Radiative processes, stellar atmospheres and winds (WS 2017/2018)

LAST problem set 10

Problem 1 [4 points] *Electrons in the solar photosphere*

According to the Holweger-Müller model of the solar photosphere, at $\tau_{5000} = 0.04$ (the optical depth at 5,000 Å) there is a temperature of 5,000 K, and a gas pressure of $2.63 \cdot 10^4$ dyn/cm^2.

Adopt LTE conditions, and calculate the corresponding electron-density n_e, by assuming that the photosphere consists of hydrogen and helium only ($\mathrm{N_{He}/N_H} = 0.1$), and that Helium is completely neutral.

Compare your result with the corresponding electron pressure of 2.54 dyn/cm^2 from the Holweger-Müller model, and try to explain the discrepancy.

Problem 2 [2.5 points] *Approximate rate equations*

NOTE: The following nomenclature refers to the script, page 119/120.

Some definitions. To quantify NLTE-effects, one introduces the so-called NLTE departure coefficient,

$$b_i = \frac{n_i}{n_i^*} \tag{1}$$

where n_i is the actual, NLTE population density of a level i (ion j), and n_i^* the corresponding LTE population, calculated from the Saha-Boltzmann equation but using the actual NLTE ion- and electron densities, n_k (ground-state of ion $k = j+1$) and n_e, respectively:

$$n_i^* = n_k \Big(\frac{n_i}{n_k}\Big)^* = n_k\, n_\mathrm{e} C_\mathrm{I} \frac{g_i}{g_k} T_\mathrm{e}^{-3/2} \exp(E_\mathrm{ion,i}/(kT_\mathrm{e})) \tag{2}$$

(cf. page 113, combining the Saha equation with Boltzmann excitation, and defining $E_\mathrm{ion,i}$ as the ionization energy from level i). When $b_i > 1$, the level is 'overpopulated', and 'underpopulated' vice versa.

The radiative bound-free and free-bound rates (ionisation and recombination) are given by

$$n_i R_{ik} = n_i 4\pi \int_{\nu_0}^{\infty} \frac{\alpha_\nu}{h\nu} J_\nu d\nu \tag{3}$$

with ionisation frequency (at ionisation threshold) ν_0 and ionisation cross-section α_ν, and by

$$n_k R_{ki} = n_k \Big(\frac{n_i}{n_k}\Big)^* 4\pi \int_{\nu_0}^{\infty} \frac{\alpha_\nu}{h\nu}\Big(\frac{2h\nu^3}{c^2} + J_\nu\Big) \exp(-h\nu/(kT_\mathrm{e})) d\nu = n_i^* 4\pi \int_{\nu_0}^{\infty} (\ldots) d\nu \tag{4}$$

a) Detailed balance in the resonance lines of a stellar wind

To estimate the occupation numbers (particularly, the ground-state) of an ion in the supersonic part of an expanding hot-star atmosphere (wind), one might apply the following approximations

(i) Because of the low densities, all collisional rates can be neglected.

(ii) The resonance lines (radiative transitions connected to the ground state) are optically thick throughout the wind, and the corresponding radiative bound-bound rates (upwards and downwards) cancel each other, i.e., $n_1 R_{1j} = n_j R_{j1}$. In other words, these rates do not appear in the rate equations.

Write down the corresponding approximate rate equations for an ion with four bound levels, in the form

$$\text{matrix} \cdot (n_1, n_2, n_3, n_4)^T = \vec{b},$$

assuming that the ground-state population of the next higher ion, n_k, is known, and that $\vec{b}$ is a vector containing all rates proportional to n_k.

Convince yourself that the ground-state population of the considered ion, n_1, is exclusively controlled by ground-state ionisation and recombination,

$$n_1 R_{1k} = n_k R_{k1} \tag{5}$$

b) Nebular approximation

The situation in a Planetary Nebula or an HII region illuminated by a hot star is similar to the conditions from a), except that because of the much lower densities the radiative bound-bound rates for the resonance lines do no longer cancel each other, and that *generally* (i.e., for all lines) only the spontaneous emission terms 'survive'. With respect to page 119/120, in this situation we then have

$$n_i R_{ij} \to 0, \text{and } n_j R_{ji} \to n_j A_{ji}, \tag{6}$$

with A_{ji} the Einstein coefficient for spontaneous decay. Moreover, the ionization rates, $n_i R_{ik}$, can be neglected for all *excited* levels, because of the very small dilution factor (sizes of order 0.1 to 1 pc for PNe, and 10 to 100 pc for HII regions).

Formulate the corresponding approximate rate equations similar to problem 2a), and compare the structure of both systems.

Problem 3 [5.5 points] *Ground-state population in expanding atmospheres*

By means of Eq. 5, the ground-state departure coefficient can be approximated by

$$b_1 \approx \frac{1}{W} \frac{T_\mathrm{e}}{T_\mathrm{rad}} \exp\left[-\frac{h\nu_0}{k}\left(\frac{1}{T_\mathrm{e}} - \frac{1}{T_\mathrm{rad}}\right)\right], \tag{7}$$

where b_1, W and T_e are depth dependent quantities. W is the dilution factor, and T_rad the radiation temperature at the ionisation threshold, such that $J_{\nu_0} = W B_{\nu_0}(T_\mathrm{rad})$ in the wind.

a) Convince yourself that $h\nu_0/(kT) \gg 1$ for typical hot star conditions, such that (i) one can approximate

$$\frac{1}{\exp\left(h\nu_0/(kT)\right) - 1} \approx \exp\left(-h\nu_0/(kT)\right),$$

and (ii) the term for stimulated emission in Eq. 4, $\propto J_\nu$, can be neglected compared to the term $\propto 2h\nu^3/c^2$, so that the recombination rate can be approximated by

$$n_k R_{ki} \approx n_i^* 4\pi \int_{\nu_0}^{\infty} \frac{\alpha_\nu}{h\nu} \frac{2h\nu^3}{c^2} \exp(-h\nu/(kT_\mathrm{e}))d\nu$$

b) Prove Eq. 7 (using all related approximations outlined so far), by assuming that the ionisation cross-section can be described by

$$\alpha_\nu = \alpha_0 \left(\frac{\nu_0}{\nu}\right)^2,$$

(which is true for most transitions), and that T_rad does not vary close to the ionization edge.

c) Show that deep in the atmosphere $b_1 \to 1$, i.e., that Eq. 7 recovers the appropriate LTE limit.

d) By using Eq. 7 and the definition of the departure coefficient, calculate the approximate NLTE ground-state occupation number, $n_1(r)$.

What is the *basic* difference between the run of the LTE occupation number, $n_1^*(r)$, and the corresponding NLTE one, $n_1(r)$? In particular, consider a region far away from the stellar surface, $r \gg R_*$. Hint: Compare the ratios n_1^*/n_k and n_1/n_k.

Have fun, and much success!