v = 2$ km/s

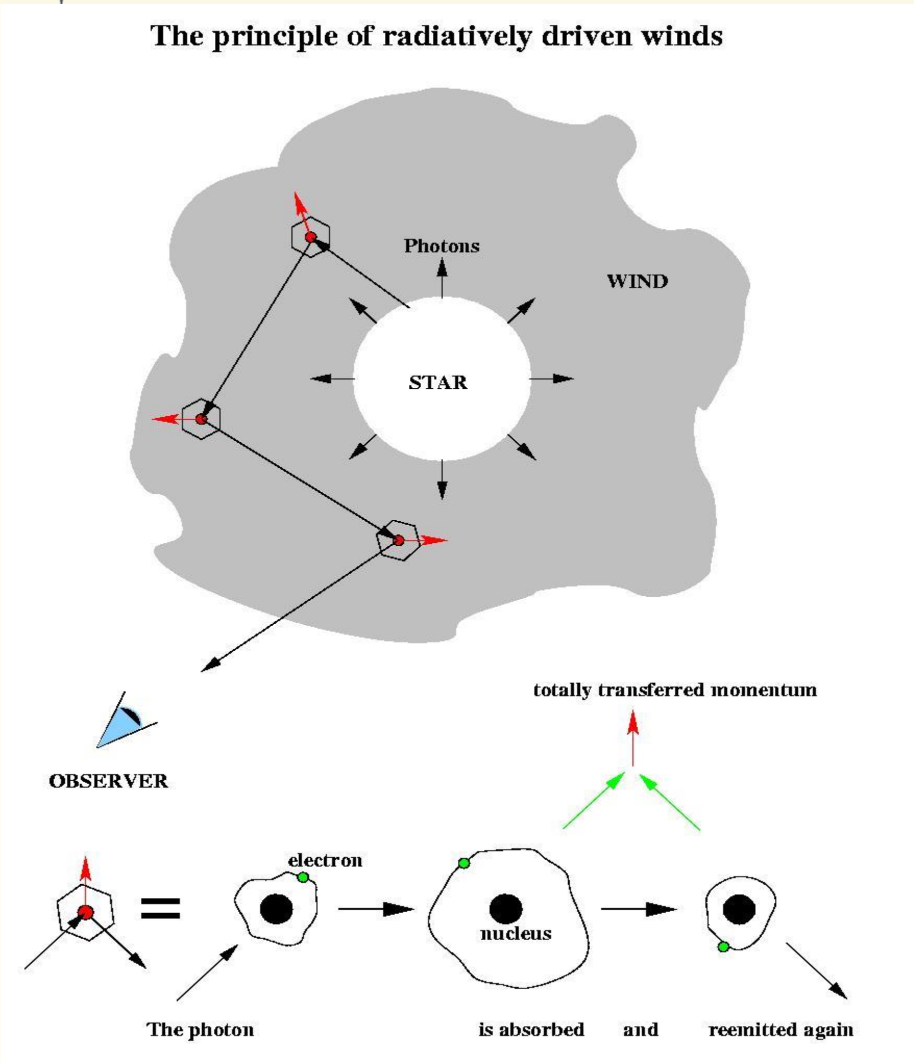
Stars and their winds - typical parameters

	The sun	Red AGB-stars	Blue supergiants
mass [$M_\odot$]	1	1 ... 3	10...100
luminosity [$L_\odot$]	1	10^4	$10^5...10^6$
stellar radius [$R_\odot$]	1	400	10...200
effective temperature [K]	5570	2500	$10^4...5\cdot10^4$
wind temperature [K]	10^6	1000	8000...40000
mass loss rate [$M_\odot$ /yr]	10^{-14}	$10^{-6}...10^{-4}$	10^{-6} ... few 10^{-5}
terminal velocity [km/s]	500	30	200...3000
life time [yr]	10^{10}	10^5	10^7
total mass loss [$M_\odot$]	10^{-4}	$\gtrsim 0.5$	up to 90% of total mass

Bubble Nebula
(NGC 7635)
in Cassiopeia

wind-blown
bubble around
BD+602522
(O6.5IIIf)

Line-driven winds: basics



- accelerated by radiation pressure in **lines**
  $\dot{M} \approx 10^{-7} \ldots 10^{-5}\ \mathrm{M_{sun}}/\mathrm{yr},\ v_\infty \approx 200 \ldots 3{,}000\ \mathrm{km/s}$
- momentum transfer from accelerated species (ions) to bulk matter (H/He) via Coulomb collisions

Prerequesites for radiative driving

- large number of photons => high luminosity
  $L \propto R_*^2\, T_{\mathrm{eff}}^4$ => supergiants or hot dwarfs
- line driving:
  large number of lines close to flux maximum
  (typically some $10^4 \ldots 10^5$ lines relevant)
  with high interaction probability
  (=> mass-loss dependent on metal abundances)

- line driven winds important for chemical evolution of (spiral) Galaxies, in particular for starbursts
- transfer of momentum (=> induces *star formation*, hot stars mostly in *associations)*, energy and nuclear processed material to surrounding environment
- dramatic impact on stellar evolution of massive stars (mass-loss rate vs. life time!)

pioneering investigations by
Lucy & Solomon, 1970, ApJ 159
Castor, Abbott & Klein, 1975, ApJ 195 (CAK)

reviews by Kudritzki & Puls, 2000, ARAA 38
Puls et al. 2008 A&Arv 16, issue 3

$g_{rad} \propto N$ (number of absorbed photons)

LINE absorption

absorption only if frequency close to a possible line transition,

$$\kappa_\nu \propto \kappa_0 \text{ if } \nu_0 \pm \delta\nu \text{ (thermal width)}$$
$$\kappa_\nu = 0 \quad \text{else}$$

- absorption always at *line frequency* ν_0 ($\pm\delta\nu$) *in frame of matter*
- matter moves at certain velocity with respect to stellar frame
- matter "sees" stellar photons at different frequency than star itself (Doppler-effect)

$$\nu_{\mathrm{CMF}} = \nu_{\mathrm{obs}} - \frac{\nu_0 \mathrm{v}(r)}{c} =: \nu_0 \quad \text{(radial photons, } \mu\text{=1, assumed)}$$

- the larger the velocity of matter, the larger the photon's stellar frame frequency must be in order to become absorbed at ν_0 (in frame of matter)

$$\left.\begin{aligned} \nu_0 &= \nu_1^{\mathrm{obs}} - \frac{\nu_0}{c}\mathrm{v}_1(r) \\ \nu_0 &= \nu_2^{\mathrm{obs}} - \frac{\nu_0}{c}\mathrm{v}_2(r) \end{aligned}\right\} \text{ if } \mathrm{v}_2(r) > \mathrm{v}_1(r), \text{ then } \nu_2^{\mathrm{obs}} > \nu_1^{\mathrm{obs}}$$

$\Rightarrow$accelerated matter "sees" photons from a considerably larger band-width than static matter, $\Delta\nu_{\mathrm{obs}} = \frac{\nu_0}{c}\Delta\mathrm{v} \gg \delta\nu$

shell of matter with spatial extent Δr,

and velocity $v_0 + \left(\frac{dv}{dr}\right)_1 \Delta r$

absorption of photons at $\nu_0 \pm \delta\nu$

in frame of matter

photons must start at higher (stellar) frequencies, are "seen" at $\nu_0 \pm \delta\nu$ in frame of matter because of Doppler-effect.

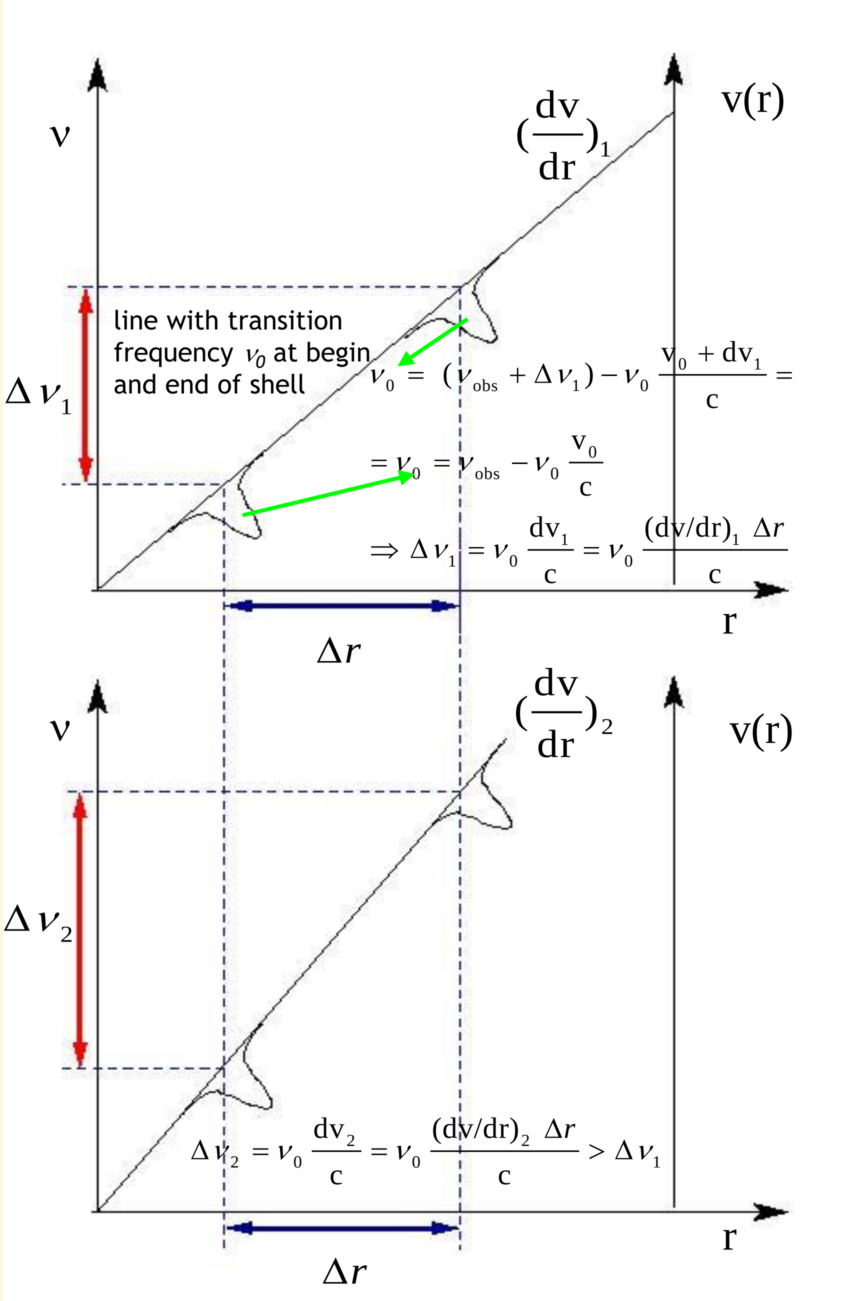
Let $\Delta\nu$ be frequency band contributing to acceleration of matter in Δr

The larger $\frac{dv}{dr}$,

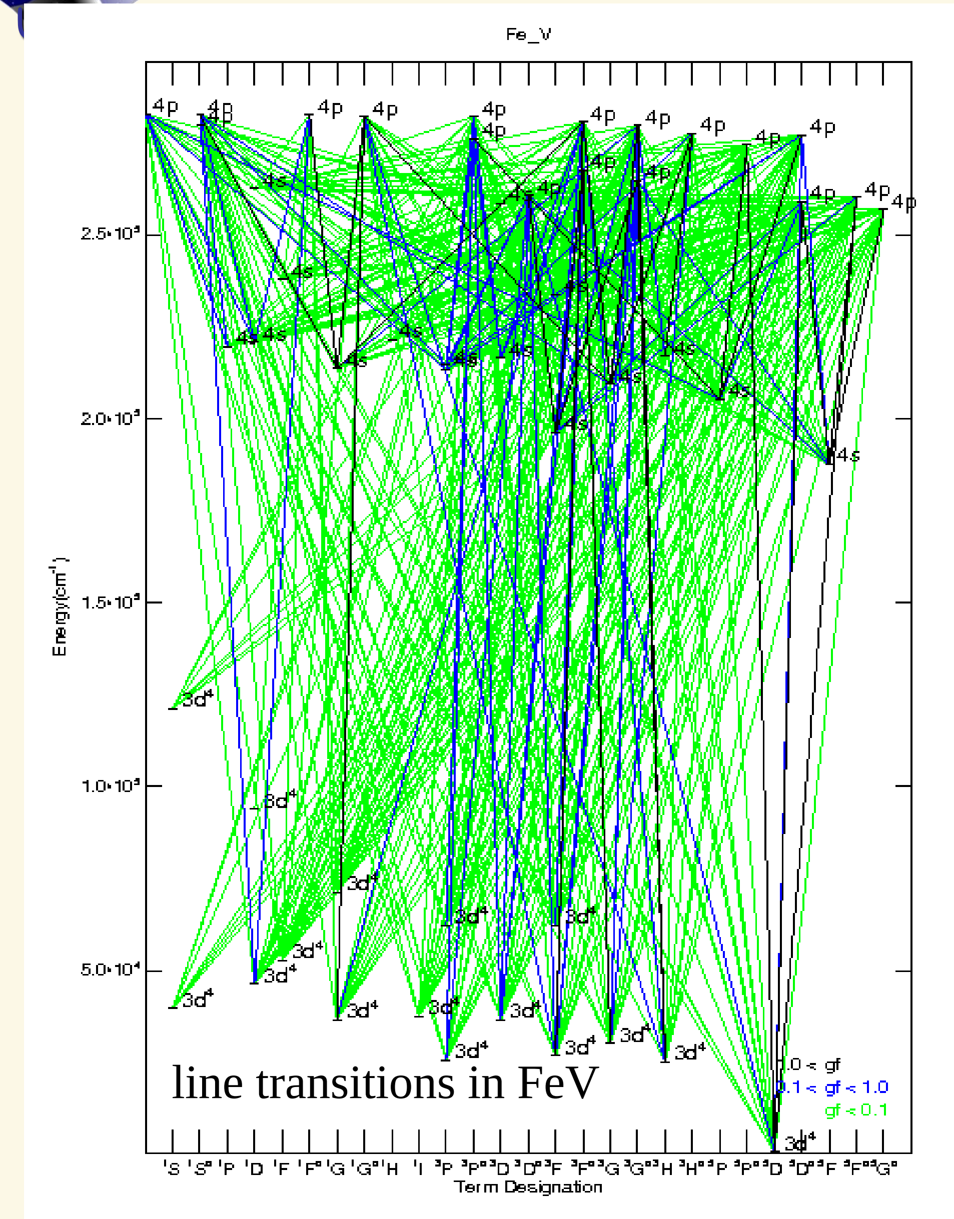
- the larger $\Delta\nu$
- the more photons can be absorbed
- the larger the acceleration

$$\boxed{g_{rad} \propto \frac{dv}{dr}}$$

(assuming that each photon is absorbed, i.e., acceleration from optically thick lines)



Millions of lines ....



line transitions in FeV

... are present
... and needed!

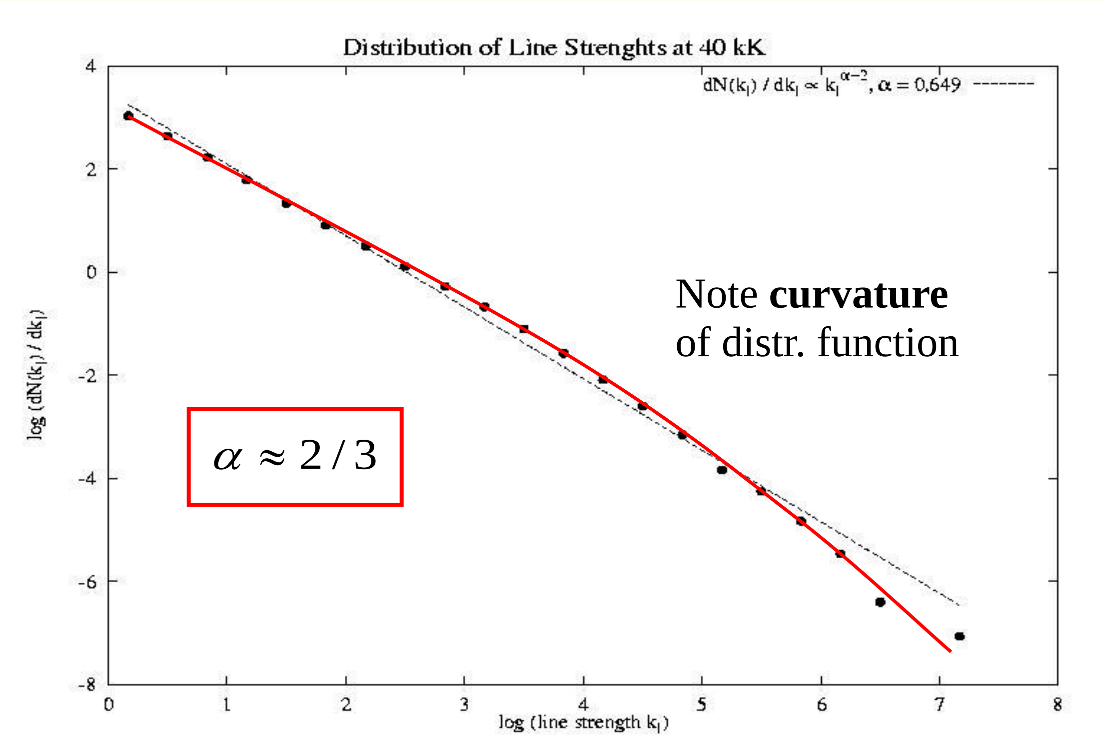
$$g_{rad}^{tot} = \sum_{\text{all lines}} g_{rad}^{i},$$

$$g_{rad}^{thin} \propto L_{\nu}^{i} k^{i}, \qquad k^{i} \propto \frac{\kappa^{i}}{\rho} \text{ (line-strength)}$$

$$g_{rad}^{thick} \propto L_{\nu}^{i} \frac{\mathrm{d}v/dr}{\rho}$$

The line distribution function

- pioneering work by Castor, Abbott & Klein (1975) and by Abbott (1982)
- first realistic line-strength distribution function by Kudritzki et al. (1988)
- NOW: 4.2 Ml (Mega lines), 150 ionization stages (H – Zn), NLTE



$$\frac{dN(k)}{dk} = k^{\alpha-2}, \quad \alpha \approx 0.6 \ldots 0.7$$

+ 2nd empirical finding:
valid in *each* frequential subinterval

$$dN(k,\nu) = -N_0 f(\nu) d\nu \, k^{\alpha-2} dk$$

Logarithmic plot of line-strength distribution function for an O-type wind at 40,000 K and corresponding power-law fit (see Puls et al. 2000, A&AS 141)

$$g_{rad}^{tot} = \sum_{\text{all lines}} g_{rad}^{i} \quad \Rightarrow \iint g_{rad}^{i}(\nu,k)\, dN(\nu,k) \quad \propto N_{\text{eff}} L \left(\frac{\mathrm{d}v/dr}{\rho} \right)^{\alpha},$$

N_{eff} "effective" number of lines

α exponent of line-strength distr. function, also: $\alpha = \dfrac{g_{rad}^{thick}}{g_{rad}^{tot}}$

Hydrodynamical descri

with

$$g_{rad} = g_{\text{cont}} + g_{\text{lines}}, \quad g_{\text{lines}} \propto N_{\text{eff}} L \left(\frac{\mathrm{d}v/dr}{\rho} \right)^{\alpha},$$

(approximate) analytical solution possible

mass-loss rate $\dot{M}$, radiu

$\dot{M} = 4\pi \mathrm{r}^2 \rho v,$ ... n of continuity

... vation of mass-flux

isothermal soundspeed

$$\left(1 - \frac{a^2}{v^2}\right) v \frac{dv}{dr} = -\frac{GM}{r^2} + g_{rad} + \frac{2a^2}{r} - \frac{da^2}{dr}$$

equation of motion
conservation of momentum-flux

velocity field	grav. accel.	radiative accel.	“pressure”
positive for v > a	inwards	outwards	outwards
negative for v < a			

Scaling relations for line-driven winds (without rotation)

$$\dot{M} \propto N_{\mathrm{eff}}^{1/\alpha'} L^{1/\alpha'} \left(M(1-\Gamma) \right)^{1-1/\alpha'}$$

$$v_\infty \approx 2.25 \frac{\alpha}{1-\alpha} v_{\mathrm{esc}}, \quad v_{\mathrm{esc}} = \left(\frac{2GM(1-\Gamma)}{R_*} \right)^{\frac{1}{2}}$$

$$v(r) = v_\infty \left(1 - \frac{R_*}{r} \right)^{\beta}, \quad \beta = 0.8 \text{ (O-stars) ... 2 (BA-SG)}$$

Γ Eddington factor, accounting for acceleration by Thomson-scattering, diminishes effective gravity

N_{eff} number of lines effectively driving the wind, corrected for ionization effects, dependent on metallicity and spectral type

α exponent of line-strength distribution function, $0 < \alpha < 1$
large value: more optically thick lines

$\alpha' = \alpha - \delta$, with δ ionization parameter, typical value for O-stars: $\alpha' \approx 0.6$

The wind-momentum luminosity relation (WLR)

- use scaling relations for $\dot{M}$ and v_∞, calculate modified wind-momentum rate

$$\dot{M}\, v_\infty R_*^{1/2} \propto N_{\text{eff}}^{1/\alpha'}\, L^{1/\alpha'} \big(M(1-\Gamma)\big)^{1-1/\alpha'} \big(M(1-\Gamma)\big)^{1/2}$$

$$(\alpha' \approx \tfrac{2}{3}) \propto N_{\text{eff}}^{1/\alpha'}\, L^{1/\alpha'}, \text{ independent of } M \text{ and } \Gamma$$

$$\Rightarrow \log(\dot{M}\, v_\infty R_*^{1/2}) \approx \frac{1}{\alpha'} \log L + const(z,\ \text{sp.type})$$

(Kudritzki, Lennon & Puls 1995)

- (at least) two applications

(1) construct observed WLR, calibrate as a function of spectral type and metallicity (N_{eff} and α' depend on both parameters)
independent tool to measure extragalactic distances
from *wind-properties*, T_{eff} and metallicity

(2) compare with theoretical WLR to test validity of radiation driven wind theory

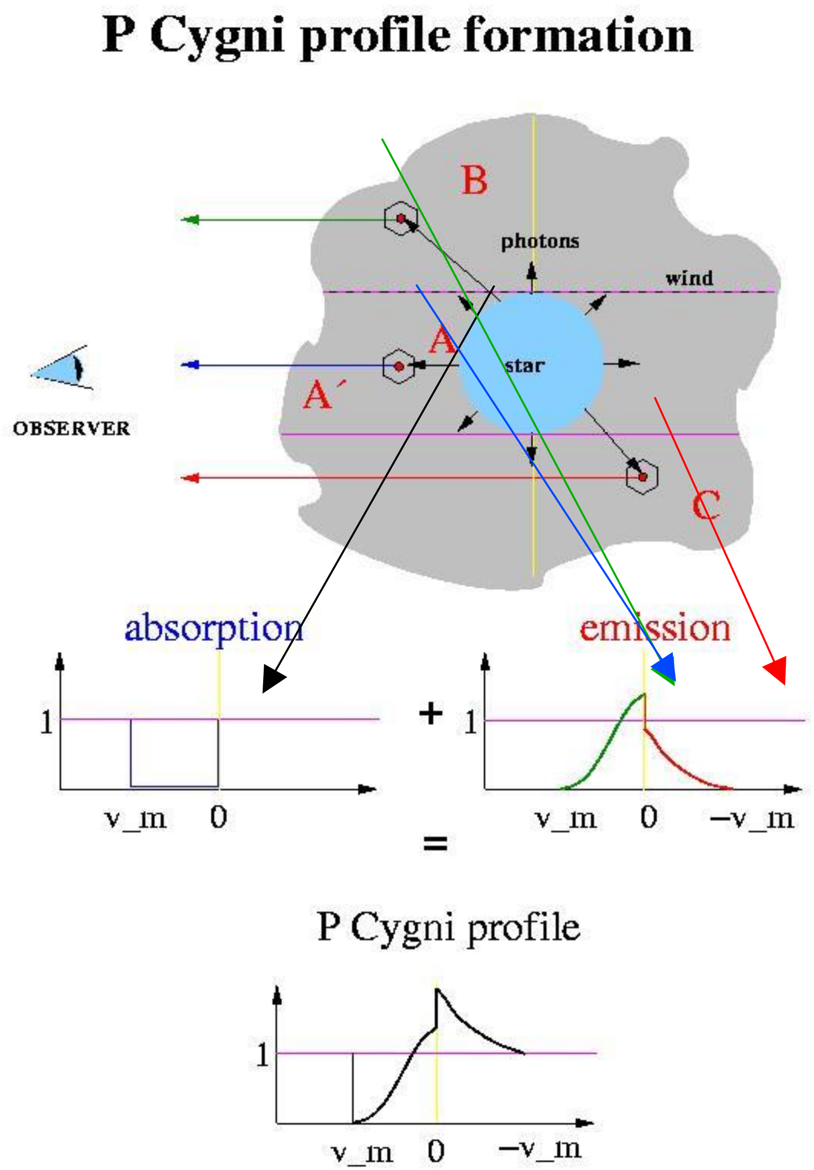
Determination of wind-parameters: v_∞

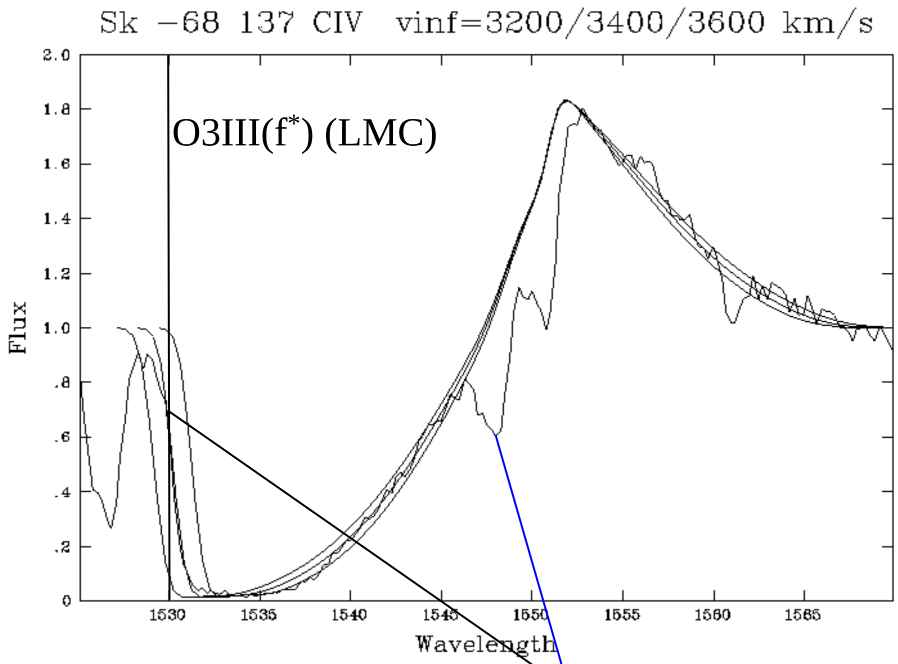
$$\nu_{\mathrm{obs}} = \nu_0 \left(1 + \frac{\mu v(r)}{c}\right); \quad \nu_0 \text{ line frequency in CMF}$$

$$\mu v(r) > 0: \quad \nu_{\mathrm{obs}} > \nu_0 \quad \text{blue side}$$

$$\mu v(r) < 0: \quad \nu_{\mathrm{obs}} < \nu_0 \quad \text{red side}$$

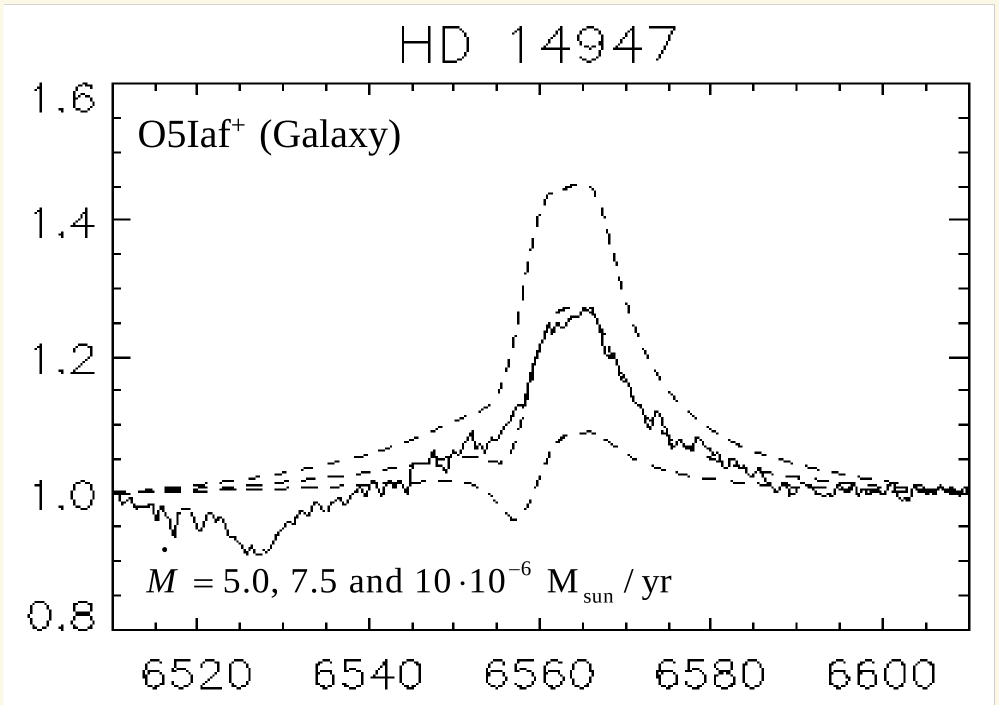
$$\frac{v_{\mathrm{m}}}{\mathrm{c}} = \frac{\nu_{\max} - \nu_0}{\nu_0} = 1 - \frac{\lambda_{\min}}{\lambda_0}$$

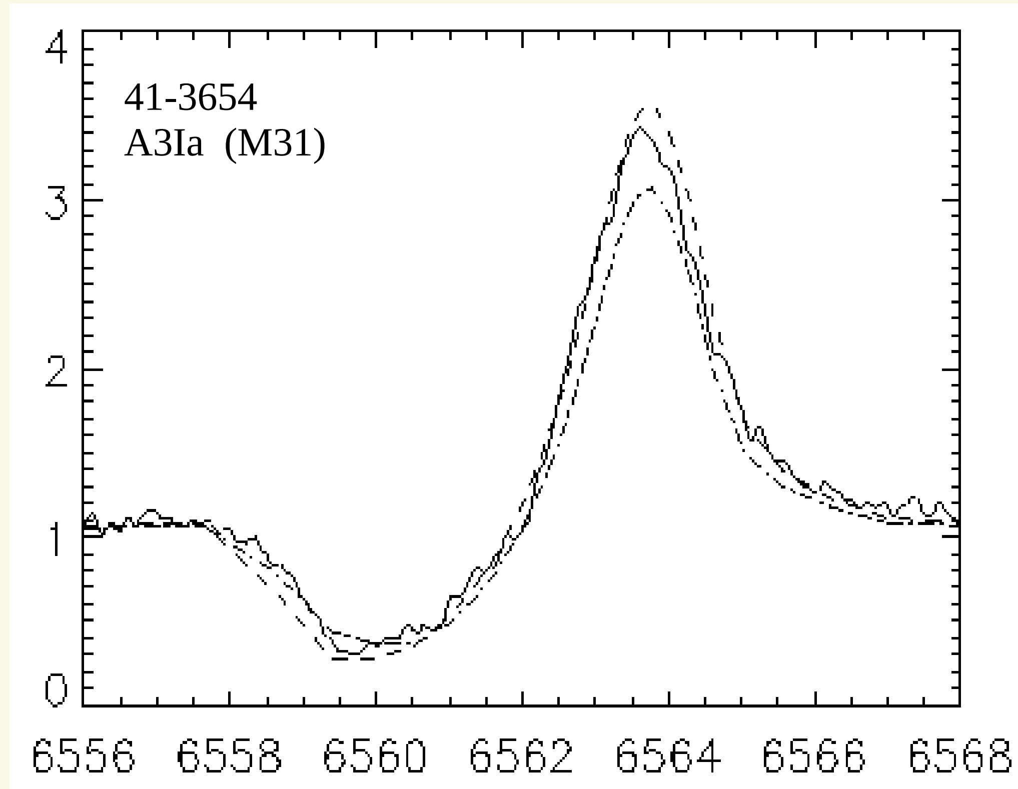




$$v_{\mathrm{m}} \approx 2.998 \cdot 10^5 \left(1 - \frac{1530}{1548}\right) \approx 3,480 \text{ km/s}$$

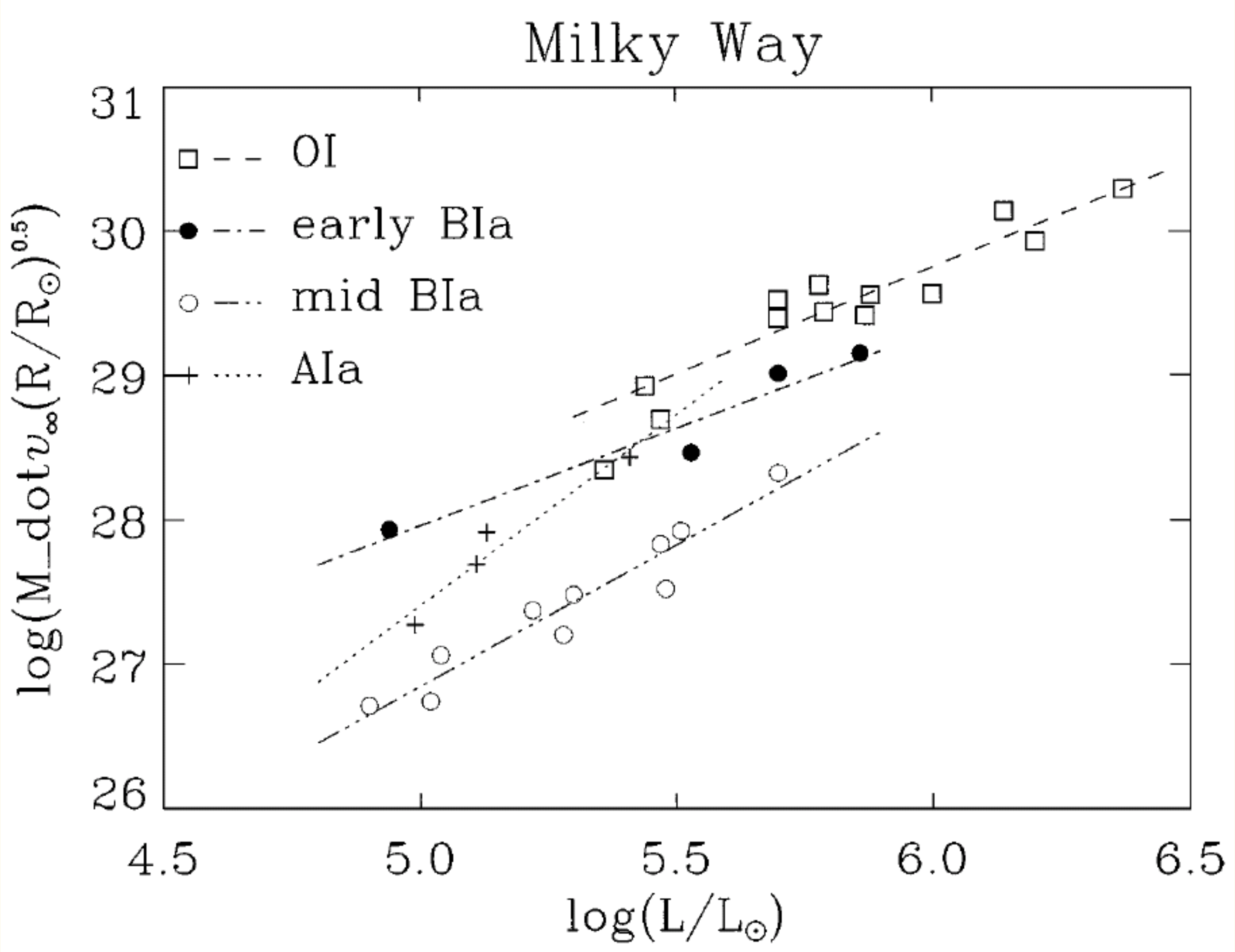
Determination of mass-loss rate from H_α





H_α taken with the Keck HIRES spectrograph, compared with two model calculations adopting $\beta = 3$, $v_\infty = 200$ km/s and *Mdot* = 1.7 and 2.1 $\times 10^{-6}$ M_{sun}/yr.

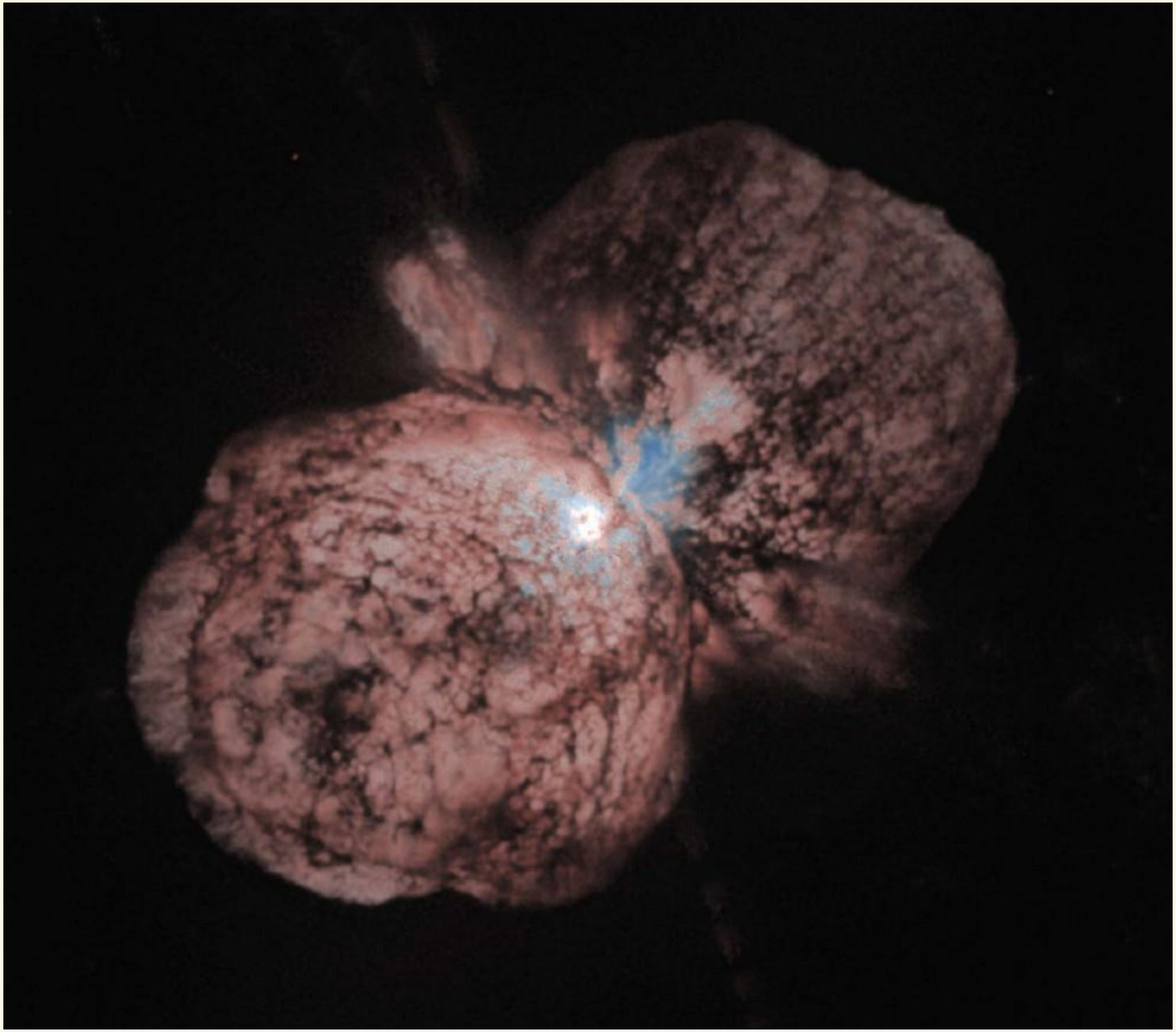
Observed WLR



Modified wind momenta of Galactic O-, early B-, mid B- and A-supergiants as a function of luminosity, together with specific WLR obtained from linear regression. (From Kudritzki & Puls, 2000, ARAA 38).

η Car: Aspherical ejecta

image by HST

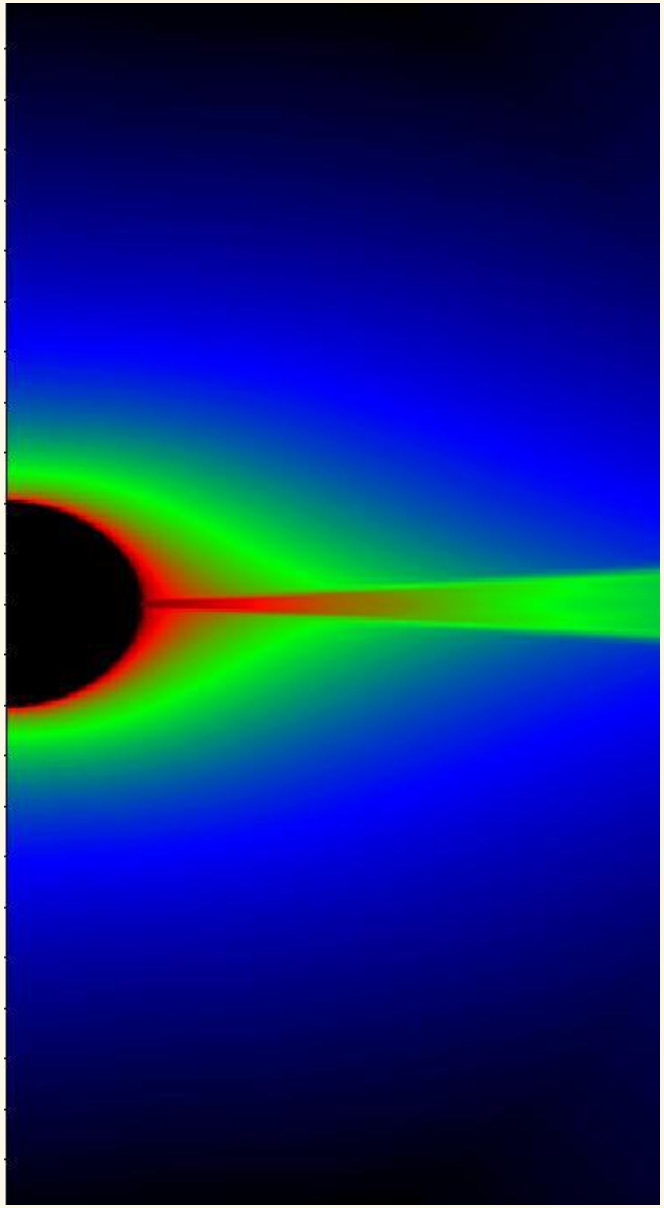


Influence of rotation

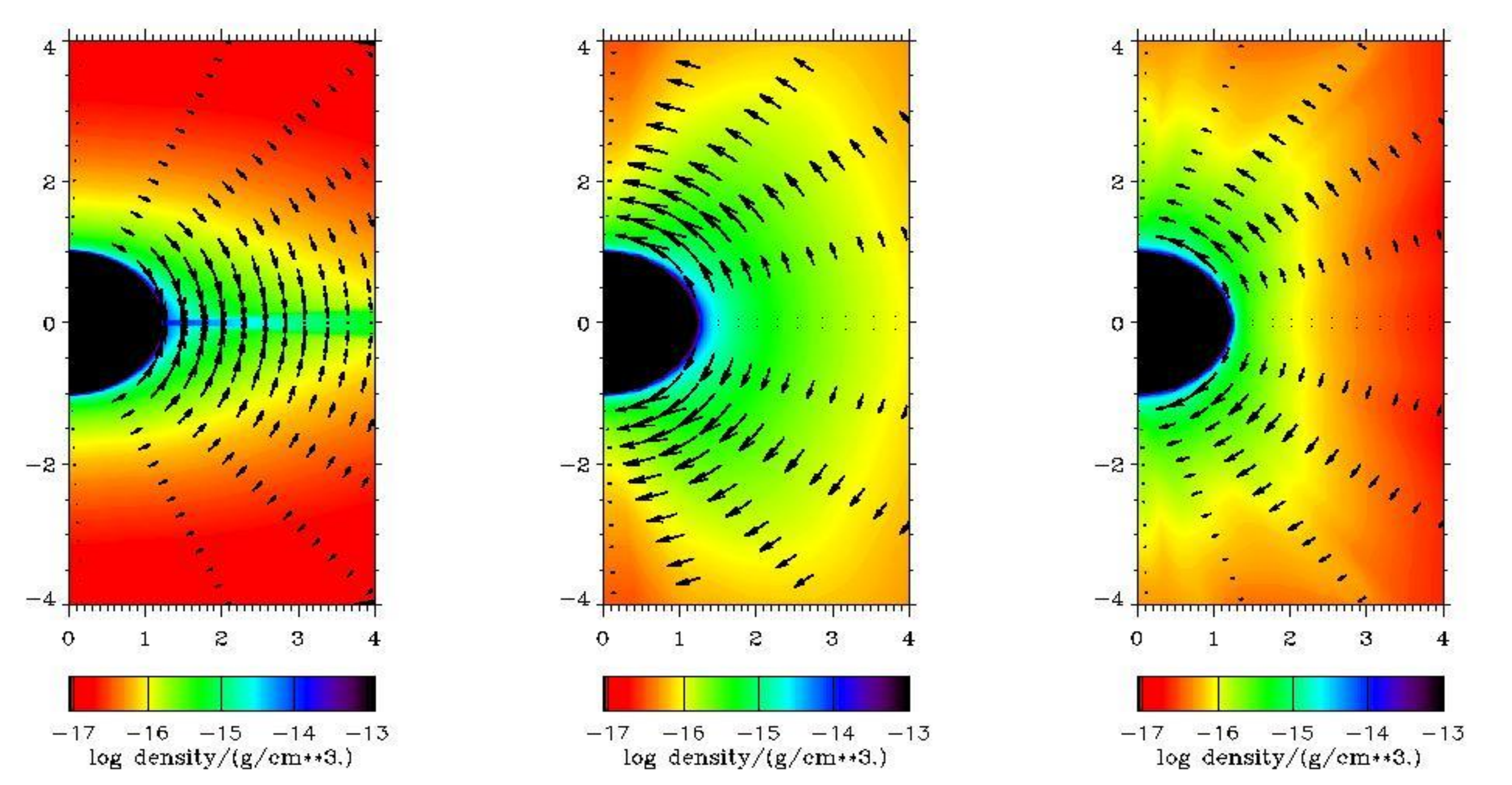
hot, massive stars = young stars

rapidly rotating (up to several 100 km/s)

twofold effect
- star becomes “oblate”
- wind has to react on additional centrifugal acceleration, large in equatorial, small in polar regions



Prolate or oblate wind structure?

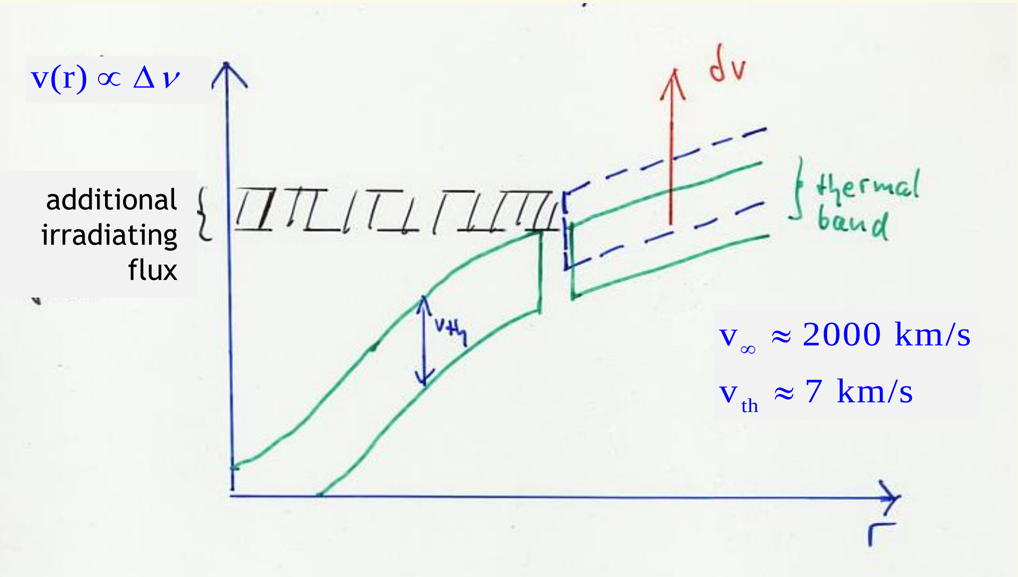


purely radial radiative acceleration:
wind-compressed disk

inclusion of non-radial component of line-acceleration (rotation breaks symmetry)

non-radial line-acceleration plus „gravity darkening“:
prolate geometry

The line-driven instability



perturbation $\delta v \uparrow$

→ profile shifted to higher freq.

→ line 'sees' more stellar flux

→ line force grows $\delta g \uparrow$

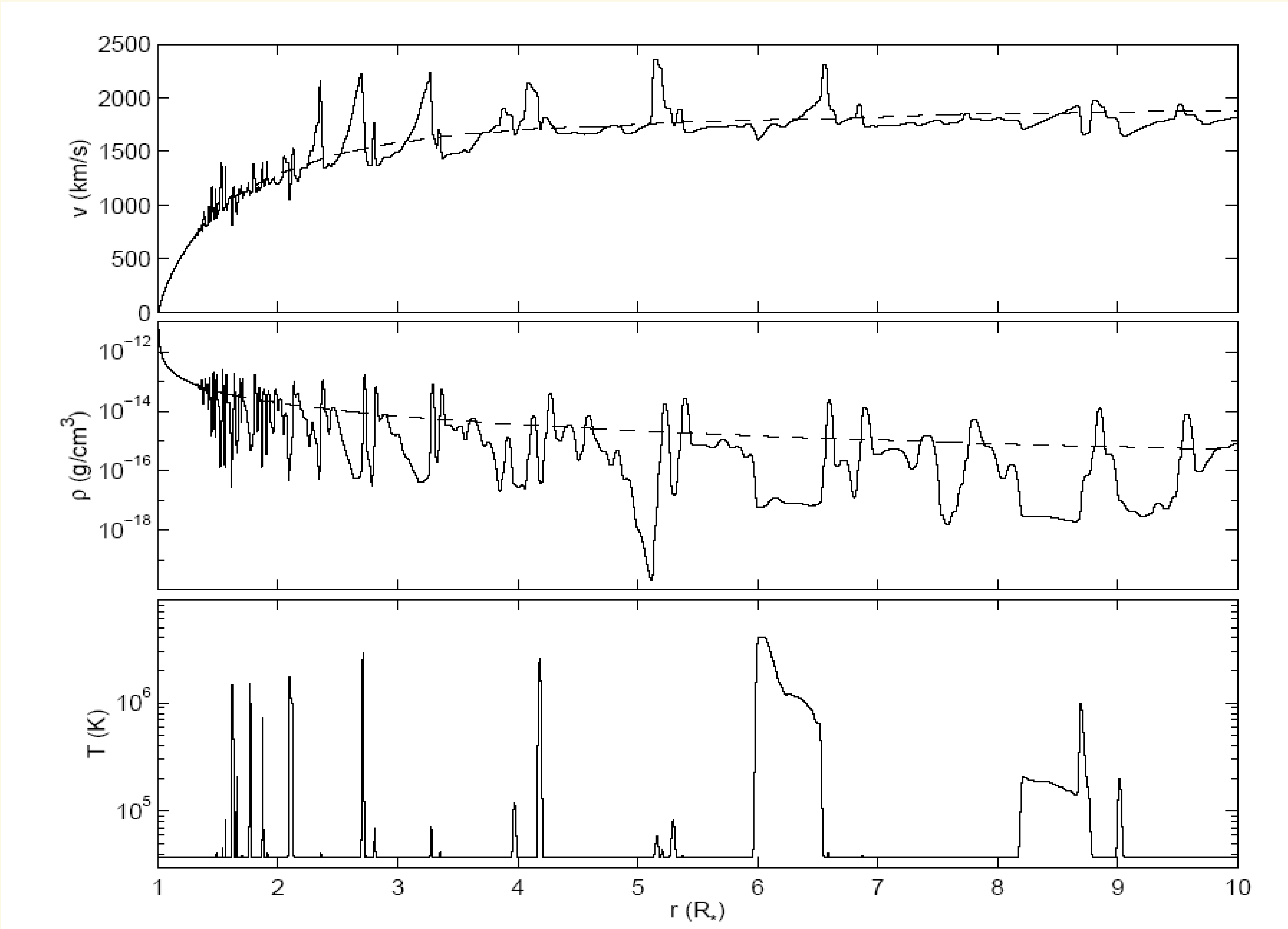
→ additional acceleration $\delta v \uparrow$

exponential growth of perturbation

$\delta g_{Rad} \propto \delta v$

[for details, see MacGregor et al.1979 and Carlberg 1980]

Time dependent hydro-simulations of line-driven winds: Snapshot of density, velocity and temperature structure

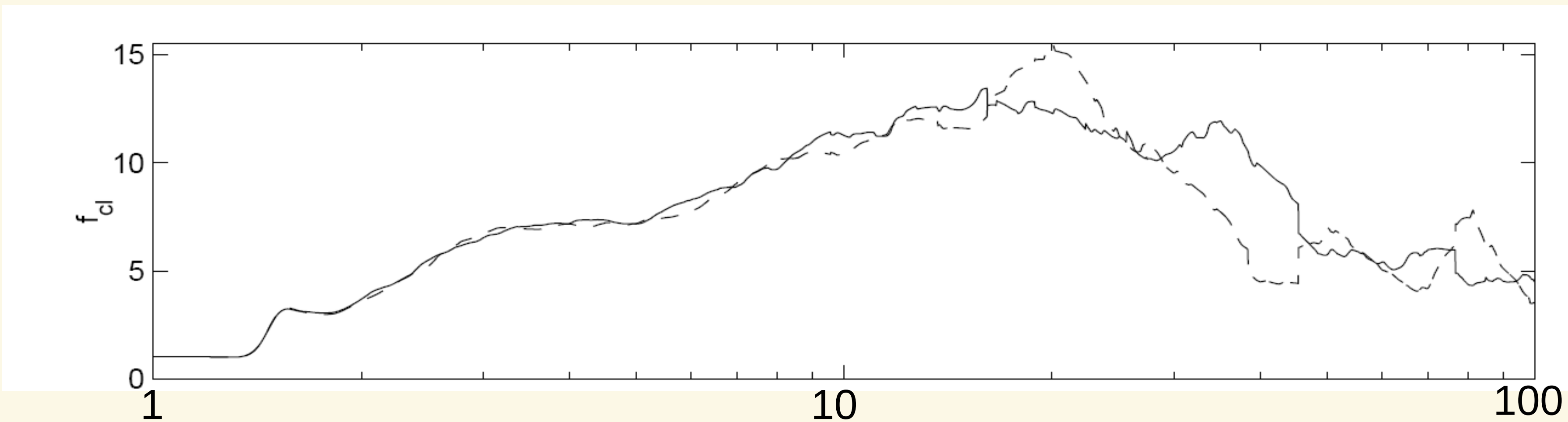


From Runacres & Owocki, 2002, A&A 381

The clumping factor

$$f_{\mathrm{cl}} = \frac{\langle \rho^2 \rangle}{\langle \rho \rangle^2} \geq 1 \text{ always! } (=1 \text{ only for smooth flows})$$

brackets denote temporal averages



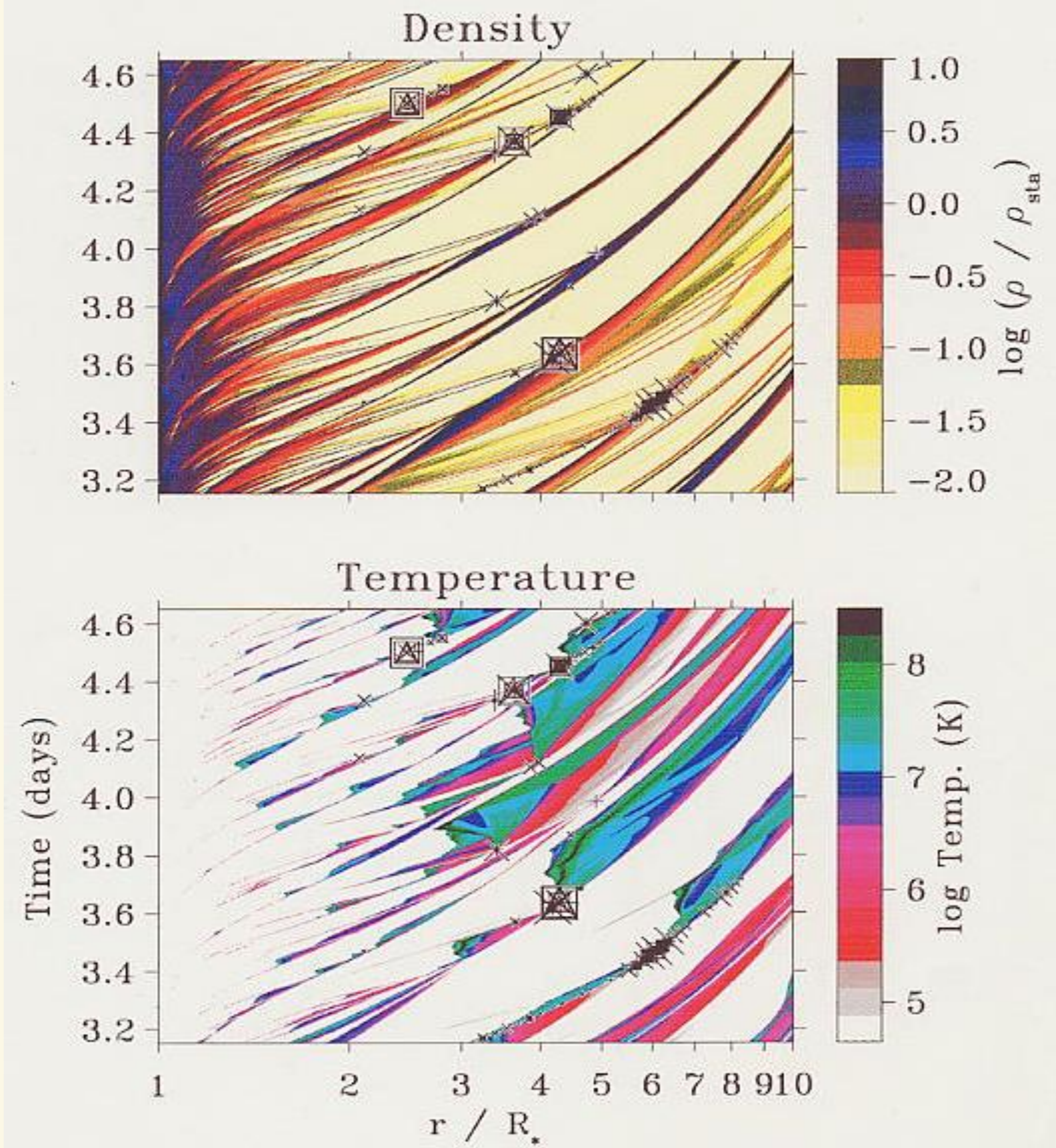
Inhomogeneities have to be accounted for in model atmospheres/spectrum synthesis!

Clumping and X-ray emission in hot stars

density and temperature evolution
as a function of time

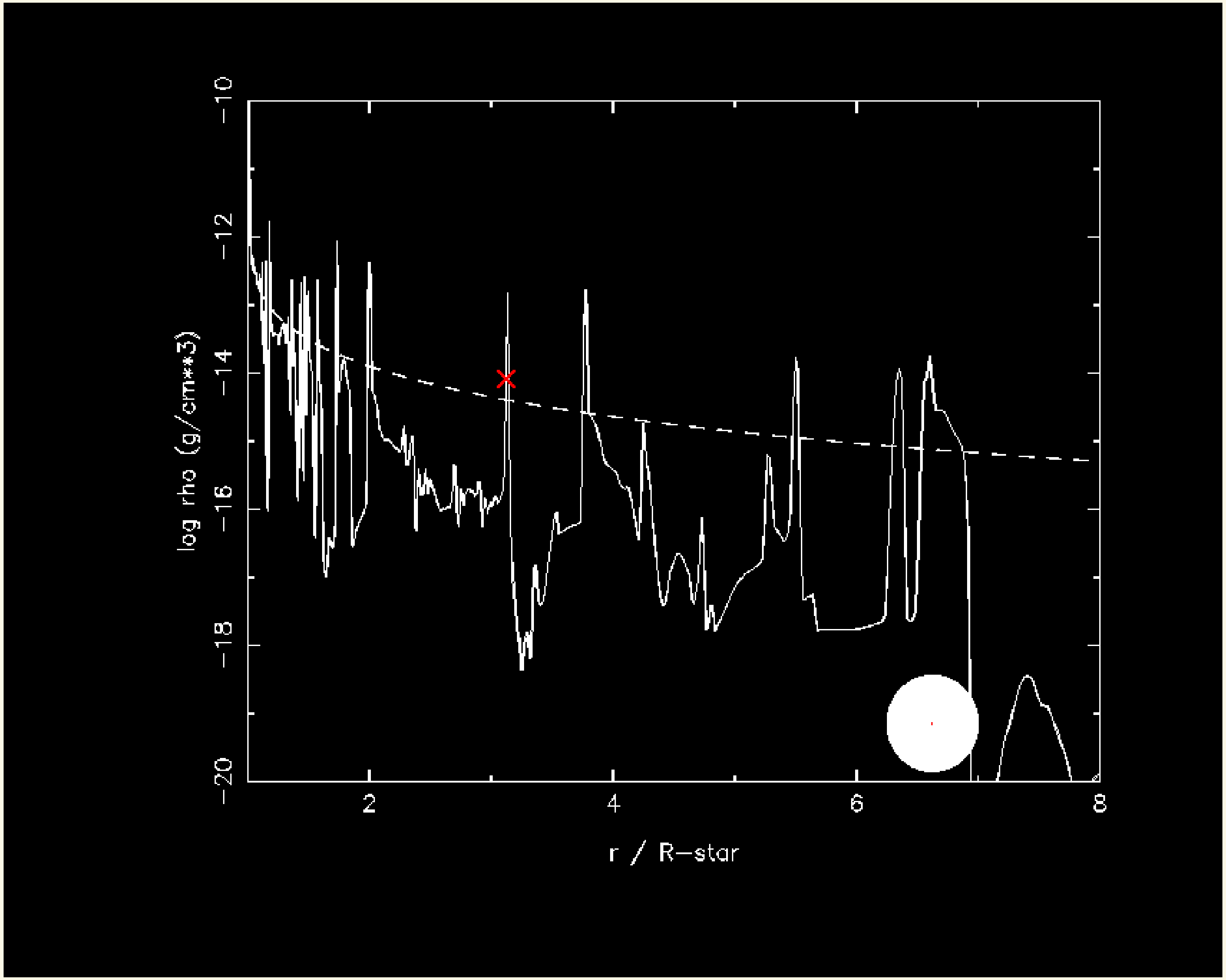
(very) hot gas
→ X-ray emission
(observed!)

hydrodynamical simulations of unstable hot star winds, from Feldmeier et al., 1997, A&A 322



Density evolution in an unstable wind

x
X-ray
"flash"



Determine atmospheric parameters from observed spectrum

Required

T_{eff}, log g, R, Y_{He}, Mdot, v_∞, β (+ metal abundances)
(R stellar radius at $\tau_R = 2/3$)

also necessary

v_{rad} (radial velocity)
v sin i (projected rotational velocity)

Given

- *reduced* optical spectra (eventually +UV, +IR, +X-ray)
- λ/Δλ, resolution of observed spectrum
- Visual brightness V
- distance d (from cluster/association membership), partly rather insecure
- NLTE-code(s), "model grid"

1. Rectify spectrum, i.e. divide by continuum (experience required)

2. Shift observed spectrum to lab wavelengths (use narrow **stellar** lines as reference):

$$\lambda_{\rm lab} \approx \lambda_{\rm obs}\left(1 - \frac{v_{\rm rad}}{c}\right), \quad v_{\rm rad} \text{ assumed as positive if object moves away from observer}$$

- **Alternative set of parameters**

  L, M, R *or*
  L, M, T_{eff} *or*
  T_{eff}, log g, R …

- **interrelations**

$$L = 4\pi R_*^2 \sigma_B T_{\rm eff}^4$$

$$g = \frac{GM}{R_*^2}$$

- **Useful scaling relations**
  If L, M, R in *solar units*, then

$$R_* = \frac{L^{0.5}}{T_{\rm eff}^2} \cdot 3.327 \cdot 10^7$$

$$\log g = \log\left(\frac{M}{R_*^2} \cdot 2.74 \cdot 10^4\right)$$

$$v_{\rm esc} = \sqrt{R_* g (1-\Gamma) \cdot 1.392 \cdot 10^{11}}$$

$$\Gamma = s_e T_{\rm eff}^4 / g \cdot 1.8913 \cdot 10^{-15}$$

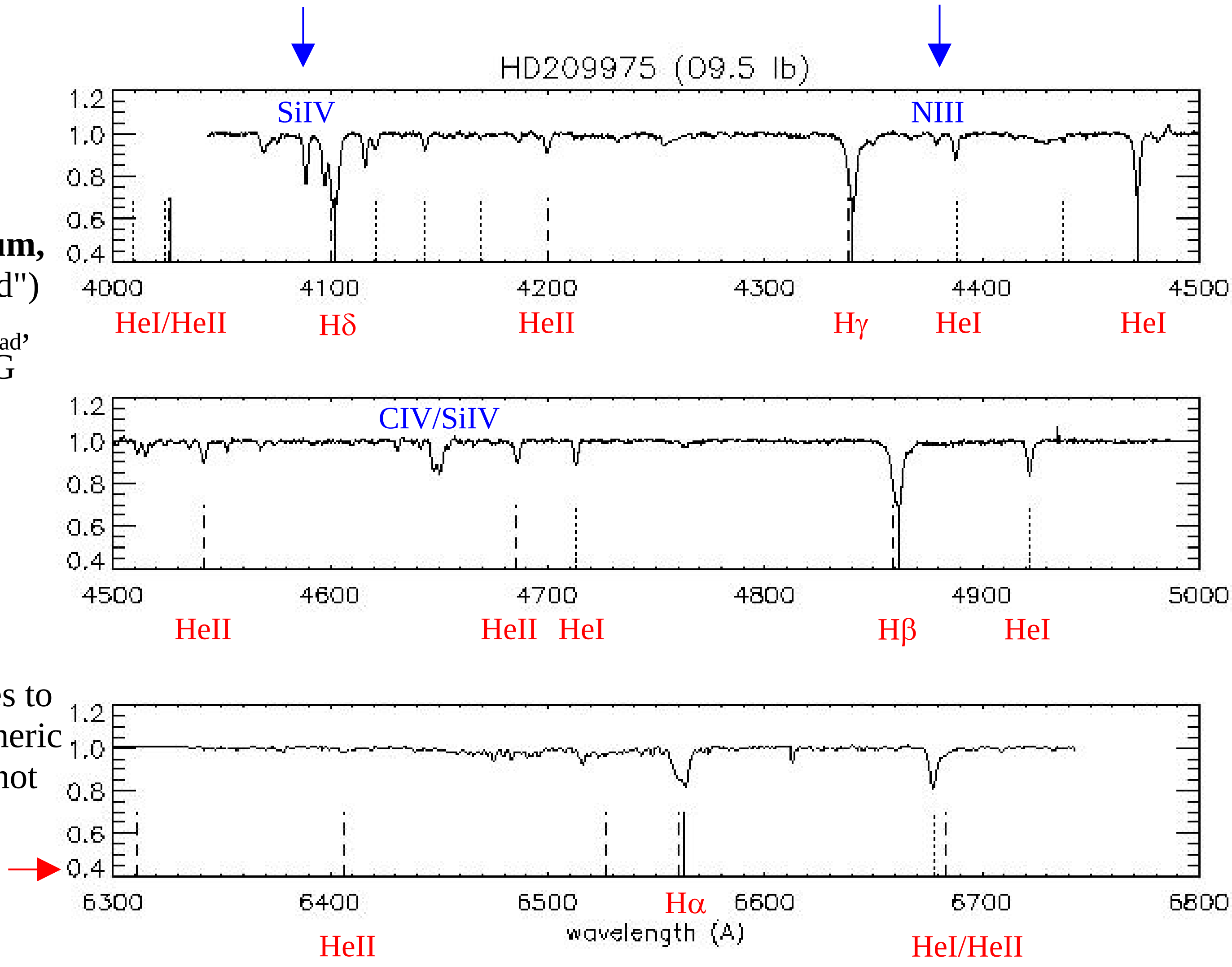
$$s_e = 0.4 \frac{1 + I_{\rm He} Y_{\rm He}}{1 + 4 Y_{\rm He}}, \quad \text{cf. page 90}$$

with I_{He} number of free electrons per Helium atom
(e.g.,=2, if completely ionized)

rectified **optical spectrum,** ("blue" and "red") corrected for v_{rad}, of the late O-SG *19 Cep*

___ Hydrogen
...... Helium I
- - - Helium II

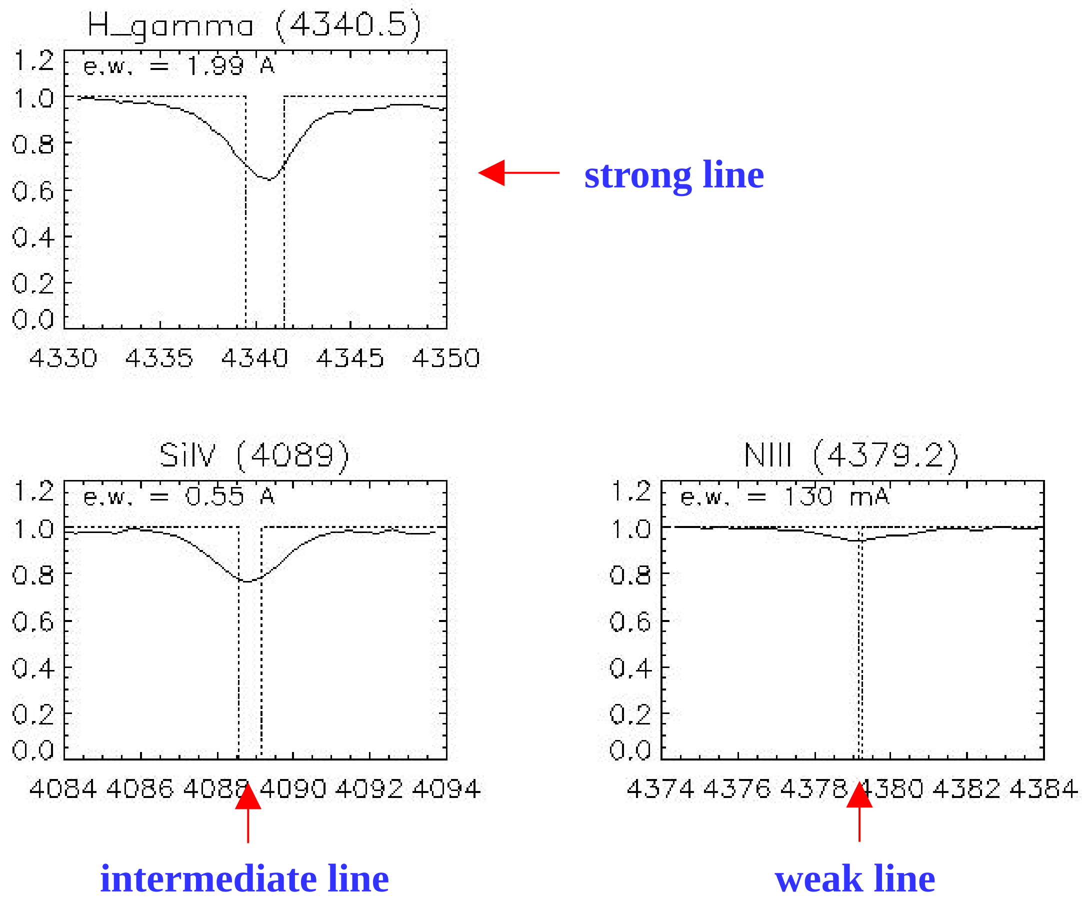
in "red":
"strategic" lines to derive atmospheric parameters in hot stars



remember equivalent width $W_\lambda = \int\limits_{\text{line}} \frac{H_{\text{cont}} - H_{\text{line}}(\lambda)}{H_{\text{cont}}} d\lambda = \int\limits_{\text{line}} \left(1 - R(\lambda)\right) d\lambda,$

area of profile under continuum, $\dim[W_\lambda]$ = Angstrom or milliAngstrom, mÅ

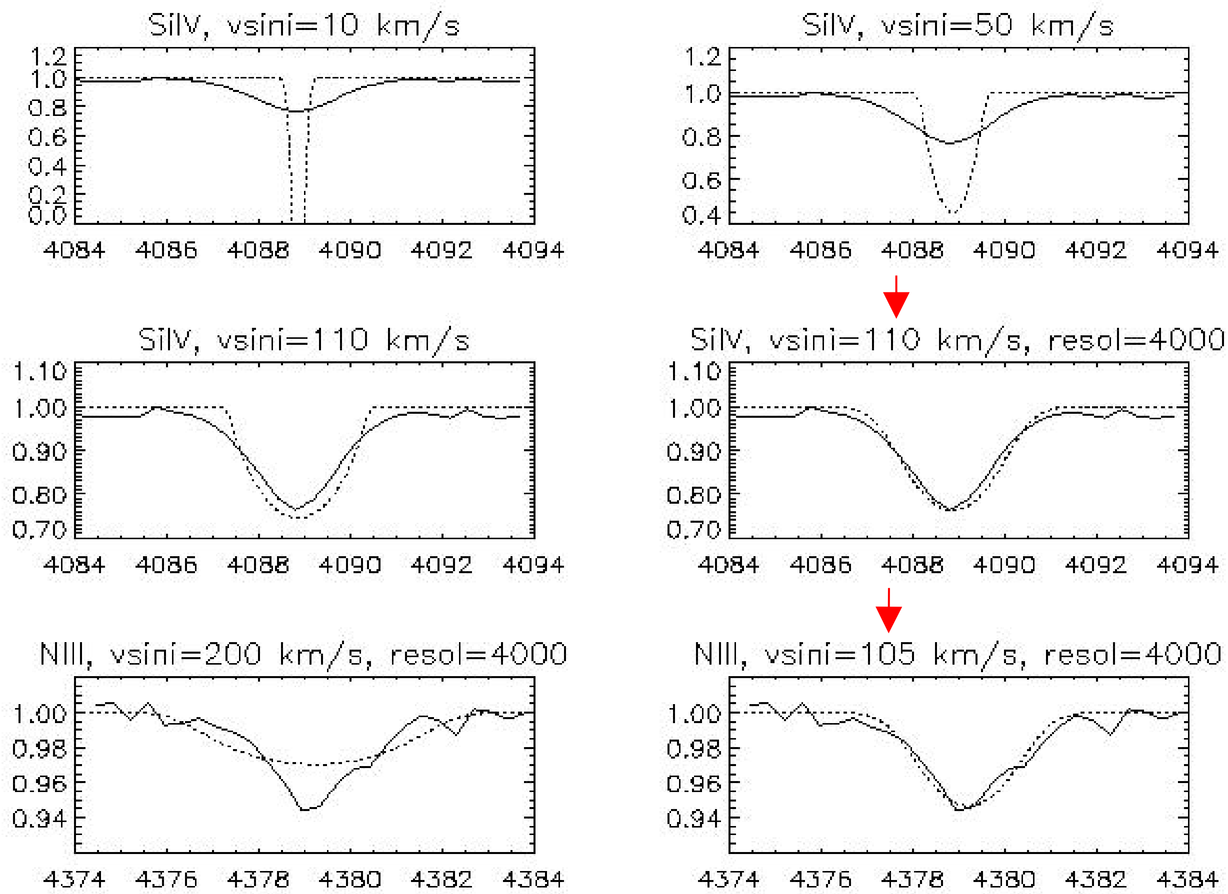
corresponds to width of saturated profile $(R(\lambda) = 0)$ with same area



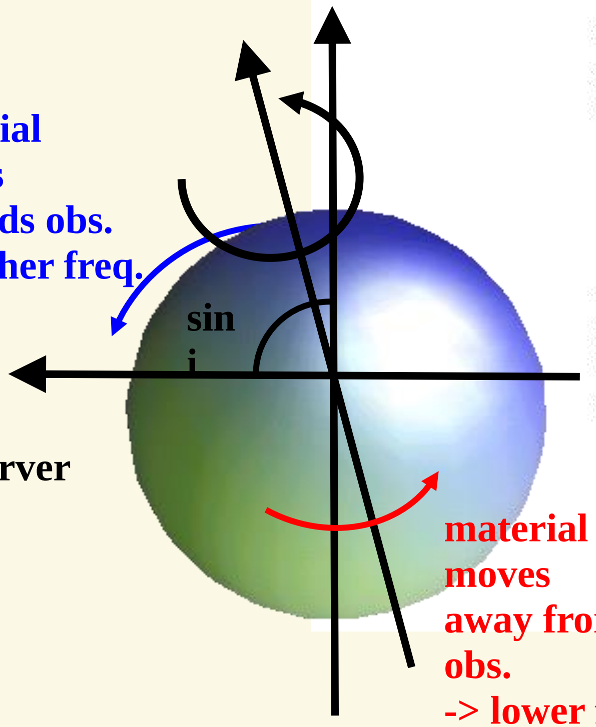
Determine projected rotational speed v sin i

Use weak metal lines
to derive v sin i:
Convolve theoretical line
with rotational profile.

Convolve finally with
instrumental profile
(~ Gauss) according
to spectral resolution



material moves towards obs. -> higher freq.

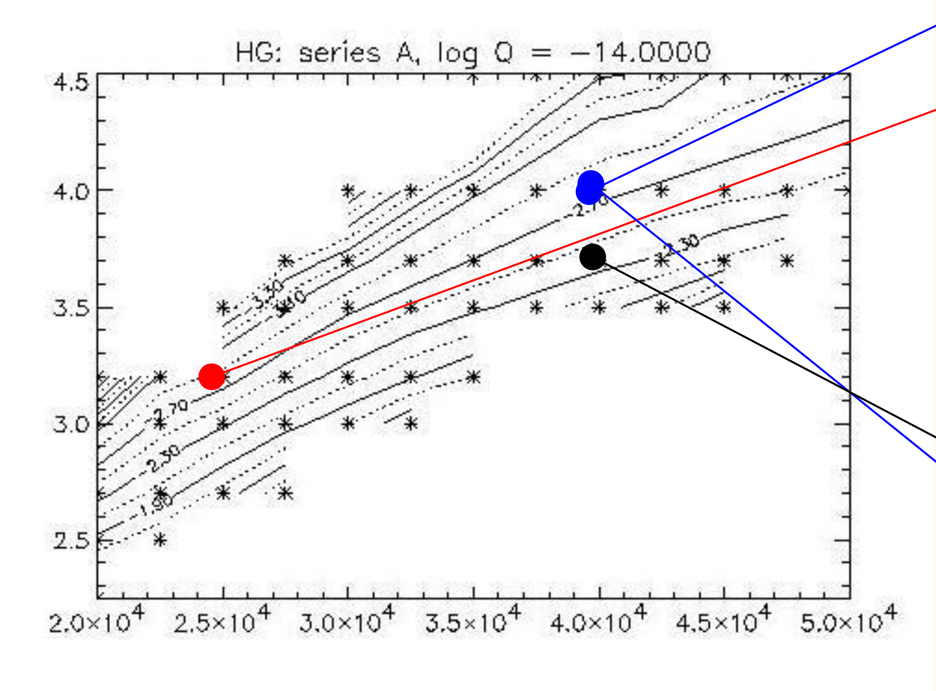


to observer

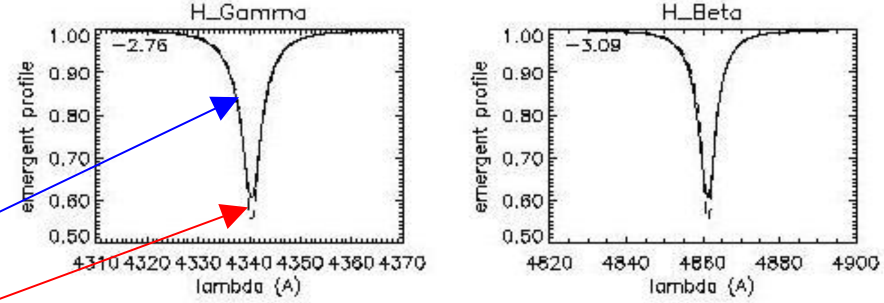
material moves away from obs. -> lower freq.

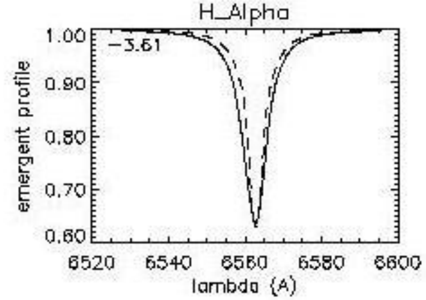
Convolution with rotational and instrumental profile conserves equivalent width!!!
Recent methods use a Fourier technique to infer vsini

Hγ - log g and T_{eff}

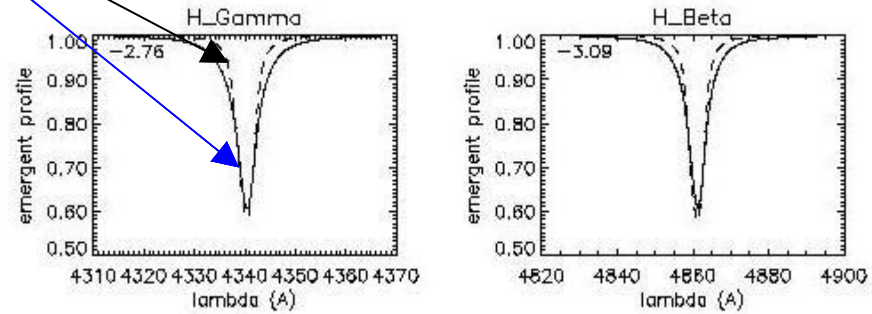


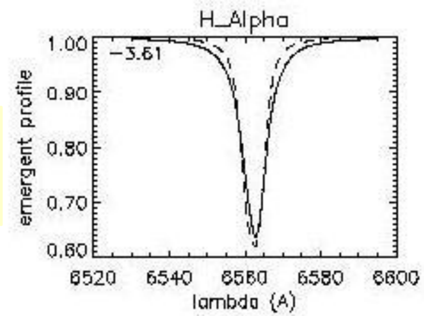
Iso-contours of equiv. widths for Hγ (from model grid), for solar Helium abundance and (very) thin winds





degeneracy of profiles:
(almost) identical lines for
T_{eff} =40,000 and log g=4.0
and
T_{eff} =25,000 and log g=3.2





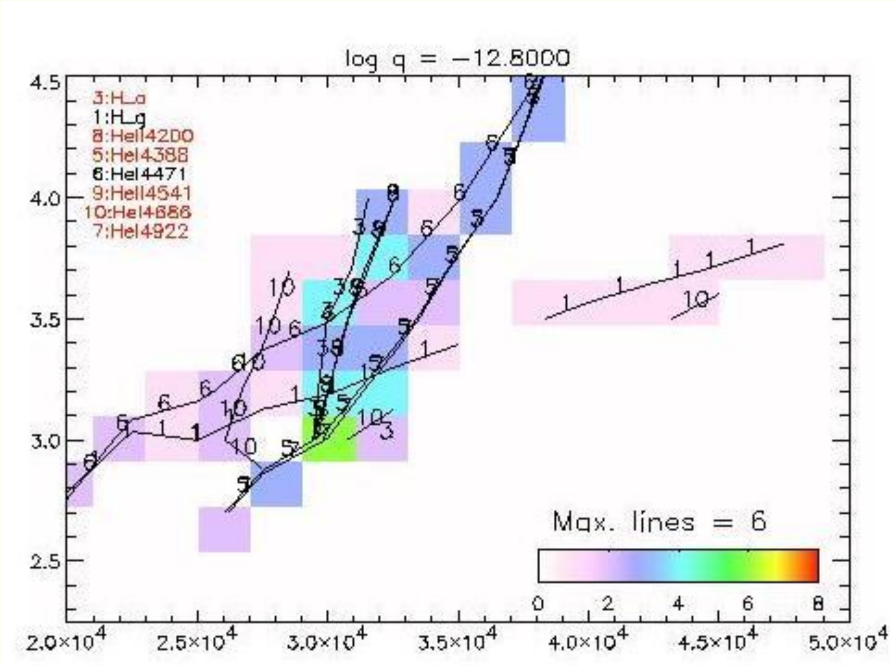
wings of Balmer lines
(Stark-broadened)
react strongly on electron-density (as a function of τ)
=> perfect gravity indicator

to derive T_{eff}, log g and Y_{He},
at least 3 lines have to be fitted in parallel
(if no wind is present):

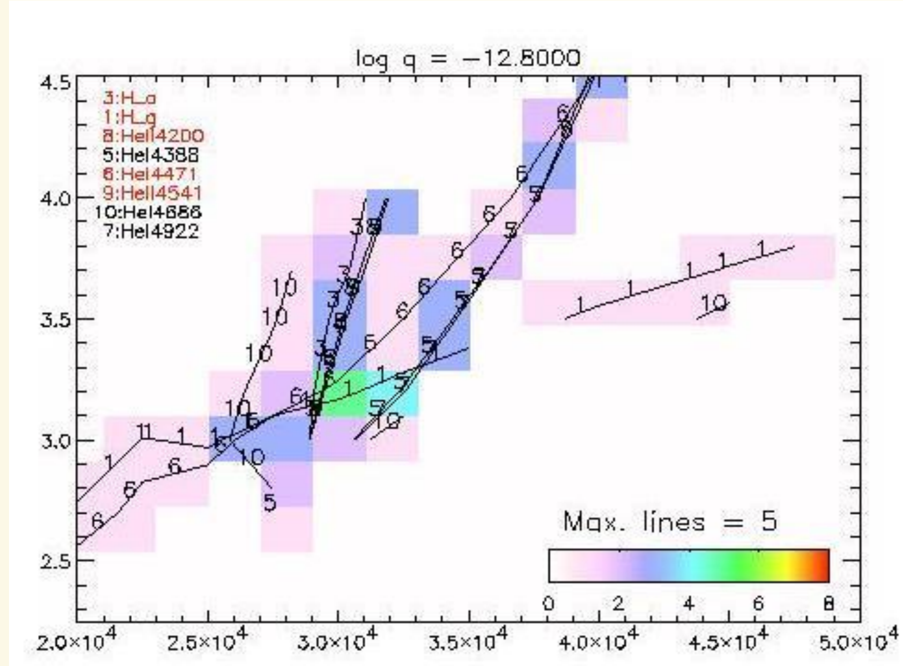
Hγ defines **log g** (for given T_{eff})
HeII/HeI define $\mathbf{T_{eff}}$ (for given log g)
absolute **strength of He lines** define $\mathbf{Y_{He}}$

usually, wind emission has to be accounted for (profiles shallower)

Coarse fit - analysis of equivalent widths



Fit diagram for Y_{He} = 0.1 (best fit at log Q = -12.8)



Fit diagram for Y_{He} = 0.15 (best fit at log Q = -12.8)

Measured equivalent widths

Balmer lines		**HeI**		**HeII**	
Hγ	1.99	4387	0.32	4200	0.25
Hα	1.33	4471	0.86	4541	0.31
		4922	0.46	4686	0.27

Note: Hα and HeII 4686 mass-loss indicators

Result: $T_{eff} \approx$ 30,000 K, log g ≈ 3.0 ...3.2, $Y_{He} \approx$ 0.10 ... 0.15, log Q ≈ -12.8

Fit diagram constructed from model grid with

$$20{,}000 \text{ K} < T_{eff} < 50{,}000 \text{ K with } \Delta T = 2{,}500 \text{ K}$$

$$2.2 < \log g < 4.5 \text{ with } \Delta\log g = 0.25$$

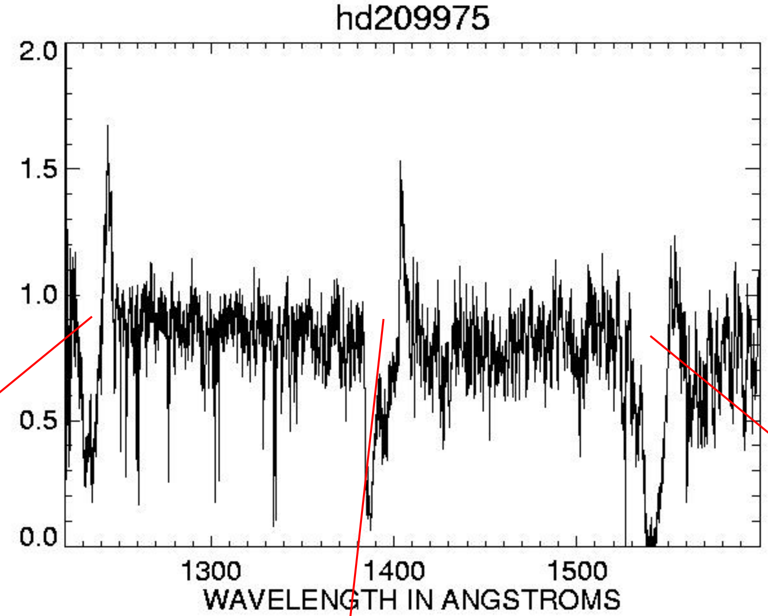
$$-14 < \log Q < -11 \text{ with } \Delta\log Q = 0.3,\ Y_{He} = 0.10, 0.15, 0.20$$

Note: Wind parameters can be cast into one quantity

$$\log Q = \frac{\dot{M}}{(R_* v_\infty)^{1.5}}$$

For same values of Q (albeit different combinations of Mdot, v_∞ and R_*), profiles look almost identical!

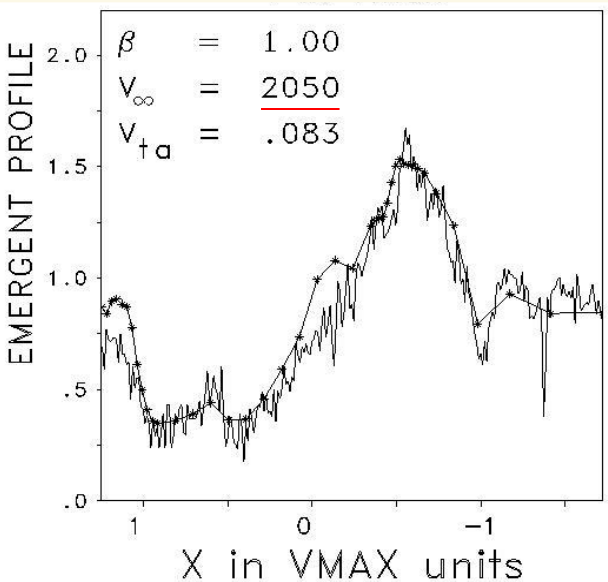
Determination of terminal velocity from UV-P Cygni profiles



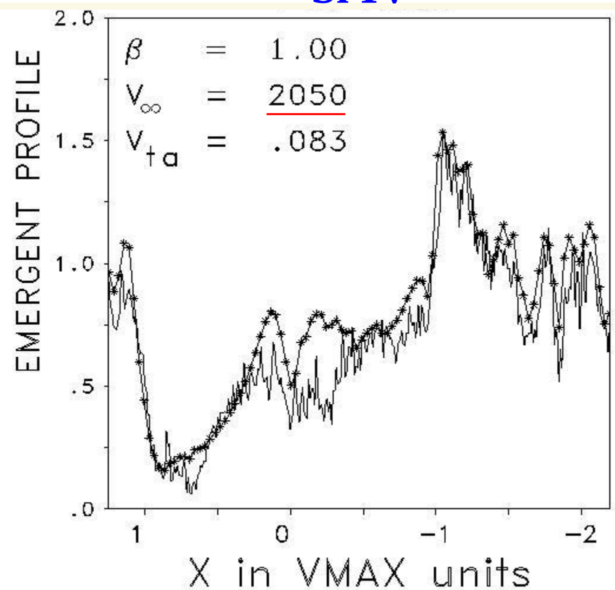
observation with IUE (International Ultra- violet Explorer)
no longer active

recent data (archive!) from HST (λ >1200 Å) and FUSE (λ > 911 Å)

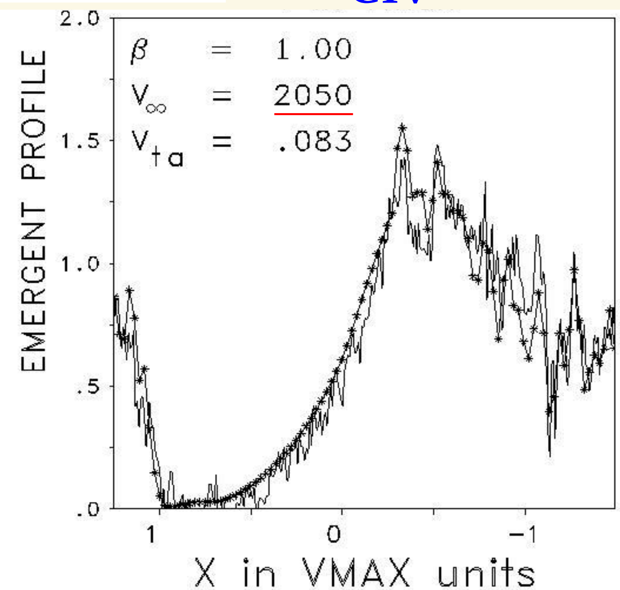
NV



Si IV



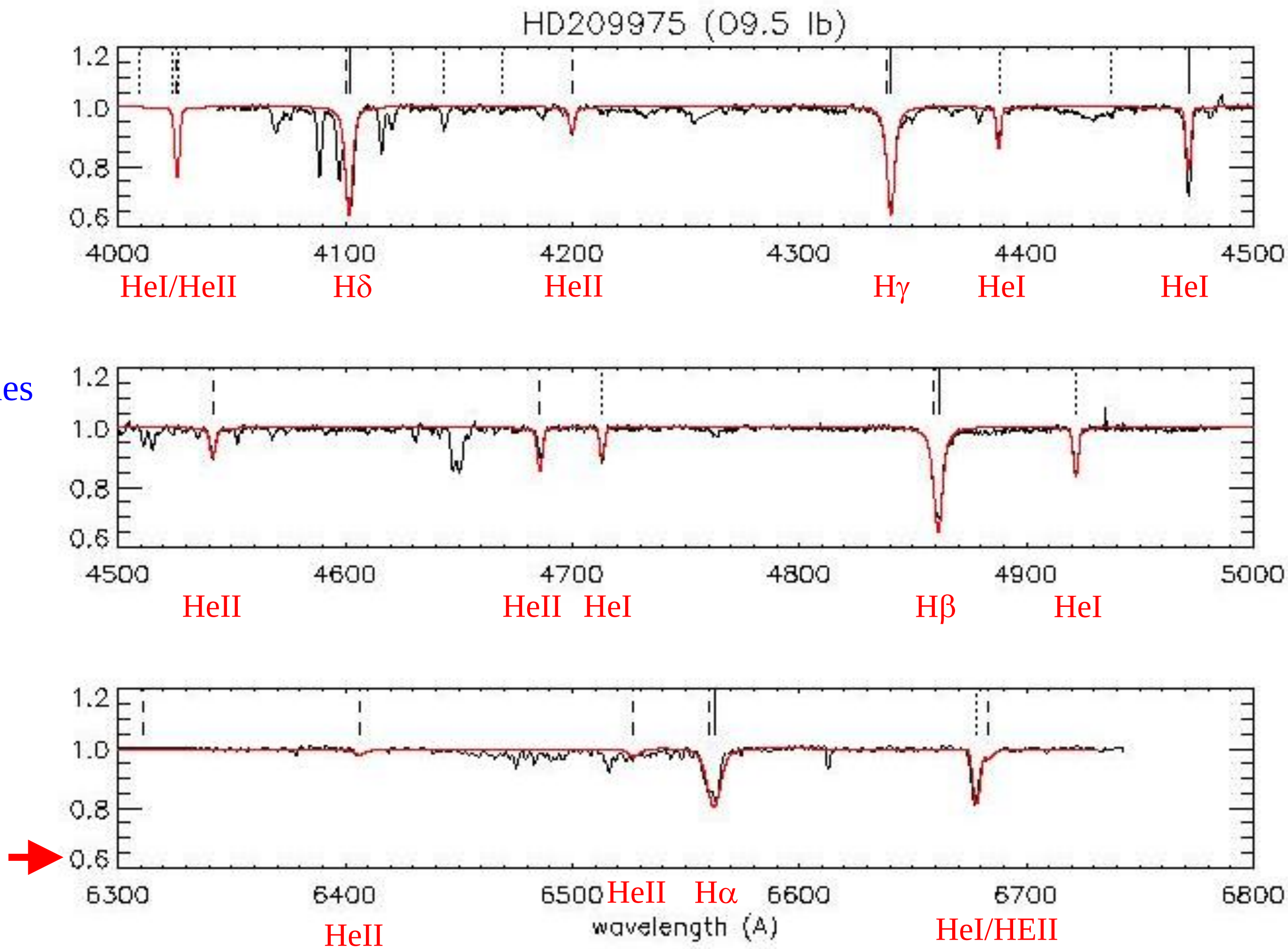
CIV



Fine fit - detailed comparison of line profiles

___ Hydrogen
...... Helium I
- - -Helium II

indicated lines used for fits



NOTE: meanwhile, (semi-) automatic analysis methods available; based on
– model grids or
– genetic algorithm
to optimize the fit to a multitude of lines

derived parameters

T_{eff} = 31,000 K
log g = 3.17
log Q = -12.87
Y_{He} = 0.10
β = 1.0

with v_∞ = 2050 km/ s we have

$$\log(\dot{M}/R_*^{1.5}) = -7.9$$

Determination of stellar radius - if it cannot be resolved

- **IF** *you believe in stellar evolution*
  - use **evolutionary tracks** to derive M from (measured) T_{eff} and log g **=> R**
  - transformation of conventional HRD into log T_{eff} - log g diagram required
  - problematic for evolved massive objects, "mass discrepancy": spectroscopic masses (see below) and evolutionary masses not consistent, inclusion of rotation into stellar models improves situation (but does not solve it)

- **IF** *you know the distance* **and** *have theoretical fluxes* (from model atmospheres), proceed as follows

- **IF** *you believe in radiation driven wind theory*
  - use **wind-momentum luminosity relation**

$$V = -2.5 \log \int_{\text{filter}} \mathcal{F}_\lambda S_\lambda d\lambda + \text{const}$$

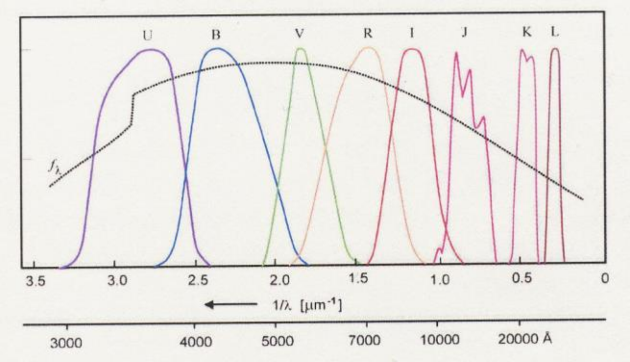
S_λ spectral response of photometric system

absolute flux calibration

$V = 0$ corresponds to $\mathcal{F}_\lambda = 3.66 \cdot 10^{-9}$ erg s^{-1} cm^{-2} Å^{-1} at $\lambda_0 = 5{,}500$ Å outside earth's atmosphere

λ_0 *isophotal* wavelength such that $\int_{\text{filter}} \mathcal{F}_\lambda S_\lambda d\lambda \approx \mathcal{F}(\lambda_0) \int_{\text{filter}} S_\lambda d\lambda$, $\int_{\text{filter}} S_\lambda d\lambda \approx 2895$ for Johnson V-filter

$\Rightarrow$



$$\text{const} = -2.5 \log(3.66 \cdot 10^{-9} \cdot 2895) = -12.437$$

$$M_V = -2.5 \log \left[\left(\frac{R_* R_{\text{sun}}}{10 \text{ pc}} \right)^2 \int_{\text{filter}} \mathcal{F}_\lambda S_\lambda d\lambda \right] + \text{const}$$

$$5 \log R_* = 29.553 + (V_{\text{theo}} - M_V)$$

if R_* in solar units, M_V the absolute visual brightness (dereddened!) and

$V_{\text{theo}} - 2.5 \log \int_{\text{filter}} 4 H_\lambda S_\lambda d\lambda$ with H_λ the *theoretical* Eddington flux in units of [erg s^{-1} cm^{-2} Å^{-1}]

Alternatively, use bolometric correction (BC)

Calibration for Galactic O-stars:

$$BC = M_{\mathrm{Bol}} - M_V \approx 27.58 - 6.8 \log(T_{\mathrm{eff}}) \quad \text{(see Martins et al. 2005, A\&A 436)}$$

and definition of M_{Bol}

$$\log \frac{L}{L_\odot} = 4 \log \frac{T_{\mathrm{eff}}}{T_{\mathrm{eff},\odot}} + 2 \log \frac{R_*}{R_\odot} = 0.4(M_{\mathrm{Bol},\odot} - M_{\mathrm{Bol}})$$

$$\log \frac{R_*}{R_\odot} = 0.2(4.74 - M_{\mathrm{Bol}}) - 2 \log \frac{T_{\mathrm{eff}}}{5770} =$$

$$= 0.2(4.74 - M_{\mathrm{V}} - 27.58 + 6.8 \log(T_{\mathrm{eff}})) - 2 \log \frac{T_{\mathrm{eff}}}{5770} =$$

$$= 2.954 - 0.2 M_{\mathrm{V}} - 0.64 \log(T_{\mathrm{eff}})$$

[valid only for O-stars with $Z \approx Z_\odot$]

remember relation between M_V and V (distance modulus)

$\boxed{M_V = V + 5(1 - \log d) - A_V}$, d distance in pc, A_V reddening

d from parallaxes (if close) or cluster/ association/ galaxy membership (hot stars)
(note: clusters/ assoc. radially extended!)

For Galactic objects, use compilation by
Roberta Humphreys, 1978, ApJS 38, 309 *and/or*
Ian Howarth & Raman Prinja, 1989, ApJS 69, 527

Back to our example

HD 209975 (19 Cep): M_v = -5.7
check: belongs to Cep OB2 Assoc., d ≈ 0.83 kpc
V = 5.11, A_V = 1.17 => M_V = -5.65, OK

From our final model, we calculate V_{theo} = -29.08 => R = 17.4 R_{sun}
(Alternatively, by using BC, M_V and T_{eff} = 31kK, we would obtain R = 16.6 R_{sun})

Finally, from the result of our fine fit, $\log(\dot{M}/R_*^{1.5}) = -7.9$, we find $\dot{M} = 0.91 \cdot 10^{-6}\ M_{sun}$ / yr

Finished, **determine metal abundances if required,**
next star ....

… but end of lecture!!!