Radiative processes, stellar atmospheres and winds (WS 2017/2018)

Problem set 9

Problem 1 [3.5 points] *Lorentz profile*

In an excited atom, energy is radiated away by an electromagnetic wave (spontaneous decay). In a classical picture, one can describe the excited electron by a damped harmonic oscillator with eigenfrequency ω_0, where a damping force is exerted on the electron by its own electromagnetic field, $F_{\mathrm{damp}} = -m\Gamma v$, with damping constant Γ and velocity $v = \dot{x}$. The corresponding equation of motion is then given by

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = 0.$$

With the Ansatz $x = \mathrm{Re}(x_o \exp(i\omega t))$ and noting that $\Gamma \ll \omega_o$ (see problem 2), the solution can be approximated by

$$x(t) \approx x_0 \exp(-(\Gamma/2)t)\cos(\omega_0 t).$$

a) Because of the decaying amplitude, the frequency of the radiated wave is no longer monochromatic as for an infinite oscillation. Calculate the frequency spectrum $H(\omega)$ from a Fourier transform of the oscillation, assuming that the decay begins at $t = 0$,

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x(t)\exp(-i\omega t)dt$$

Your solution should involve two terms, one dependent on $(\omega - \omega_0)$ and another one dependent on $(\omega + \omega_0)$. In the following, we will study the behaviour close to the resonance frequency, i.e., for $(\omega - \omega_0) \ll \omega_0$. In this case, the second term involving $(\omega + \omega_0)$ can be neglected, and you should have obtained

$$H(\omega) \approx \frac{x_0}{\sqrt{8\pi}} \frac{1}{\Gamma/2 + i(\omega - \omega_0)}$$

b) The radiated spectral power, $P(\omega)$, results from a multiplication with the complex conjugate of $H(\omega)$,

$$P(\omega) = H(\omega)H^*(\omega),$$

and $P(\omega)$ is proportional to the line profile. Convert ω and ω_0 to ν and ν_0, and show by normalizing

$$\int_0^\infty \mathrm{const}\, P(\nu)d\nu = 1$$

that the line profile due to radiation damping (natural line-broadening) is given by the Lorentz profile quoted in the script (page 103). Summarize the assumptions regarding the validity of this expression.
Hint: At some point in your calculation, you might approximate $-\nu_0$ by $-\infty$.

Problem 2 [4 points] *Natural line width*

One can show (e.g., by using the equation of motion), that the radiated power from an excited atom decays with $\exp(-\Gamma t)$.

a) Interpret this as a probability distribution function, $p(t) \propto \exp(-\Gamma t)$, perform the missing normalization ($\int p(t)dt = 1$), and show that the mean life time of an excited atom, $\tau = <t> = \int tp(t)dt = 1/\Gamma$.

b) Convince yourself that $\Gamma \ll \omega_0$ under typical conditions (see Problem 1).

c) Assume a transition between two excited states with energies E_i and E_f, and mean life times τ_i and τ_f. Calculate the corresponding line-width (with respect to frequency and wavelength) from the uncertainty principle.

d) Compare the result from 2c) with the full-width at half maximum from a corresponding Lorentz profile.

e) Calculate the natural line-width (see 2c/d) for the Balmer-α transition of hydrogen in units of Å, assuming $\tau_{n=2} = \tau_{n=3} = 10^{-8}$s.

Problem 3 [4.5 points] *Doppler broadening*

For the following problem and nomenclature, see script page 104.

In order to account for the thermal velocities of the radiating atoms, we have to convolve the 'atomic' profile function with the corresponding velocity distribution, $P(v_x, v_y, v_z)$ (Dopplershifts!). Thus, if the emission is isotropic, we need to evaluate

$$\Phi(\nu) = \int\int\int P(v_x, v_y, v_z)\phi(\nu' - \nu_0)dv_x, dv_y, dv_z.$$

$\Phi(\nu)$ is the resulting profile function at observer's (rest) frequency ν, and ϕ is the intrinsic ('atomic') profile in dependence of $(\nu' - \nu_0)$, with $\nu' = \nu'(\nu, \vec{n} \cdot \vec{v})$ the frequency in the atomic frame and ν_0 the transition frequency.

a) Derive the equation for $\Phi(\nu)$ as quoted on page 104.

   Hint: Assume (without loss of generality) that the x-axis of the $\vec{v}$ coordinate system is aligned with $\vec{n}$ (direction from atom towards observer), and that in this geometry only the v_x components contribute to the Dopplershifts (no transversal Dopplershift, because $v \ll c$).

b) Assume that the intrinsic profile, $\phi(\nu' - \nu_0)$, is a delta function, and derive the Doppler-profile quoted on page 105.

c) Assume now that the intrinsic profile, $\phi(\nu' - \nu_0)$, is a Lorentzian, and derive the Voigt-profile quoted on page 105.

d) Compare the natural line-width from problem 2e) (actually, half of this width) with the corresponding thermal Dopplerwidth, $\Delta\lambda_{\rm D}$ (also in Å), assuming a thermal speed of 10 km/s. In view of this result, interpret the parameter a appearing in the Voigt-profile.

Have fun, and much success!