Statistical methods – an introduction (SS 2016)

Problem set 8

1. In this problem set, the (measurement) error Δx of a quantity x is, as usual, an abbreviation for σ_x, i.e., if the actual measurement is $x = x_m + \delta x$, then $E(x) = x_m$, and $\Delta x = \sqrt{Var(x)} = \sqrt{E\{(\delta x)^2\}}$.

2. For calculating variances and covariances, use the calculation rules provided in Sect. 6 resulting from a linearization. In particular, remember the last equation on page 134.

Problem 1 [3 points] *Wind-momentum and luminosity*

[Note: In the following, 'logarithmic' refers to the logarithm with base 10.]

The theory of radiation driven winds predicts a linear relation between the logarithmic luminosity, $\log L'$, and the logarithmic 'modified' wind-momentum rate, $\log D'$, of a hot star ($T_{\rm eff} >$ 10,000 K), where

$$\begin{aligned} L' &= T_{\rm eff}'^4 R_*'^2 \\ D' &= \dot{M}' v_\infty' R_*'^{0.5}, \end{aligned}$$

and the primes indicate convenient normalizations (such that all quantities are dimensionless). $T_{\rm eff}$ and R_* are stellar effective temperature and radius, and $\dot{M}$ and v_∞ are the mass-loss rate and the terminal velocity of the corresponding wind, respectively. Unfortunately, typical diagnostics does not directly yield the mass-loss rate, but a related quantity,

$$Q' = \frac{\dot{M}'}{R_*'^{3/2}}.$$

Note: $Q', T_{\rm eff}', R_*', v_\infty'$ are independent quantities!

a) Which normalization quantities would be useful for the first of the above equations ($L' = \ldots$)?

b) In order to check the theoretical predictions or to use the observed relation, one has to plot $\log D'$ vs. $\log L'$, and to perform a linear regression. To obtain meaningful results and errors, the covariance between both quantities needs to be accounted for! Calculate the covariance matrix for ($\log L'$, $\log D'$) in terms of $Var(\log T_{\rm eff}')$, $Var(\log R_*')$, $Var(\log Q')$ and $Var(\log v_\infty')$.

c) In the above considerations, $Var(\log x)$ corresponds to

$$Var(\log x) = \sigma^2(\log x) \approx \big(\Delta \log(x)\big)^2 = \big(\log(x + \Delta x) - \log(x)\big)^2.$$

Show that for $\Delta x / x \ll 1$, $Var(\log x)$ can be approximated by

$$Var(\log x) \approx \big(0.43 \frac{\Delta x}{x}\big)^2.$$

Convince yourself about the validity of this approximation, by using appropriate examples (numbers).

Table 1: Parameters of HD 16 691 required to calculate M_V and error.

quantity	value	error
V	8.69	0.02
B	9.10	0.04
$(B-V)_0$	-0.28	0.03
R	3.1	0.2
d(pc)	2290	200

Problem 2 [5 points] *Absolute magnitude of a star – error propagation*

a) Suppose a function

$$f(x,y) = x + y,$$

where x and y are *not* independent. What is the corresponding squared error, $(\Delta f)^2$? Now, let $y = g(x)$ and re-calculate $(\Delta f)^2$ as a function of $(\Delta x)^2$ by determining $cov(x, g(x))$.

Show that the result is identical with the squared error $(\Delta f)^2$ when *directly* calculating this quantity for the expression

$$f(x) = x + g(x).$$

b) The absolute visual magnitude of an object, M_V, can be calculated via

$$M_V = V + 5(1 - \log_{10} d) - A_V,$$

with apparent visual magnitude, V, distance d (in pc) and visual reddening (extinction) A_V. Calculate $(\Delta M_V)^2$ as a function of ΔV, Δd and ΔA_V. Assume that all variables are independent.

c) The reddening parameter, A_V, can be obtained from the so-called color excess, E_{B-V}, and the extinction ratio, R (under most conditions, $R \approx 3.1$), via

$$A_V = R\, E_{B-V} = R\big((B-V) - (B-V)_0\big)$$

with V and B the apparent magnitudes in the visual (see above) and in the blue, respectively, and $(B-V)_0$ the *intrinsic* (i.e., actual) color of the object (from calibrations or from theory). Determine the squared error, $(\Delta A_V)^2$, as a function of ΔV, ΔB, $\Delta(B-V)_0$ and ΔR. Assume that all variables are independent.

d) Since both M_V and A_V depend on V, the error ΔM_V needs to be augmented by the corresponding covariance.

Re-calculate $(\Delta M_V)^2$ accounting for this fact, and express the latter quantity in terms of *all* potential errors. Use your knowledge from problem 1a), i.e., do *not* calculate ΔM_V from the complete expression, but use the individual errors from 1b) and 1c) + the covariance term! (Now, $V, B, d, R, (B-V)_0$ are independent variables).

e) Calculate M_V and ΔM_V for the O4If star HD 16 691 (in the PerOB1 association), for the values as quoted in Table. 1. Compare with the ΔM_V you would have obtained when forgetting about the correlation!

   What would be the error on M_V if you would only account for the uncertainty in distance?

Problem 3 [4 points] *Covariance matrix*

Assume that you can independently measure the mass, m, and velocity, v, of a particle, and that there are no systematic errors. The relative measurement errors shall be known and constant, i.e., loosely speaking, $\Delta m/m = a$ and $\Delta v/v = b$. More precisely, we assume $\sigma_m/m_m = a$ and $\sigma_v/v_m = b$ (see top of this problem set).

a) Write down the covariance matrix for m and v.

b) Calculate the corresponding (transformed) covariance matrix for the momentum p and kinetic energy E of the particle, and express the matrix elements in terms of p_m, E_m, a and b, where p_m and E_m are the 'true' values of p and E

c) Determine the (linear) correlation coefficient, $\rho(p, E)$, in dependence of a, b, and derive its value for the three cases $a = b$, $a = 0$, and $b = 0$, respectively. What do you conclude, particularly from the results for $a = 0$ and $b = 0$?

Have fun, and much success!