Statistical methods – an introduction (SS 2016)

Problem set 6

Problem 1 [6 points] *Detector efficiency*

Solve the following problems using a binomial and a Poisson-distribution, and compare the results!

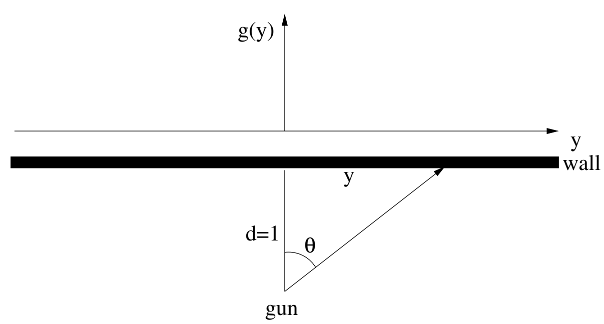
a) A detector system is 98% efficient in registering the incidence of cosmic ray particles. What is the probability that it will register all of 100 incident particules? (Hint: What is the mean number of misses?)

b) How many particles must pass the detector to have a better than even ($> 50\%$) chance that one or more particles are missed?

c) How many particles must pass the detector to have a better than 90% chance that two or more particles are missed?

Problem 2 [6 points] *Cauchy distribution*

Assume a gun in front of a long wall, at unit distance. The gun shoots at the wall, with angles θ randomly chosen from a uniform distribution within [-$\pi/2 \ldots \pi/2$], $f(x) = \frac{1}{\pi}$ for $-\pi/2 \leq x \leq \pi/2$ and $f(x) = 0$ else.



a) Determine the distribution (pdf) $g(y)$ of the holes shot in the wall, by an appropriate transformation of variables.

b) Confirm that the resulting *standard Cauchy* distribution is normalized within $[-\infty, \infty]$. (If you could not solve problem 1a), consult the literature for a definition of the Cauchy distribution).

c) Show that the Cauchy distribution has undefined variance, by calculating this quantity from the 2nd moment and the expectation value.

d) The so-called Lorentz distribution (sometimes also called Breit-Wigner distribution) used to describe the profile functions of certain spectral lines is defined as

$$\Phi(\nu)\mathrm{d}\nu = \frac{\Gamma}{4\pi^2} \frac{1}{(\nu - \nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2} \mathrm{d}\nu,$$

with frequency ν, central=transition frequency ν_0 and 'damping parameter' Γ. Calculate the corresponding distribution function for the angular frequency $\omega = 2\pi\nu$. Show that the Cauchy distribution can be generated from a Lorentzian one (or vice versa) if $y = (\omega - \omega_0)/(\Gamma/2)$.

Note (this is just a comment, nothing to do here): It is easy to show that the damping parameter Γ describes the FWHM of the distribution w.r.t. ω.

e) Show that the characteristic function of a Cauchy distribution (according to problem a)) is $\Phi_y(t) = \exp(-|t|)$.

Use this result to prove that the arithmetic mean of Cauchy-distributed variates,

$$\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i$$

is also Cauchy-distributed.

Note: At first glance, this result seems to contradict the Central Limit Theorem; however, the CLT can be only applied for distributions with well-defined expectation value and variance!!!

Have fun, and much success!