Radiative processes, stellar atmospheres and winds (WS 2017)

Problem set 4

Problem 1 [9 points] *Radiation pressure*

a) Prove that the components of the radiation pressure tensor (with respect to Cartesian or spherical coordinates) for the general case, i.e., when the specific intensity depends both on θ and ϕ, can be expressed by the following relations (index '1,2,3' refering to x, y, z or Θ, Φ, r):

$$
\begin{aligned}
P_{11} &= \frac{1}{c}\int_0^{2\pi} \cos^2\phi d\phi \int_{-1}^{+1} (1-\mu^2) I_\nu(\vec{r},\mu,\phi,t) d\mu \\
P_{22} &= \frac{1}{c}\int_0^{2\pi} \sin^2\phi d\phi \int_{-1}^{+1} (1-\mu^2) I_\nu(\vec{r},\mu,\phi,t) d\mu \\
P_{33} &= \frac{1}{c}\int_0^{2\pi} d\phi \int_{-1}^{+1} \mu^2 I_\nu(\vec{r},\mu,\phi,t) d\mu \\
P_{12} = P_{21} &= \frac{1}{c}\int_0^{2\pi} \sin\phi\cos\phi d\phi \int_{-1}^{+1} (1-\mu^2) I_\nu(\vec{r},\mu,\phi,t) d\mu \\
P_{13} = P_{31} &= \frac{1}{c}\int_0^{2\pi} \cos\phi d\phi \int_{-1}^{+1} (1-\mu^2)^{\frac{1}{2}} \mu I_\nu(\vec{r},\mu,\phi,t) d\mu \\
P_{23} = P_{32} &= \frac{1}{c}\int_0^{2\pi} \sin\phi d\phi \int_{-1}^{+1} (1-\mu^2)^{\frac{1}{2}} \mu I_\nu(\vec{r},\mu,\phi,t) d\mu
\end{aligned}
$$

b) Prove the result provided on page 64 of the script (bottom of left part) for the radiation pressure tensor in plane-parallel/spherical symmetry.

c) Show that the radiation pressure tensor is not only isotropic for an isotropic radiation field, but also if the angular dependence of the radiation field is given by $I(\mu) = I_0 + I_1\mu$. This result is important because such a description of the radiation field is appropriate in the so-called diffusion limit, reached at large depths in the atmosphere.

d) Show that for an isotropic radiation field in thermal equilibrium, the total (i.e., frequency integrated) radiation pressure scalar is given by

$$p_R^*(r,t) = \frac{1}{3} a T^4(r)$$

where a is the so-called radiation constant. How does a relate to the Stefan-Boltzmann constant, σ_B?

e) The most extreme case of an anisotropic radiation field is associated with a planar wave, where we can write (see exercise 3.1) $I_\nu(\mu, \phi) = I_0 \delta(\mu - \mu_0)\delta(\phi - \phi_0)$. Calculate the individual components of the radiation pressure tensor, for arbitrary μ_0 ($-1 < \mu_0 < 1$) and ϕ_0 ($0 \le \phi_0 < 2\pi$). Show that the radiation pressure tensor (expressed w.r.t. Cartesian coordinates) has only one non-vanishing component when the wave moves along one of the coordinate axes (for all three axes).

f) An extreme anisotropy of the radiation field is reached far away from a star, e.g., in a nebula, when the radiation field originates from a stellar surface that occupies only a very small solid angle. To show this explicitly, suppose on observer located at distance r from the center of a star with radius R_*. Assume that the (spherically symmetric) star has a uniformly bright surface, i.e., I shall be independent of μ, and that there is negligible absorption between stellar surface and observer. Show that the corresponding (half) opening angle of the star as seen from the observer, μ_*, is (exactly) given by

$$\mu_* = \left(1 - \left(\frac{R_*}{r}\right)^2\right)^{1/2},$$

and that all moments of the radiation field, $1/2 \int_{-1}^{1} I \mu^n \mathrm{d}\mu$, with $n = 0, 1, 2, \ldots$ become almost identical for $r \gg R_*$. Prove that in this situation the radiation pressure tensor has only one non-vanishing component w.r.t. spherical coordinates.

Problem 2 [3 points] *Solar sail*

a) A 10-tons spacecraft is launched from Earth, and is to be accelerated radially away from the sun by using a circular solar sail, where the weight of the sail is included in the above mass. The initial acceleration of the spacecraft shall be 1 g (when it is roughly at 1 AU away from the sun). Assuming a flat sail, determine the required radius of the sail if it is black, so that it absorbs all of the sun's light (for a reflective sail, the force becomes a factor of two larger).

   Hint(1): In view of the results from exercise 1f), the force, F, on the sail can be computed, as in hydrostatics, from $F = pA$, with pressure p and area A. Show that the radiative acceleration is given by $g_{\mathrm{rad}} = (f_\odot/c)\, A/m$, when $f_\odot$ is the solar irradiance.

   Hint(2): The spacecraft, like the earth, is orbiting the sun. Should you include the sun's gravity in your calculations?

b) Derive the radial dependence of the radiative acceleration (initially 1 g) as the spacecraft propagates away from the earth and sun.

Have fun, and much success!